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対称性の入った順序保存力学系における安定性解析
(Stability analysis in order-preserving systems in the presence of symmetry)

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1. Introduction

This note is a summary of my recent work [8] with Professor Hiroshi Matano (University of Tokyo).

Many mathematical models in physics, biology or other fields possess some kind of symmetry, such as symmetry with respect to reflection, rotation, translation, dilation, gauge transformation, and so on. Given an equation with certain symmetry, it is important, from the point of view of applications, to study whether or not its solutions inherit the same type of symmetry as the equation. As is well-known, the answer is generally negative unless we impose additional conditions on the equation or on the solutions. We will henceforth restrict our attention to solutions that are 'stable' in a certain sense and discuss the relation between stability and symmetry, or stability and some kind of monotonicity.

In the area of nonlinear diffusion equations or heat equations, one of the early studies in this direction can be found in Casten-Holland [1], and Matano [6]. Among many other things, they showed that if a bounded domain Ω is rotationally symmetric then any stable equilibrium solution of a semilinear diffusion equation

$$u_t = \Delta u + f(u), \qquad x \in \Omega,\ t > 0$$

inherits the same symmetry as that of Ω. Later, it was discovered that the same result holds in a much more general framework, namely in the class of equations in which the comparison principle holds in a certain strong sense. Such a class of equations form the so-called 'strongly order-preserving dynamical systems'. Mierczyński–Poláčik [9] (for the time-continuous case) and Takáč [10] (for time-discrete case) showed that, in a strongly order-preserving dynamical system having a symmetry property corresponding to the action of a compact connected group G, any stable equilibrium point or stable periodic point is G-invariant.

The aim of this note is to establish a theory analogous to [9] and [10] for a wider class of systems. To be more precise, we will relax the requirement that the dynamical system be strongly order-preserving. This will allows us to deal with degenerate diffusion equations and equations on an unbounded domain. Secondly, we will relax the requirement that the acting group be compact. This will allow us to discuss the symmetry or monotonicity properties with respect to translation. The result will then be applied to the stability analysis of travelling waves of reaction-diffusion equations and that of stationary solutions of an evolution equation of surfaces.

2. Notation and main results

Let X be an ordered complete metric space, that is, a complete metric space with a closed partial order relation denoted by $\preceq$. Here, we say that a partial order relation in X is closed if any converging sequences $\{u_n\}$, $\{v_n\} \subset X$ with $u_n \preceq v_n$ for each $n \in \mathbb{N}$ satisfy $\lim_{n\to\infty} u_n \preceq \lim_{n\to\infty} v_n$. We also assume that, for any u, $v \in X$, the greatest lower bound of $\{u, v\}$ —denoted by $u \wedge v$— exists and that $(u, v) \mapsto u \wedge v$ is a continuous mapping from $X \times X$ into X. We write $u \prec v$ if $u \preceq v$ and $u \neq v$, and denote by d the metric of X.

Let $\{\Phi_t\}_{t\geq 0}$ be a semigroup of mappings Φ_t from X to X satisfying the following conditions (Φ1), (Φ2), (Φ3):

(Φ1) Φ_t is order-preserving (that is, $u \preceq v$ implies $\Phi_t u \preceq \Phi_t v$ for all $u, v \in X$) for all $t \geq 0$,

(Φ2) Φ_t is upper semicontinuous (that is, if a sequence $\{u_n\}$ in X converges to a point $u_\infty \in X$ and if the corresponding sequence $\{\Phi_t u_n\}$ also converges to some point $w \in X$, then $w \preceq \Phi_t(u_\infty))$ for all $t \geq 0$,

(Φ3) any bounded monotone decreasing orbit (a bounded orbit $\{\Phi_t u\}_{t\geq 0}$ satisfying $\Phi_t u \succeq \Phi_{t'} u$ for $t \leq t'$) is relatively compact.

Let G be a metrizable topological group acting on X. We say G acts on X if there exists a continuous mapping γ: $G \times X \to X$ such that $g \mapsto \gamma(g, \cdot)$ is a group homomorphism of G into $Hom(X)$, the group of homeomorphisms of X onto itself. For brevity, we write $\gamma(g, u) = gu$ and identify the element $g \in G$ with its action $\gamma(g, \cdot)$. We assume that

(G1) γ is order-preserving (that is, $u \preceq v$ implies $gu \preceq gv$ for any $g \in G$),

(G2) γ commutes with Φ_t (that is, $g\Phi_t(u) = \Phi_t(gu)$ for all $g \in G$, $u \in X$) for all $t \geq 0$.

(G3) G is connected.

In what follows, e will denote the unit element of G, and $B_\delta(e)$ the δ-neighborhood of e.

Definition 2.1. An equilibrium point $u \in X$ of $\{\Phi_t\}_{t\geq 0}$ is *lower stable* if, for any $\epsilon > 0$, there exists some $\delta > 0$ such that

$$d(\Phi_t v, u) < \epsilon$$

for any $t \geq 0$, $v \in X$ satisfying $v \preceq u$ and $d(v, u) < \delta$.

Remark 2.2. It is easily seen that if u is stable in the sense of Ljapunov, then it is lower stable.

Main Theorem. *Let $\overline{u}$ be an equilibrium point of $\{\Phi_t\}_{t\geq 0}$ satisfying the following conditions*: (1) *$\overline{u}$ is lower stable*; (2) *for any equilibrium point $u \prec \overline{u}$, there exists some $\delta > 0$ such that $gu \prec \overline{u}$ for any $g \in B_\delta(e)$. Then, for any $g \in G$, the inequality $g\overline{u} \succeq \overline{u}$ or $g\overline{u} \preceq \overline{u}$ holds.*

If the group G is compact, one can easily show that the inequality $g\overline{u} \succ \overline{u}$ or $g\overline{u} \prec \overline{u}$ never holds (see Takač [20]). Thus we have the following corollary.

Corollary 2.3. *Under the hypotheses of Main Theorem, assume further that G is a compact group. Then $\overline{u}$ is G-invariant, that is, $\overline{u}$ is symmetric.*

Now let us consider the case where G is isomorphic to the additive group $\mathbb{R}$:

$$G = \{g_a \mid a \in \mathbb{R}\}, \quad g_a + g_b = g_{a+b}.$$

Then the following holds:

Corollary 2.4. *Under the hypotheses of Main Theorem, assume further that G is isomorphic to $\mathbb{R}$. Then one of the following holds:*

(i) $\overline{u}$ *is G-invariant;*

(ii) $g_a\overline{u}$ *is strictly monotone increasing in a ($a < b$ implies $g_a\overline{u} \prec g_b\overline{u}$);*

(iii) $g_a\overline{u}$ *is strictly monotone decreasing in a ($a < b$ implies $g_a\overline{u} \succ g_b\overline{u}$).*

Remark 2.5. If the mapping Φ_t is strongly order-preserving for some $t > 0$ (that is, $u \prec v$ implies $\Phi_t B_\delta(u) \preceq \Phi_t B_\delta(v)$ for sufficiently small $\delta > 0$), then clearly the assumption (2) in Main Theorem is automatically fulfilled.

Remark 2.6. If G is not connected, then the conclusion of Main Theorem does not necessarily hold. See [15], [16] for detail.

3. Applications —Rotational symmetry of stable equilibria

First we consider an initial boundary value problem for a nonlinear diffusion equation of the form

$$\begin{cases} u_t = \Delta(u^m) + f(u), & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) = u_0, & x \in \Omega, \end{cases} \tag{3.1}$$

where $m \geq 1$ is a constant, and the domain $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. We assume that $f\colon [0,\infty) \to \mathbb{R}$ is a C^1 function satisfying $f(0) = 0$, $f'(0) \neq 0$. In the case of $m > 1$, we consider only bounded nonnegative solutions.

Given an equilibrium solution $\overline{u}$ of (3.1), we set

$$X = \begin{cases} C_0(\overline{\Omega}) = \{w \in C(\overline{\Omega}) \mid w = 0 \text{ on } \partial\Omega\} & \text{if } m = 1, \\ \{u \in L^1(\Omega) \mid \text{for some } g \in G,\ 0 \leq u(x) \leq \overline{u}(gx) \text{ a.e. } x \in \Omega\} & \text{if } m > 1. \end{cases}$$

The following theorem holds:

Theorem 3.1. *If a bounded domain Ω is rotationally symmetric then any stable equilibrium solution of* (3.1) *is rotationally symmetric.*

Outline of the proof. Define an order relation in X by

$$u_1 \preceq u_2 \quad \text{if} \quad u_1(x) \leq u_2(x) \text{ a.e. } x \in \Omega.$$

Then, letting G be the rotation group and applying Corollary 2.3, we obtain this theorem. □

The result in Theorem 3.1 has been already known for the case where $m = 1$, namely, for the problem

$$\begin{cases} u_t = \Delta u + f(u), & x \in \Omega,\ t > 0, \\ u = 0, & x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) = u_0, & x \in \Omega, \end{cases} \tag{3.2}$$

([2], [15]). Our theory in Section 2 is also applicable to the case where $m > 1$. Furthermore, our theory allows us to treat the case where Ω is not bounded. To be more precise, under the additional condition that $f'(0) < 0$, we obtain the following:

Theorem 3.2. *If an unbounded domain Ω is rotationally symmetric, then any stable equilibrium solution $\overline{u}$ of* (3.2) *satisfying*

$$\overline{u}(x) \to 0 \quad \text{as} \quad |x| \to \infty.$$

is rotationally symmetric.

By the same argument as in the proof of Theorem 3.1, we obtain the above theorem. Here we set $X = C_0(\overline{\Omega})$.

4. Applications —Instability of solitary waves

We apply our theory to the so-called travelling waves for systems of equations of the form

$$\begin{cases} u_t = \quad u_{xx} + f(u, v), & x \in \mathbb{R},\ t > 0, \\ v_t = \quad dv_{xx} + g(u, v), & x \in \mathbb{R},\ t > 0, \end{cases} \tag{4.1}$$

where $d > 0$ is a constant. Here $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that there exists some $M_0 > 0$ satisfying

$$|f_u(u,p)| < M_0, \quad |f_p(u,p)| < M_0$$

for all $u, p \in \mathbb{R}$, and $g\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is also a C^1 function satisfying precisely the same condition.

Here we assume $f_v \leq 0$, $g_u \leq 0$ so that the system (4.1) be of competition type.

A solution (u,v) of (4.1) is called a travelling wave with the speed $c \in \mathbb{R}$ if it can be written in the form

$$(u(x,t), v(x,t)) = (\phi(x-ct), \psi(x-ct)),$$

where $\phi(y)$, $\psi(y)$ are some functions. Here we restrict our attention to the travelling waves that satisfy the condition

$$\lim_{x\to\pm\infty} (u(x,0), v(x,0)) = (u_\pm, v_\pm),$$

where u_+, u_-, v_+ and v_- are constants. A travelling wave is called a solitary wave (a travelling pulse) if $(u_+, v_+) = (u_-, v_-)$, and a travelling front if $(u_+, v_+) \neq (u_-, v_-)$. We assume that $(u_\pm, v_\pm)$ are both stable equilibrium solutions of the ordinary differential equation corresponding to (4.1), namely,

$$\begin{cases} u_t = & f(u,v), \qquad t > 0, \\ v_t = & g(u,v), \qquad t > 0. \end{cases}$$

Given a travelling wave $(\overline{u}, \overline{v})$ with the speed c, let us define a metric space X by

$$X = \{(\overline{u}(\cdot,0) + w_1, \overline{v}(\cdot,0) + w_2) \mid w_1, w_2 \in H^1(\mathbb{R})\}.$$

Then, define a semigroup of mappings $\{\Phi_t\}_{t\geq 0}$ on X by

$$\Phi_t(u(x), v(x)) = \Psi_t(u(x+ct), v(x+ct))$$

with $\{\Psi_t\}_{t\geq 0}$ being the semiflow that equation (4.1) defines in X. It is easily seen that $\{\Phi_t\}_{t\geq 0}$ is the semiflow defined by the equation

$$\begin{cases} u_t = & u_{xx} + cu_x + f(u,v), \qquad x \in \mathbb{R},\ t > 0, \\ v_t = & dv_{xx} + cv_x + g(u,v), \qquad x \in \mathbb{R},\ t > 0. \end{cases}$$

Clearly $(\overline{u}(\cdot,0),\overline{v}(\cdot,0))$ is an equilibrium point of the system $\{\Phi_t\}_{t\geq 0}$. A travelling wave $(\overline{u},\overline{v})$ is called stable if $(\overline{u}(\cdot,0),\overline{v}(\cdot,0))$ is a stable equilibrium point of $\{\Phi_t\}_{t\geq 0}$.

We say that a travelling wave (u,v) is *monotone* if $u(x,0)$ and $-v(x,0)$ are both nonincreasing functions or both nondecreasing functions.

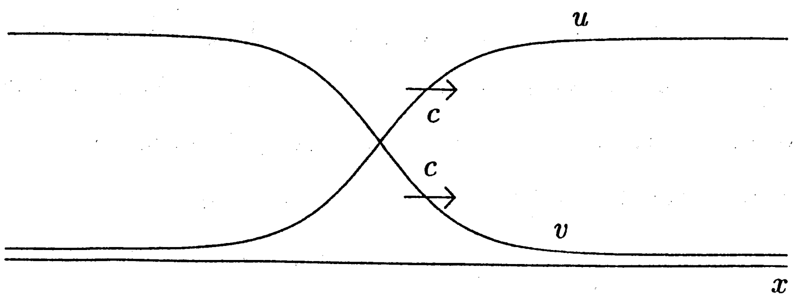
Theorem 4.1. *Any stable travelling wave of* (4.1) *is monotone.*

Corollary 4.2. *Solitary waves of* (4.1) *are unstable.*

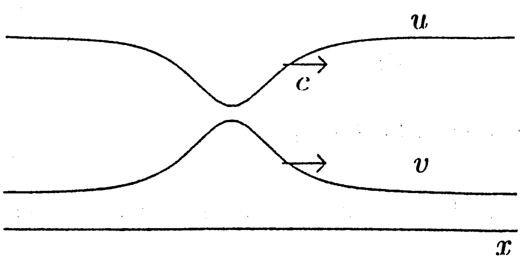
Outline of the proof of Theorem 4.1. Define an order relation in X by

$$(u_1,v_1)\preceq(u_2,v_2)\quad\text{if}\quad u_1(x)\leq u_2(x),\ \ v_1(x)\geq v_2(x)\ \text{a.e. } x\in\mathbb{R}.$$

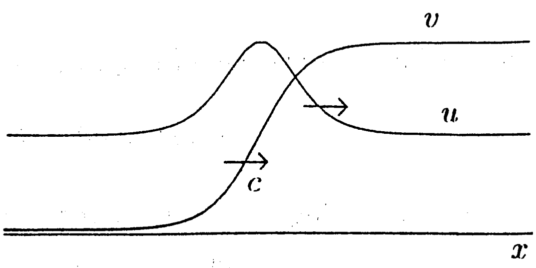
Letting G be the group of translations ($\cong\mathbb{R}$) and applying Corollary 2.4, we obtain this theorem. □



(monotone travelling wave)



(solitary wave)



(another type of travelling wave)

5. Applications —Instability of stationary surfaces

Let $\{\gamma(t)\}_{t\geq 0}$ be a family of time-dependent hypersurfaces embedded in $\mathbb{R}^n$. We assume that the motion of $\gamma(t)$ is subject to

$$V = f(\mathbf{n}, \nabla \mathbf{n}), \tag{5.1}$$

where $\mathbf{n} = \mathbf{n(x,t)}$ is the outward unit normal vector at each point of $\gamma(t)$ and V denotes the normal velocity of $\gamma(t)$ in the outward direction. A typical example of (5.1) is

$$V = \alpha(\mathbf{n})\kappa + \mathbf{g}(\mathbf{n})$$

where $\kappa = (1/(n-1))\,\mathrm{trace}\nabla\mathbf{n}$ is the mean curvature at each point of $\gamma(t)$. In the case where $\alpha(\mathbf{n}) \equiv \mathbf{1}$ and $g(\mathbf{n}) \equiv \mathbf{0}$, this equation is known as the mean curvature flow equation.

We consider (5.1) in the framework of generalized solutions. The notion of such solutions was introduced by Evans and Spruck [4] and independently by Chen, Giga and Goto [2].

We assume that f is a smooth function and that the equation (5.1) is strictly parabolic.

Let us define a metric space X by

$$X = \left\{ (\Gamma, D) \;\middle|\; \begin{array}{l} D \text{ is a bounded open set in } \mathbb{R}^n \text{ and} \\ \Gamma(\subset \mathbb{R}^n \setminus D) \text{ is a compact set containing } \partial D \end{array} \right\}$$

equipped with the metric d defined by

$$d((\Gamma, D), (\Gamma', D')) = h(\Gamma, \Gamma') + h(D \cup \Gamma, D' \cup \Gamma').$$

Here, for compact sets K_1 and K_2, $h(K_1, K_2)$ means the Hausdorff metric between K_1 and K_2 if $K_1, K_2 \neq \emptyset$, $h(K_1, K_2) = \infty$ if $K_1 \neq \emptyset$ and $K_2 = \emptyset$, and $h(K_1, K_2) = 0$ if $K_1, K_2 = \emptyset$. Then, define a mapping Φ_t on X by

$$\Phi_t(\Gamma, D) = (\Gamma_t, D_t),$$

where $(\Gamma_t, D_t)_{t\geq 0}$ is a solution of (5.1) with the initial data $(\Gamma_0, D_0) = (\Gamma, D)$.

In this note, we will call a family of surfaces $\{\gamma(t)\}_{t\geq 0}$ *compact* if $\gamma(t)$ is a compact for each $t \geq 0$, and *smooth* if $\gamma(t)$ is a smooth hypersurface for each $t \geq 0$.

Theorem 5.1. *Any smooth compact stationary surface is unstable.*

Outline of the proof. Define an order relation in X by

$$(\Gamma_1, D_1) \preceq (\Gamma_2, D_2) \quad \text{if} \quad D_1 \subset D_2 \text{ and } D_1 \cup \Gamma_1 \subset D_2 \cup \Gamma_2.$$

Letting G be the group of translations and applying Main Theorem, we obtain this theorem. □

Remark 5.2. Giga and Yama-uchi [5], Ei and Yanagida [3] have shown the above result by using methods different from ours. However, our arguments are much simpler and give deeper perspective than their methods. Furthermore, unlike the methods in theirs, which depend on linearization arguments or distant function arguments (thus smoothness assumptions are essential), our method may be extendable to generalized solutions of (5.1) if one can check the condition (2) of Main Theorem holds for generalized solutions (which remains to be checked).

Remark 5.3. With minor modifications, most of the results in Section 2 carry over to time-discrete systems. Thus the results in Theorems 3.1–5.1 can be extended to nonautonomous equations (equations that are periodic in t). For example, an analogy of Theorem 5.1 holds for periodic solutions of

$$V = f(\boldsymbol{n}, \nabla \boldsymbol{n}, t) \qquad (f \text{ is periodic in } t).$$

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