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Cauchy Problems for Sheaves and its Applications

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1 Introduction

— The study of the solvability of partial differential operators has a long history. When the operator is simple characteristic, Nirenberg-Treves [13], Kawai [12], S-K-K [15] studied the local solvability very precisely. But if the characteristic variety of the operator has singular points, this problem becomes more difficult. One of the advantage of the employment of the hyperfunction theory is that we can sometimes treat the operators with multiple characteristics very neatly. For example, Bony-Schapira [1] showed that the Cauchy problems for general hyperbolic operators are always solvable in the framework of the hyperfunction theory and such operators are solvable in the sheaf $\mathcal{B}_M$ of Sato's hyperfunctions. This result has been extended by many authors (Kashiwara-Kawai [6], Kashiwara-Schapira [7], Kaneko [4], Oaku [14]) and now we have a general theory for micro-hyperbolic systems ([7] and [8]). Bony-Schapira also proved the solvability of partially elliptic operators in the sheaf $\mathcal{C}_M$ of Sato's microfunctions in another paper [2].

In this talk, we prove the solvability of a class of operators which are not (micro-) hyperbolic nor partially elliptic by making use of the theory of bimicrolocalization developed in [18] and [20]. Let $M = M' \times M''$ be a product of two real analytic manifolds and $X = X' \times X''$ a complexification of M. We denote by $\mathcal{D}_X$ (resp. $\mathcal{D}_{X'}$) the sheaf of ring of holomorphic differential operators on X (resp. X'). Then we prove the following theorem in Section 4.

Theorem : Let E (resp. Q) $\in \mathcal{D}_{X'}$ be an elliptic differential operator (resp. a hyperbolic differential operator with constant coefficients) on X' and set :

$$P := E \cdot Q + (\text{lower order terms}) \tag{1.1}$$

by taking arbitrary lower order terms from $\mathcal{D}_X$. Then

(i) $P : \mathcal{B}_M \longrightarrow \mathcal{B}_M$ is surjective.

(ii) $P : \mathcal{C}_M \longrightarrow \mathcal{C}_M$ is surjective at any $p \in \dot{T}^*_M X$.

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Note that Kashiwara-Kawai [6] showed that $P := E \cdot Q +$(lower) is solvable in $\mathcal{B}_M$ when X'' reduces to a point and Q is an arbitrary hyperbolic operator (Theorem 6.5 of [6]). Our theorem above can be considered as a relative (second microlocal) version of the theorem of [6]. Therefore in Section 2, we give a new and purely algebraic proof of their theorem generalizing it to systems of partial differential equations. We prove the theorem above by combining this reduction with the solution to the Cauchy problem in the sheaves of bimicrofunctions. Roughly speaking, we perform a kind of "blow-up" along the singular locus of the characteristic variety $\{\sigma(P) = 0\} \subset T^*X$ of P to separate the partially elliptic factor $\{\sigma(E) = 0\}$ and the hyperbolic one $\{\sigma(Q) = 0\}$. The theory of [18] and [20] and the flabbiness of the sheaf $\mathcal{C}_{ML}$ of second microfunctions proved in Kataoka-Tose [11] will be essentially used in its proof.

In Section 5, we also prove the solvability in the sheaf $\hat{\mathcal{B}}_{N|\Omega}$ of mild hyperfunctions for microlocally semi-hyperbolic differential operators (see Definition 5.7) and extend a general theorem of Oaku [14] on the solvability of homogeneous boundary value problems (Theorem 3 of [14]) to inhomogeneous cases. For this purpose, we generalize in Theorem 5.4 a well-known result "the hyperfunction solutions to non-characteristic differential equations are always mild" of Kataoka [9] to systems and to higher cohomologies at the same time.

2 Basic ideas to prove the solvability for $\mathcal{D}_X$-modules

— Let M be a real analytic manifold and $N = \{x_1 = 0\}$ its closed submanifold of codimension one. We denote by $Y \subset X$ a complexification of $N \subset M$, $\mathcal{B}_M$ the sheaf of Sato's hyperfunctions on M, $\mathcal{C}_M$ and $\mathcal{C}_{N|X}$ the sheaves of microfunctions of Sato [15] which are associated to M and N respectively.

We will generalize a theorem on the solvability of single differential equations (Theorem 6.5 of Kashiwara-Kawai [6]) to the systems of differential equations, that is, to $\mathcal{D}_X$-modules.

Theorem 2.1 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module for which Y is noncharacteristic. Assume that $\mathcal{M}$ is micro-hyperbolic in the directions $\pm dx_1 \in \dot{T}^*_N M$ on $N \times_M \dot{T}^*_M X$. Then we have the vanishing of cohomologies :*

$$\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)\,|_N \simeq 0 \quad \text{for} \quad j > d = \text{proj.dim}\, \mathcal{M}_Y. \tag{2.1}$$

Proof: First of all, there is a distinguished triangle :

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)\,|_M \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \longrightarrow R\dot{\pi}_{M*}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M) \longrightarrow +1, \tag{2.2}$$

where $\dot{\pi}_M : \dot{T}^*_M X \longrightarrow M$ is the natural projection. By Cauchy-Kowalevski-Kashiwara's theorem, $\mathcal{E}xt^j_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) |_N \simeq 0$ for $j > d$. Hence it is enough to show for any open subset $U \subset N$:

$$H^j(U; R\dot{\pi}_{M*}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M) |_N) = 0 \quad \text{for} \quad j > d. \tag{2.3}$$

Thanks to the micro-hyperbolicity of $\mathcal{M}$ and the division theorem of Kashiwara-Kawai [5], the complex $\mathrm{R}\Gamma(U; R\dot{\pi}_{M*}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M) |_N)$ is a direct summand of the complex $\mathrm{R}\Gamma(\dot{T}^*_N Y; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N))$. Hence the assertion follows from the flabbiness of the sheaf $\mathcal{C}_N$ of the microfunctions on the initial hypersurface N. ■

Corollary 2.2 *(Theorem 6.5 of [6]) Let E (resp. Q) $\in \mathcal{D}_X$ be an elliptic differential operator (resp. a hyperbolic operator in the directions $\pm dx_1 \in \dot{T}^*_N M$) on X, and set $P = E \cdot Q+$ (lower order terms). Then the coherent $\mathcal{D}_X$-module $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$ satisfies the assumptions of Theorem 2.1 and $P : \mathcal{B}_M \longrightarrow \mathcal{B}_M$ is surjective.*

3 Cauchy problems in bimicrofunctions

— In this section, we essentially employ the terminology of [8] and [18]. Let $X \supset L \supset M$ be a sequence of C^∞-manifolds. Let us denote it by (X, L, M) and call it a triplet of manifolds. First recall the construction of the binormal deformation of X along (L, M) in [18]. We shall denote it by $\tilde{X}_{ML}$ and let $t, s \in \mathbb{R}$ be the deformation parameters. Then we have the commutative diagram below:

$$\begin{tikzcd} T_M L \times_L T_L X \arrow[r, "s_X"] \arrow[d, "\tau_X"'] & \tilde{X}_{ML} \arrow[d, "p_X"'] & \Omega_X := \{t > 0, s > 0\} \arrow[l, "j_X"'] \arrow[dl, "\tilde{p}_X"] \\ M \arrow[r, "i_X"] & X. & \end{tikzcd} \tag{3.1}$$

By the immersion s_X in (3.1), $T_M L \times_L T_L X$ is identified with $\tilde{X}_{ML} \cap \{t = s = 0\}$. If we choose a local coordinate system $x = (x', x'', x''')$ of X such that

$$\begin{cases} L = \{x' = 0\} \\ M = \{x' = 0, x'' = 0\}, \end{cases} \tag{3.2}$$

then the morphism p_X in(3.1) is described by

$$(x', x'', x''', t, s) \longmapsto (tsx', tx'', x'''). \tag{3.3}$$

Let $\mathbf{D}^\mathrm{b}(*)$ be the derived category of $\mathbb{C}$- vector spaces on a topological space with bounded cohomologies. In [18] we defined the functor of bispecialization ν_{ML} : $\mathbf{D}^\mathrm{b}(X) \longrightarrow \mathbf{D}^\mathrm{b}(T_M L \times_L T_L X)$ by

$$\nu_{ML}(F) := s_X^{-1} Rj_{X*} \tilde{p}_X^{-1} F. \tag{3.4}$$

We also defined two functors

$$\begin{cases} \nu\mu_{ML} : \mathbf{D}^{\mathrm{b}}(X) \longrightarrow \mathbf{D}^{\mathrm{b}}(T_M L \times_L T_L^* X) \\ \mu_{ML} : \mathbf{D}^{\mathrm{b}}(X) \longrightarrow \mathbf{D}^{\mathrm{b}}(T_M^* L \times_L T_L^* X) \end{cases} \tag{3.5}$$

as the Fourier-Sato transformations of $\nu_{ML}(*)$. For $F \in \mathbf{D}^{\mathrm{b}}(X), \nu_{ML}(F), \nu\mu_{ML}(F)$ and $\mu_{ML}(F)$ are biconic objects. We can give an estimation of the support of the complex $\mu_{ML}(F)$ (Funakoshi [3]) :

$$\operatorname{supp} \mu_{ML}(F) \subset T^*_{(M \times_L T_L^* X)}(T_L^* X) \cap C_{T_L^* X}(\mathrm{SS}(F)), \tag{3.6}$$

where we used the natural isomorphism $T_M^* L \times_L T_L^* X \simeq T^*_{(M \times_L T_L^* X)}(T_L^* X)$ and the Hamilton isomorphism $-H : T^*(T_L^* X) \simeq T_{(T_L^* X)}(T^* X)$.

Let $g : M \to M''$ be a smooth morphism of real analytic manifolds and $g_{\mathbb{C}} : X \to X''$ a complexification of g. Set $L := g_{\mathbb{C}}^{-1}(M'')$ and assume $\dim^{\mathbb{R}} M = n$. It follows from Kashiwara's abstract edge of the wedge theorem that the complex $\mu_{ML}(\mathcal{O}_X)[n]$ on $T_M^* L \times_L T_L^* X$ is concentrated in degree 0. Hence we set :

$$\mathcal{C}_{ML} := \mu_{ML}(\mathcal{O}_X) \otimes \mathrm{or}_M[n] \tag{3.7}$$

and call it the sheaf of second microfunctions along L (Kataoka-Tose [11] and [18]).

From now on, to the end of this section, we study the Cauchy problem in the framework of sheaves of second microfunctions reviewed above. Let $N \subset M = M' \times M'' = \mathbb{R}^d \times M''$ a submanifold of codimension one such that $g|_N : N \longrightarrow M''$ is also smooth. We denote by $g_{\mathbb{C}} : X = X' \times X'' = \mathbb{C}^d \times X'' \longrightarrow X''$ (resp. $(g|_N)_{\mathbb{C}} : Y \longrightarrow X''$) a complexification of $g : M \longrightarrow M''$ (resp. $g|_N : N \longrightarrow M''$). Finally set $L := g_{\mathbb{C}}^{-1}(M'') \subset X$ and $H := (g|_N)_{\mathbb{C}}^{-1}(M'') \subset Y$. Then we have the canonical injections :

$$T_N^* L \times_L T_L^* X \xleftarrow{\delta} (N \times_M T_M^* X) \times_L T_L^* X \xrightarrow{\varpi_0} T_M^* L \times_L T_L^* X. \tag{3.8}$$

The next theorem is a second microlocal version of a result of Kashiwara-Schapira (Theorem 6.7.1 of [8]).

Theorem 3.1 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Suppose there exists a coherent $\mathcal{D}_{X'}$-module $\mathcal{M}'$ which satisfies :*

(i) $\mathcal{M}'$ is hyperbolic in the directions $\pm dx_1 \in \dot{T}_N^* M$ with constant coefficients.

(ii) $Ch\mathcal{M} \subset Ch\mathcal{M}' \times T^* X''$.

Then we have the isomorphism :

$$\delta_* \varpi_0^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{ML}) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{NL})[1]. \tag{3.9}$$

Since $V := X \times_{X''} T^*X''$ be a regular involutive submanifold of T^*X, we can define the natural injection :

$$T_Y^*X \longrightarrow T_Y^*X \times_X V \longrightarrow T^*(X/X'') \times_X V \simeq T_V(T^*X) \tag{3.10}$$

by using the zero-section of V. We also use the projection $\rho_N : T_N^*L \times_L T_L^*X \longrightarrow T_N^*H \times_H T_H^*Y$.

Proposition 3.2 *([21]) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module which satisfies the non-microcharacteristic condition:*

$$\dot{T}_Y^*X \cap C_V(Ch\mathcal{M}) = \emptyset. \tag{3.11}$$

Then we have the canonical isomorphism :

$$R\rho_{N*} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{NL})[1] \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_{NH}). \tag{3.12}$$

By using the projection $\rho_0 := \rho_N \circ \delta : (N \times_M T_M^*L) \times_L T_L^*X \longrightarrow T_N^*H \times_H T_H^*Y$, the next result follows from Theorem 3.1 and Proposition 3.2 :

Theorem 3.3 *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module which satisfy the conditions (i)(ii) of Theorem 3.1. Then we have the isomorphism :*

$$R\rho_{0*} \varpi_0^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{ML}) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_{NH}) \tag{3.13}$$

on $T_N^*H \times_H T_H^*Y$.

Remark 3.4 If we impose the conditions of Theorem 4.1 (in the following section) on P, we can also prove the isomorphism in Theorem 3.1 on $\{\xi_1 = 0\} \cap (\dot{T}_N^*L \times_L T_L^*X) = (N \times_M \dot{T}_M^*L) \times_L T_L^*X$.

4 Solvability of operators with multiple characteristics

— In this section, we shall give two results which can not be covered by Theorem 6.5 of Kashiwara-Kawai [6]. We consider the same situation as in Section 3 and inherit the notations in it. For example, suppose that locally $M = M' \times M'' = \mathbb{R}^d \times \mathbb{R}^{n-d}$, $X = X' \times X'' = \mathbb{C}^d \times \mathbb{C}^{n-d}$ and $N = \{x_1 = 0\} = \mathbb{R}^{d-1} \times \mathbb{R}^{n-d} \subset M$.

First we consider the following case by making use of the theory of bimicrolocalization.

Theorem 4.1 *Let $E, Q \in \mathcal{D}_{X'}$ be differential operators on X'. Assume that E is elliptic and Q is hyperbolic in $\pm dx_1$-direction with constant coefficients. We set $P := E \cdot Q +$ (lower order terms) $\in \mathcal{D}_X$ by taking arbitrary lower order terms from the differential operators on the total space X. Then we have :*

(i) $P : \mathcal{B}_M \longrightarrow \mathcal{B}_M$ is surjective.

(ii) $P : \mathcal{C}_M \longrightarrow \mathcal{C}_M$ is surjective at any point in $\dot{T}^*_M X$.

Proof: For the sake of the simplicity, set $\dot{\Lambda}_N := N \times_L \dot{T}^*_L X$ and $\dot{\Sigma} := (N \times_M \dot{T}^*_M L) \times_L \dot{T}^*_L X$. Considering Sato's exact sequence, it suffices to show that

$$P : \Gamma(N \times_M \dot{T}^*_M X; \mathcal{C}_M) \longrightarrow \Gamma(N \times_M \dot{T}^*_M X; \mathcal{C}_M) \tag{4.1}$$

is surjective. But the local solvability in the sheaf $\mathcal{C}_M$ of P is trivial on $N \times_M \dot{T}^*_M X - \dot{\Lambda}_N$ by its micro-hyperbolicity there. Hence to prove the theorem, it is enough to show the surjectivity of the morphism $P : \Gamma(\dot{\Lambda}_N; \mathcal{C}_M) \longrightarrow \Gamma(\dot{\Lambda}_N; \mathcal{C}_M)$. Now consider the following commutative diagram with exact rows :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\dot{\Lambda}_N; \mathcal{CO}) & \longrightarrow & \Gamma(\dot{\Lambda}_N; \mathcal{C}_M) & \longrightarrow & \Gamma(\dot{\Sigma}; \mathcal{C}_{ML}) & \longrightarrow & 0 \\
 & & {\scriptstyle P\times}\big\uparrow & & {\scriptstyle P\times}\big\uparrow & & {\scriptstyle P\times}\big\uparrow & & \\
0 & \longrightarrow & \Gamma(\dot{\Lambda}_N; \mathcal{CO}) & \longrightarrow & \Gamma(\dot{\Lambda}_N; \mathcal{C}_M) & \longrightarrow & \Gamma(\dot{\Sigma}; \mathcal{C}_{ML}) & \longrightarrow & 0,
\end{array} \tag{4.2}$$

where $\mathcal{CO} := \mu_L(\mathcal{O}_X) \otimes \mathrm{or}_L[n-d]$ is the sheaf of microfunctions with holomorphic parameters and the exactitude follows from the vanishing of the global cohomology $H^1(\dot{\Lambda}_N; \mathcal{CO})$ shown by Theorem 3.1 of Kataoka-Tose [10]. Set $z' = (z_1, ..., z_d) = (z_1, \hat{z})$ and $\mathcal{CO}_{\hat{z}} := \mu_H(\mathcal{O}_Y) \otimes \mathrm{or}_H[n-d]$. Then by Schapira's Cauchy-Kowalevski type theorem of [2] and [16], we have by setting $\mathcal{M} = \mathcal{D}_X / \mathcal{D}_X P$:

$$\begin{aligned} &\mathrm{R}\Gamma(\dot{\Lambda}_N; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{CO})) \\ &\simeq \mathrm{R}\Gamma(\dot{\Lambda}_N; R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{CO}_{\hat{z}})) = \mathrm{R}\Gamma(\dot{\Lambda}_N; \mathcal{CO}_{\hat{z}})^{\oplus ordP}. \end{aligned} \tag{4.3}$$

By virtue of Theorem 3.1 of Kataoka-Tose [10] again, it implies that the first vertical arrow

$$P : \Gamma(\Lambda_N; \mathcal{CO}\,|_{\Lambda_N}) \longrightarrow \Gamma(\Lambda_N; \mathcal{CO}\,|_{\Lambda_N}) \tag{4.4}$$

of the diagram (4.2) is surjective. We can also show the surjectivity of the third vertical arrow $P : \Gamma(\dot{\Sigma}; \mathcal{C}_{ML}) \longrightarrow \Gamma(\dot{\Sigma}; \mathcal{C}_{ML})$ by applying the same arguments as in the proof of Theorem 2.1 to (the proofs of) Theorem 3.1 and Proposition 3.2. In this case, we require also the flabbiness of the sheaf $\mathcal{C}_{NH}$ of second microfunctions on $\dot{T}^*_N H \times_H \dot{T}^*_H Y$ proved by Kataoka-Tose [11]. It completes the proof. ■

Next take an elliptic operator E on $X' = \mathbb{C}^d_{z'}$ (that is, $\mathrm{Ch}(\mathcal{D}_{X'}/\mathcal{D}_{X'}E) \cap \dot{T}^*_{M'}X' = \emptyset$) and a hyperbolic operator Q in $\pm dx_1$-direction on the total space $X = X' \times X''$. We set :

$$P = E \cdot Q + (\text{lower order terms}). \tag{4.5}$$

If we consider E as a differential operator on X, we have :

$$\mathrm{Ch}(\mathcal{D}_X/\mathcal{D}_X E) \cap T^*_M X = M' \times T^*_{M''}X'' = M \times_L T^*_L X. \tag{4.6}$$

Theorem 4.2 *Assume the separation condition :*

$$Ch(\mathcal{D}_X/\mathcal{D}_X E) \cap Ch(\mathcal{D}_X/\mathcal{D}_X Q) \cap \dot{T}^*_M X = Ch(\mathcal{D}_X/\mathcal{D}_X Q) \cap (M \times_L \dot{T}^*_L X) = \emptyset. \tag{4.7}$$

Then $P : \mathcal{B}_M \to \mathcal{B}_M$ is surjective.

Proof: By virtue of Sato's exact sequence and Cauchy-Kowalevski-Kashiwara's theorem, it suffices to show the surjectivity of the morphism :

$$P : \Gamma(N \times_M \dot{T}^*_M X; \mathcal{C}_M) \longrightarrow \Gamma(N \times_M \dot{T}^*_M X; \mathcal{C}_M). \tag{4.8}$$

We know from the proof of Theorem 2.1 that P is globally solvable in a neighborhood of $\mathrm{Ch}(\mathcal{D}_X/\mathcal{D}_X Q) \cap \dot{T}^*_M X$ in $\dot{T}^*_M X$. The problem is to show the surjectivity of the morphism:

$$P : \Gamma(\dot{\Lambda}_N; \mathcal{C}_M) \longrightarrow \Gamma(\dot{\Lambda}_N; \mathcal{C}_M) \tag{4.9}$$

for $\dot{\Lambda}_N := N \times_L \dot{T}^*_L X$. This is equivalent to the vanishing for the $\mathcal{D}_X$-module $\mathcal{M} := \mathcal{D}_X/\mathcal{D}_X P$:

$$H^1 \mathrm{R}\Gamma(\dot{\Lambda}_N; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M)) \simeq 0. \tag{4.10}$$

Since the $\mathcal{D}_X$-module $\mathcal{M}$ is partially elliptic along $\dot{V} = X' \times \dot{T}^*X'' \subset T^*X$ in the sense of Bony-Schapira [2], we have the chain of isomorphisms :

$$\begin{aligned} &\mathrm{R}\Gamma(\dot{\Lambda}_N; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M)) \\ &\xleftarrow{\sim} \mathrm{R}\Gamma(\dot{\Lambda}_N; R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{CO})) \simeq \mathrm{R}\Gamma(\dot{\Lambda}_N; R\mathcal{H}om_{\mathcal{E}_Y}(\mathcal{E}_Y^{\oplus ordE}, \mathcal{CO}_{\hat{z}})) \\ &= \mathrm{R}\Gamma(\dot{\Lambda}_N; \mathcal{CO}_{\hat{z}})^{\oplus ordE}, \end{aligned} \tag{4.11}$$

where we used Schapira's Cauchy-Kowalevski type theorem of [16] to show the second isomorphism. The first cohomology group $H^1(\dot{\Lambda}_N; \mathcal{CO}_{\hat{z}}^{\oplus ordE})$ of the last term vanishes by Theorem 3.1 of Kataoka-Tose [10] and it completes the proof. ∎

5 Solvability of boundary value problems

— In this section, we apply the methods in Theorem 2.1 to boundary value problems. In particular, we extend a result of Oaku [14] to inhomogeneous boundary value problems.

First we shall recall some basic notions concerning boundary value problems. Let $N = \{x_1 = 0\} \subset M$ be a real analytic submanifold of codimension one as before and $\Omega = \{x_1 > 0\} \subset M$ an open subset such that $N = \partial\Omega$. We take a complexification $Y \subset X$ of $N \subset M$ as usual.

Definition 5.1 (Schapira[17]) We define the complex $\mathcal{C}_{\Omega|X}$ by $\mathcal{C}_{\Omega|X} := \mu hom(\mathbb{C}_\Omega, \mathcal{O}_X) \otimes \mathrm{or}_M[n]$, where $\mu hom(\cdot,\cdot) : \mathrm{D^b}(X)^{op} \times \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(T^*X)$ is a bifunctor introduced in [8] and $n = \dim^{\mathbb{R}} M$.

We set $M_+ := \{x_1 \geq 0\} = \bar{\Omega}$ and we can also define Kataoka's sheaf $\mathcal{C}_{M_+|X}$ by replacing $\mathbb{C}_\Omega$ with $\mathbb{C}_{M_+}$ in the definition above. Recall the following results proved by Schapira-Zampieri.

Theorem 5.2 *(Schapira-Zampieri [19]) Let* $T^*X \xleftarrow{\rho} Y \times_X T^*X \xrightarrow{\varpi} T^*X$ *be natural morphisms. Then the complex* $R\rho_!\varpi^{-1}\mathcal{C}_{\Omega|X}$ *of sheaves on* T^*Y *is concentrated in degree 0 and coincides with Kataoka's sheaf* $\hat{\mathcal{C}}_{N|\Omega}$ *of mild microfunctions ([9]).*

Remark 5.3 The sheaf $\hat{\mathcal{C}}_{N|\Omega}$ is supported by $T^*_N Y$ and $\hat{\mathcal{B}}_{N|\Omega} := \hat{\mathcal{C}}_{N|\Omega} |_{T^*_Y Y}$ was called the sheaf of mild hyperfunctions in Kataoka [9]. Since supp $\hat{\mathcal{B}}_{N|\Omega} \subset N$, we sometimes consider it a sheaf on N.

The sheaf $\hat{\mathcal{B}}_{N|\Omega}$ of mild hyperfunctions is a subsheaf of $\Gamma_\Omega \mathcal{B}_M |_N$ and we can explicitly take the boundary values to N of mild hyperfunctions by restricting their defining holomorphic functions in the complex domain. Kataoka [9] found that the hyperfunction solutions $u \in \Gamma_\Omega \mathcal{B}_M |_N$ of the single differential equations for which Y is noncharacteristic are always mild and made the defintion of the boundary value more explicit. The next proposition extends this classical result to systems.

Theorem 5.4 *Let* $\mathcal{M}$ *be a coherent* $\mathcal{D}_X$*-module for which* Y *is noncharacteristic. Then we have the isomorphism :*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_\Omega \mathcal{B}_M) |_N \xleftarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \hat{\mathcal{B}}_{N|\Omega}). \tag{5.1}$$

To define the semi-hyperbolicity of differential operators, we take a coordinate system $(z,\zeta), z = x+iy, \zeta = \xi + i\eta$ of T^*X such that $T^*_M X = \{y = 0, \xi = 0\}$, and $N = \{x_1 = 0\}$, $\Omega = \{x_1 > 0\} \subset M$. If we take a point $p = (x'_0; i\eta'_0) \in T^*_N Y \subset T^*Y$, the fiber $\rho^{-1}(p) \simeq \mathbb{C}^1_{\zeta_1}$ is isomorphic to a complex plane. Let P be a differential operator or a pseudo-differential operator defined on a neighborhood of $\rho^{-1}(p) \simeq \mathbb{C}^1_{\zeta_1} \subset T^*X$.

Definition 5.5 (Kaneko [4], Kataoka [9]) We say P is semi-hyperbolic (resp. microlocally semi-hyperbolic) in $+dx_1$-direction at p, if there exists a constant $\varepsilon > 0$ such that

$$\sigma(P)(x_1, x'; \xi_1 + i\eta_1, i\eta') \neq 0 \tag{5.2}$$

for $0 \leq x_1 < \varepsilon$, $|x' - x'_0| < \varepsilon$, $0 < \xi_1$ (resp. $0 < \xi_1 < \varepsilon$), $\eta_1 \in \mathbb{R}$, $|\eta' - \eta'_0| < \varepsilon$.

Remark 5.6 Asumme $p \in N \subset T^*_N Y$ and P is a differential operator defined on a neighborhood of $p \in N \subset X$. Then the definition above coincides with that of Kaneko [4].

For example, the operator $Q = D_1^2 - x_1^k D_{x'}^2 (k \in \mathbb{N})$ is semi-hyperbolic in $+dx_1$-direction at any $p \in T^*_N Y$. If we take an elliptic operator E and set :

$$P = E \cdot Q + \text{(lower order terms)}, \tag{5.3}$$

then P is microlocally semi-hyperbolic at any $p \in \dot{T}^*_N Y$. As we will show later, such operators P are solvable in the sheaf $\hat{\mathcal{B}}_{N|\Omega}$ of mild hyperfunctions. Hence we define a family of differential operators to include these examples.

Definition 5.7 We say $P \in \mathcal{D}_X$ is microlocally semi-hyperbolic in $+dx_1$-direction, if P is so at any $p \in \dot{T}^*_N Y$.

Remark 5.8 A similar but stronger condition "microlocally hyperbolic" was introduced in [21].

Theorem 5.9 *Let P be a differential operator for which Y is noncharacteristic. Assume P is microlocally semi-hyperbolic in $+dx_1$-direction. Then :*

(i) $P : \Gamma_\Omega \mathcal{B}_M |_N \to \Gamma_\Omega \mathcal{B}_M |_N$ *is surjective.*

(ii) $P : \hat{\mathcal{B}}_{N|\Omega} \to \hat{\mathcal{B}}_{N|\Omega}$ *is surjective.*

(iii) $P : \hat{\mathcal{C}}_{N|\Omega} \to \hat{\mathcal{C}}_{N|\Omega}$ *is surjective at any* $p \in \dot{T}^*_N Y$.

Remark 5.10 The proof of (i) is reduced to show :

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P, \mathcal{C}_{M_+|X}) \simeq 0 \tag{5.4}$$

on $N \times_M \dot{T}^*_M X$ by using the same arguments as in the proof of Theorem 2.1. (5.4) follows from Kataoka's result on the solvability of semi-hyperbolic pseudo-differential operators (Corollary 1.9 of [9]). The part (ii) is a direct consequence from (i) and Theorem 5.4.

As an application of Theorem 5.9 (ii), we get a result which extends Theorem 3 of Oaku [14] to inhomogeneous boundary value problems.

Corollary 5.11 *Let P be a differential operator for which Y is noncharacteristic. Assume P is microlocally semi-hyperbolic in $+dx_1$-direction and*

$$\sharp \quad [\{q \in \rho^{-1}(p); \sigma(P)(q) = 0\} \cap \{\xi_1 \leq 0\}] \geq m' \tag{5.5}$$

holds for any $p \in \dot{T}^*_N Y$, where $0 \leq m' \leq m = \mathrm{ord}P$. Then there always exists a mild hyperfunction solution $u \in \hat{\mathcal{B}}_{N|\Omega}$ to the inhomogeneous boundary value problem:

$$\begin{cases} Pu = f, \\ D_1^j u \mid_{x_1 \to +0} = v_j \quad (j = 0, ..., m' - 1) \end{cases} \tag{5.6}$$

for any $f \in \hat{\mathcal{B}}_{N|\Omega}$ and any $v_j \in \mathcal{B}_N$ $(j = 0, ..., m' - 1)$.

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