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MOURRE Theory for Time-Periodic Systems

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We consider the following Schrödinger equation with time-dependent Hamiltonian on $\mathbb{R}^{\nu}$,

(1) \[ i \frac{\partial}{\partial t} u(t, x) = H(t)u(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^{\nu}, \]

(2) \[ H(t) = -\Delta_x + V(t), \]

where $V(t)$ is a multiplicative operator by a function $V(t, x)$ which is periodic in $t$ with period $2\pi$:

(3) \[ V(t + 2\pi, x) = V(t, x). \]

As is well-known, with some suitable conditions on $V(t, x)$, $H(t)$ generates a unique unitary propagator $\{U_1(t, s)\}_{-\infty < s < \infty}$. For $H_0 = -\Delta_x$, the associated unitary propagator is denoted by $U_0(t, s) = e^{-i(t-s)H_0}$. A traditional way to study the temporal asymptotics as $t \to \pm \infty$ of $U_1(t, s)$ is to introduce an operator $K = -i \frac{d}{dt} + H(t)$ on $\mathbb{T} \times \mathbb{R}^{\nu}$, where $\mathbb{T} = \mathbb{R}/2\pi \mathbb{Z}$, and to investigate the asymptotic behavior of $e^{-i\sigma K}$. They are mutually related through the following formula

(4) \[ (e^{-i\sigma K} f)(t, x) = (U_1(t, t-\sigma) f(t-\sigma, \cdot))(x), \quad f \in \mathbb{H} = L^2(\mathbb{T} \times \mathbb{R}^{\nu}). \]

Let

(5) \[ K_0 = -i \frac{d}{dt} + H_0. \]

Definition 1 (conjugate operator).

(6) \[ A = \frac{1}{2}(L_D \cdot x + x \cdot L_D) \]

where \( D_x = \frac{i}{\imath} \nabla_x \) and $L_D = (L_j)_{1 \leq j \leq \nu}$ with $L_j = D_{x_j} < D_x >^{-2}$.

The following assumption is imposed on $V(t)$.

Assumption 1. Let $V$ be the operator of multiplication by the function $V(t, x)$ on $\mathbb{H}$. We assume that

(i) $V$, $[V, A]$ are extended to $K_0$-compact operators.

(ii) $[[V, A], A]$ is extended to a $K_0$-bounded operator.
We denote the extension of the form $[K, A]$ as $[K, A]^0$.

**Theorem 1.** Suppose Assumption 1 is satisfied. For $\lambda \in \mathbb{R} \setminus \mathbb{Z}$, let $d(\lambda, \mathbb{Z})$ denote the distance from $\lambda$ to $\mathbb{Z}$. Then, Eigenvalues of $K$ (the set of which are denoted by $\sigma_{pp}(K)$) are discrete with possible accumulation points in $\mathbb{Z}$. If $\lambda \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$, for each $\epsilon > 0$ there exists $0 < \delta < d(\lambda, \mathbb{Z})$ such that

$$f(K)i[K, A]^0f(K) \geq (\frac{2d(I, Z)}{d(I, Z) + 1} - \epsilon)f(K)^2$$

for all $f \in C_0^\infty([\lambda - \delta, \lambda + \delta])$.

Let $\mathfrak{B}(\mathbb{H})$ be the set of bounded operators on $\mathbb{H}$.

**Theorem 2.** Suppose $\alpha > 1/2$.

(i) For each closed interval $I \subset \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$ the following inequalities hold:

$$\sup_{\text{Im}\,z \neq 0, \text{Re}z \in I} \|< x >^{-\alpha} (K - z)^{-1} < x >^{-\alpha} \|_{\mathfrak{B}(\mathbb{H})} < \infty.$$  

(ii) There exist the norm limits in $\mathfrak{B}(\mathbb{H})$.

$$\lim_{\text{Im}z \to 0, \text{Re}z \in I} < x >^{-\alpha} (K - z)^{-1} < x >^{-\alpha}$$

is H"older continuous with respect to $\lambda \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$.

Next we proceed to the propagation estimates. We need the following stronger assumption on the potential.

**Assumption 2.** There exists $\delta_0 > 0$ such that

$$\sup_{\text{Im}z \neq 0, \text{Re}z \in I} \|< x >^{-\alpha} (K - z)^{-1} < x >^{-\alpha} \|_{\mathfrak{B}(\mathbb{H})} < \infty.$$  

**Theorem 3.** Suppose Assumption 2 is satisfied. Let $E \in \mathbb{R} \setminus (\mathbb{Z} \cup \sigma_{pp}(K))$, and $\epsilon > 0$ be given. Then there exists a small open interval $I$ containing $E$ such that for any $f \in C_0^\infty(I)$ and $s' > s > 0$,

$$\|\chi(\frac{|x|^2}{4\sigma^2} - \frac{d(I, Z)}{d(I, Z) + 1} - \epsilon)e^{-i\sigma K}f(K) < x >^{-s'} \|_{\mathfrak{B}(\mathbb{H})} = O(\sigma^{-s}) \quad \text{as} \quad \sigma \to \infty$$

where $\chi(x < a)$ denotes the characteristic function of the interval $(-\infty, a)$. 