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Kyoto University
A CHARACTERISTIC CAUCHY PROBLEM OF NON-LERAY TYPE IN THE COMPLEX DOMAIN

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§0. Introduction

We consider a Cauchy problem in the complex domain. It is assumed to be a character-istic problem in the sense that the characteristic points form a submanifold $T$ (of codimension 1) of the initial hypersurface $S$.

Since Leray, the studies on this subject dealt with the cases where the solution is singular on a characteristic hypersurface tangent to $S$ along $T$. See [L], [G-K-L], [H], [D], [O-Y] and [Y].

In the present paper, we consider a totally different situation: all the characteristic hypersurfaces issuing from $T$ are transversal with $S$.

First we give two examples to show that in this kind of characteristic Cauchy problem, the solution can be singular on the above-mentioned characteristic hypersurfaces even when all the Cauchy data are regular. Next, we consider a (ramified) Cauchy problem for a certain class of operators including the examples. We perform a singular change of coordinates and reduce our problem to results of Wagschal.

§1. Examples with holomorphic data

In a neighborhood of the origin of $\mathbb{C}_{t} \times \mathbb{C}_{x} \times \mathbb{C}_{z}$, let us consider Cauchy problems for the operators $Q_1$ and $Q_2$ defined by

$$Q_1 = (xD_t + tD_x)D_t, \quad Q_2 = Q_1 - xt^2D_z^2.$$
We are going to solve, for $j = 1$ or $2$,

$$
\begin{align*}
Q_j u(t, x, z) &= 0 \\
u|_S &= -\frac{\pi i}{2} x^2 \\
D_t u|_S &= i x \\
S &= \{ t = 0 \}
\end{align*}
$$

On the initial hypersurface $S$, the characteristic points form a submanifold $T = \{ t = x = 0 \}$. The hypersurfaces $\{ x = 0 \}$, $\{ x = t \}$ and $\{ x = -t \}$ are characteristic hypersurfaces issuing from $T$. They are transversal with $S$. Although the data are holomorphic in a neighborhood of the origin, the solution $u$ is singular on the three characteristic hypersurfaces. In fact, we have

$$u = \frac{x^2}{2} \left( \frac{t}{x} \sqrt{\left( \frac{t}{x} \right)^2 - 1} - \log \left( \frac{t}{x} + \sqrt{\left( \frac{t}{x} \right)^2 - 1} \right) \right).$$

Since we are dealing with a multi-valued function, we have to clarify the definition of the restriction on $S$. Its precise meaning is that we choose a point $p$ of $S$ and that the initial condition is satisfied by the germ of $u$ at $p$.

We will prove for a class of operators including $Q_1$ and $Q_2$ that the singular support of the solution is contained in this kind of characteristic hypersurfaces when the data are arbitrary holomorphic functions. As a matter of fact, we can generalize this result to the case of ramified data.

**§2. Main result**

In a neighborhood of the origin of $\mathbb{C}_t \times \mathbb{C}_y \times \mathbb{C}_z^n$, let us consider a second order operator $P(t, y, z; D_t, D_y, D_z)$ with holomorphic coefficients whose principal symbol $\sigma(P)$ is factorized into the form

$$\sigma(P)(t, y, z; \tau, \eta, \zeta) = \prod_{i=0,1} \left( \tau - \lambda_i(t, y, z; \eta, \zeta) \right),$$

where $\tau$, $\eta$ and $\zeta = (\zeta_1, \ldots, \zeta_n)$ are the dual variables of $t$, $y$ and $z$ respectively.

We assume the following two conditions (1) and (2).

$$
\begin{align*}
\lambda_0(t, 0, z; 1, 0, \ldots, 0) &= 0 \\
\lambda_1(t, y, z; 1, 0, \ldots, 0) &= -qt^{q-1}, \\
q & \text{is an integer } \geq 2.
\end{align*}
$$
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(2) For \( i = 0,1 \), the function \( (\eta, \zeta) \mapsto \lambda_i(t, y, z; \eta, \zeta) \) is linear.

The most simple example is

\[ \lambda_0 = 0 \text{ or } y \eta, \quad \lambda_1 = -qt^{q-1} \eta. \]

Now we consider, in a neighborhood of the origin of \( \mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^n \), an operator \( Q \) with holomorphic coefficients defined by

\[ Q(t, x, z; D_t, D_x, D_z) = x^{2q-1} P(t, x^q, z; D_t, D_x) \frac{1}{q^{x^{q-1}} D_x, D_z}. \]

Sometimes the exponent \( 2q - 1 \) is larger than necessary to erase negative powers of \( x \). For example, if

\[ P(t, y, z; D_t, D_y, D_z) = P(t, y; D_t, D_y) = (D_t + qt^{q-1} D_y) D_t, \]

then

\[ x^{q-1} P(t, x^q; D_x) \frac{1}{q^{x^{q-1}}} D_x) = (x^{q-1} D_t + t^{q-1} D_x) D_t. \]

When \( q = 2 \), this is nothing but \( Q_1 \) which we studied before.

For the purpose of formulating a Cauchy problem, put \( S = \{ t = 0 \} \subset \mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^n \), which is the initial hypersurface. It is easy to see that \( T = \{ t = x = 0 \} \) is formed by the characteristic points of \( Q \) on \( S \). By the condition (1), the hypersurfaces

\[ K_j = \{ x = \exp(j \frac{2\pi i}{q}) \cdot t \} \quad (j = 0, \ldots, q - 1), \quad K_q = \{ x = 0 \} \]

are characteristic hypersurfaces of \( Q \) issuing from \( T \).

We then consider a ramified characteristic Cauchy problem in an open connected neighborhood \( \Omega \) of the origin of \( \mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^n \):

\[ \left\{ \begin{array}{l}
Q(t, x, z; D_t, D_x, D_z) u(t, x, z) = 0, \\
D^h_t u(t, x, z) |_{S} = w_h(x, z), \quad h = 0, 1.
\end{array} \right. \]

Here we assume that there exists a point \( p \in \Omega \cap (S \setminus T) \) such that for \( h = 0, 1 \), the function \( w_h \) is holomorphic in a neighborhood (relative to \( S \)) of the point \( p \) and can be analytically continued along all the paths from \( p \) in \( \Omega \cap (S \setminus T) \) (that is, \( w_h \) is holomorphic in the universal covering space of \( \Omega \cap (S \setminus T) \)).

Since \( p \not\in T \), the usual Cauchy-Kowalevski theorem is valid there. (CP) admits a unique holomorphic solution \( u \) in a neighborhood of the point \( p \).

We are going to prove the
Theorem 1.

There exists an open connected neighborhood $\Omega'$ of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^n$ such that the solution $u$ of (CP) can be analytically continued to the universal covering space of $\Omega' \setminus \cup_{j=0}^{q} K_j$.

Of course this conclusion holds true when all the data are regular.

Proof.

Put $x = y^{1/q}$. Then $D_y = \frac{1}{qx^{q-1}}D_x$. Therefore

$$Q(t, x, z; D_t, D_x, D_z) = y^{\frac{2q-1}{q}} P(t, y, z; D_t, D_y, D_z).$$

We reduce (CP) to the following noncharacteristic ramified Cauchy problem, which has been solved by Wagschal in [W2].

$$(CP')\left\{\begin{array}{l}
P(t, y, z; D_t, D_y, D_z)u(t, y^{1/q}, z) = 0, \\
D^h_t u(t, y^{1/q}, z)|_{t=0} = w_h(y^{1/q}, z), \quad h = 0, 1.
\end{array}\right.$$\]

The function $w_h(y^{1/q}, z)$ is holomorphic in the universal covering space of $\{(y, z) \in \mathbb{C} \times \mathbb{C}^n; 0 < |y| \ll 1, |z| \ll 1\}$. ($a \ll 1$ means that $a \geq 0$ is sufficiently small).

Let $p' \in (\{0\} \times \mathbb{C}_y \times \mathbb{C}_z^n) \setminus \{y = 0\}$ be the point corresponding to $p$. Then (CP') admits a unique holomorphic solution $u(t, y^{1/q}, z)$ near $p'$. According to [W2], $u(t, y^{1/q}, z)$ can be analytically continued to the universal covering space of

$$\{(t, y, z) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n; |(t, y, z)| \ll 1\} \setminus (\{y = 0\} \cup \{y = t^q\}).$$

We finish the proof by coming back to the $(t, x, z)$-space. \hfill \Box

Example.

We saw before that $Q_1$ was not quite the same as $Q$, but this does not cause any difficulty. The equation $Q_1u = 0$ is equivalent to $x^2Q_1u = 0$. The operator $x^2Q_1$ is nothing but $Q$.

This example suggests that the choice of the exponent of $x$ in the definition of $Q$ is not essential.

Remark 1.
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For convenience, put \( y = z_0, \eta = \zeta_0 \). Then, by virtue of Remarque 3.1 of [W2], (2) can be replaced by the following condition:

(3) There exists an integer \( k, 0 \leq k \leq n \), such that for \( i = 0,1 \), the function \( \lambda_i(t, z_0, \zeta_0, \ldots, \zeta_k, 0, \ldots, 0) \) is linear in \((\zeta_0, \ldots, \zeta_k)\) and does not depend on the variables \((z_{k+1}, \ldots, z_n)\).

This enables us to treat \( Q_2 \). In fact, when \( n = 1, q = 2 \), put

\[
P(t, y, z; D_t, D_y, D_z) = D_t^2 + 2tD_tD_y - t^2D_z^2.
\]

Then

\[
\sigma(P) = \tau^2 + 2t\tau \eta - t^2\zeta^2
\]

\[
= (\tau + t\eta)^2 - t^2(\eta^2 + \zeta^2)
\]

\[
= \{\tau + t(\eta + \sqrt{\eta^2 + \zeta^2})\}(\tau + t(\eta - \sqrt{\eta^2 + \zeta^2})\}
\]

\[
Q_2 = xP(t, x^2, z; D_t, \frac{1}{2x}D_x, D_z).
\]

**Remark 2.**

A singular change of coordinates was useful in some papers mentioned in the introduction ([L], [D], [O-Y] and [Y]). One introduces a new variable \( w \) by setting \( w = (t - x^l)^{1/l} \) for some positive integer \( l \). In the present paper, we have performed a different kind of singular change of coordinates.

§3. Inhomogeneous problem

If we choose a special class of \( P \), we can treat an inhomogeneous problem. Assume that

\[
\sigma(P)(t, y, z; \tau, \eta, \zeta) = \tau(\tau + qt^{q-1}\eta).
\]

We employ the same notation as in §2. Let us consider:

\[
(\text{CP}^i)\left\{ \begin{array}{l}
Q(t, x, z; D_t, D_x, D_z)u(t, x, z) = v(t, x, z), \\
D_th u(t, x, z)|_\delta = w_h(x, z), \quad h = 0,1.
\end{array} \right.
\]

Here we assume that the function \( v \) is holomorphic in a neighborhood of \( p \) and can be analytically continued along all the paths from \( p \) in \( \Omega \backslash \bigcup_{j=0}^{q} K_j \) (that is, \( v \) is holomorphic in the universal covering space of \( \Omega \backslash \bigcup_{j=0}^{q} K_j \)). Then we have
Theorem 2.

There exists an open connected neighborhood $\Omega'$ of the origin of $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^n$ such that the solution $u$ of $(CP^i)$ can be analytically continued to the universal covering space of $\Omega' \setminus \cup_{j=0}^{q} K_j$.

Of course this conclusion holds true when all the data are regular.

Proof.

We have to solve

\[
\begin{align*}
P(t, y, z; D_t, D_y, D_z)u(t, y^{1/q}, z) &= y^{-\frac{2q-1}{q}} v(t, y^{1/q}, z), \\
P &= D_t(D_t + qt^{q-1}D_y) + \text{lower}, \\
D_t^hu(t, y^{1/q}, z)|_{t=0} &= w_h(y^{1/q}, z), \quad h = 0, 1.
\end{align*}
\]

Since $v(t, x, z)$ is holomorphic in the universal covering space of $\Omega \setminus \cup_{j=0}^{q} K_j$, the function $y^{-\frac{2q-1}{q}} v(t, y^{1/q}, z)$ is holomorphic in the universal covering space of

\[
\{(t, y, z) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n; |(t, y, z)| \ll 1\} \setminus (\{y=0\} \cup \{y=t^q\}).
\]

This noncharacteristic inhomogeneous problem has been solved in [W1]. \qed

§4. Geometry

What distinguishes the present study from conventional ones is the absence of singularities on a hypersurface tangent to the initial hypersurface $S$. It is explained by the following

Proposition.

Under the assumption (1), there is no characteristic hypersurface of $Q$ that is tangent to $S$ along $T$.

Proof.

We have

\[
\sigma(Q)(t, x, z; \tau, \xi, \zeta) = x^{q-1} \prod_{i=0,1} \{\tau - \lambda_i(t, x^q, z; \frac{1}{q} \xi, x^q \zeta)\}
\]

\[
= x \prod_{i=0,1} \{x^{q-1} \tau - \lambda_i(t, x^q, z; \frac{1}{q} \xi, x^q \zeta)\}.
\]
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It is easy to see that $S$ itself is not a characteristic hypersurface. A hypersurface $\neq S$ which is tangent to $S$ along $T$ has an expression of the form:

\[ \varphi = t + x^N \psi(x, z) = 0, \quad N \geq 2 \]

where $\psi$ is a holomorphic function with $\psi(0, z) \neq 0$.

We have

\[
\sigma(Q)(t, x, z; \text{grad } \varphi) = x \prod_{i=0,1} [x^{q-1} - \frac{1}{q} N x^{N-1} \psi(x, z) + \frac{1}{q} x^N D_x \psi, x^{N+q-1} D_z \psi)].
\]

For a generic $z$ we have $\psi(0, z) \neq 0$. We fix such a $z$. Obviously $\psi(x, z) \neq 0$ holds if $|x| \ll 1$. Then it follows that

\[
\sigma(Q)(t, x, z; \text{grad } \varphi) = x \prod_{i=0,1} [x^{q-1} - \frac{1}{q} N x^{N-1} \psi(1 + \frac{x}{N \psi} D_x \psi) \lambda_i(t, x^q, z; 1 + (1 + \frac{x}{N \psi} D_x \psi)^{-1} \frac{q x^q}{N \psi} D_z \psi)].
\]

The assumption (1) implies that as $x$ tends to zero

\[
\lambda_0(t, x^q, z; 1 + \frac{x}{N \psi} D_x \psi)^{-1} \frac{q x^q}{N \psi} D_z \psi) = O(x^q)
\]

\[
\lambda_1(t, x^q, z; 1 + \frac{x}{N \psi} D_x \psi)^{-1} \frac{q x^q}{N \psi} D_z \psi) = -q t^{q-1} + O(x^q).
\]

Therefore by restricting them on the hypersurface $\{ \varphi = 0 \}$, we obtain

\[
\lambda_0(t, x^q, z; 1 + \frac{x}{N \psi} D_x \psi)^{-1} \frac{q x^q}{N \psi} D_z \psi)|_{\varphi=0} = O(x^q)
\]

\[
\lambda_1(t, x^q, z; 1 + \frac{x}{N \psi} D_x \psi)^{-1} \frac{q x^q}{N \psi} D_z \psi)|_{\varphi=0} = -q(-x N \psi)^{q-1} + O(x^q) = O(x^q).
\]

Hence $\sigma(Q)|_{\varphi=0}$ is different from zero if $0 < |x| \ll 1$. Thus $\{ \varphi = 0 \}$ is not a characteristic hypersurface. $\square$
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References


