Holonomic deformation of linear differential equations of the $A_g$ type (Study of Partial Differential Equations by means of Functional Analysis)

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Holonomic deformation of linear differential equations of the $A_g$ type

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0 Introduction.

In this paper, we consider linear differential equations of the form:

\begin{equation}
\frac{d^2 y}{dx^2} + p_1(x, t) \frac{dy}{dx} + p_2(x, t)y = 0,
\end{equation}

defined on the Riemann sphere $\mathbb{P}^1$, with the coefficients:

\begin{align*}
P_1(x, t) &= -2x^{g+1} - \sum_{j=1}^g j t_j x^{j-1} - \sum_{k=1}^g \frac{1}{x - \lambda_k}, \\
P_2(x, t) &= -(2\alpha + 1)x^g - 2\sum_{j=1}^g H_j x^{g-j} + \sum_{k=1}^g \frac{\mu_k}{x - \lambda_k}.
\end{align*}

The Riemann scheme of this equation reads:

\begin{equation}
\begin{pmatrix}
x = \lambda_k \\
0 & 0 & 0 & 0 & \cdots & 0 & \alpha + \frac{1}{2}
\end{pmatrix}
\end{equation}

Here the symbol in (0.3) means that, at the irregular point $x = \infty$, the equation (0.1)-(0.2) admits a system of formal solutions of the form:

\begin{equation}
\begin{aligned}
\hat{y}_1 &= x^{-\alpha - \frac{1}{2}}(1 + \sum_{i \geq 1} h_i^1 x^{-i}), \\
\hat{y}_2 &= x^{\alpha - \frac{1}{2}} \exp\left[\frac{2}{g + 2} x^{g+2} + t_g x^g + \cdots + t_1 x\right](1 + \sum_{i \geq 1} h_i^2 x^{-i}).
\end{aligned}
\end{equation}

Note that the Poincaré rank at $x = \infty$ of the linear equation (0.1)-(0.2) is $g+2$. The principal parts of these formal solutions are given by the primitive
function of the polynomials representing the versal deformation of the simple singularity of the $A_\nu$ type, so we call the linear equation (0.1)-(0.2) as the equation of the $A_\nu$-type.

When considering the holonomic deformation of equations of the $A_1$-type, we obtain the Hamiltonian structure:

$$(\lambda_1, \mu_1, H_1, t_1),$$

which determines the Hamiltonian system, equivalent to the second Painlevé equation, see [6]. C. H. LIN and Y. SIBUYA studied on the holonomic deformation of linear equations; having the irregular singularity at $x = \infty$ with higher Poincaré rank and admitting a non-logarithmic singular point $x = \lambda$, see [4].

When $g = 2$, it is known ([3]) that the holonomic deformation is governed by the Hamiltonian system with respect to the canonical variables:

$$(\lambda_1, \lambda_2, \mu_1, \mu_2, H_1, H_2, t_1, t_2).$$

On the other hand, in the case $g \geq 3$, the quantities $H = (H_1, \cdots, H_g)$ and $t = (t_1, \cdots, t_g)$ do not compose the Hamiltonian structure. In fact, in the case of $g = 3$, we have to determine the variables $s = (s_1, s_2, s_3)$ such that

$$(0.5) \quad s_1 = t_1 - \frac{3}{4}t_3^2, \quad s_2 = t_2, \quad s_3 = t_3,$$

and then obtain the Hamiltonian structure: (see [5])

$$(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, H_1, H_2, H_3, s_1, s_2, s_3).$$

As to general natural number $g$, in order to determine the Hamiltonian structure, we have tried to find the transformation of qualities $t = (t_1, \cdots, t_g)$ such as (0.5), but we didn’t succeed. So we determine the following new qualities instead.

$$(0.6) \quad \overline{H}_j = 2 \sum_{i=0}^{j-1} a_{i+1}^{(j)}(t) (H_{j-i} + T_{j-i}^*), \quad (j = 1, \cdots, g),$$

where

$$(0.7) \quad T_j^* = \frac{1}{4}(j-1)T_{g+2-j} + \frac{1}{8} \sum_{l=1}^{g} T_l T_{g+2-j-l} \quad (1 \leq j \leq g), \quad T_j = j t_j \quad (1 \leq j \leq g), \quad T_j = 0 \quad (j < 1) \text{ or } (j > g),$$
$a_{i+1}(t)$ is given by

$$a_1(t) = \frac{1}{2}, \quad a_2(t) = 0, \quad a_{i+1}(t) = \sum_{m=1}^{[\frac{i}{2}]} M^{(m,2m-i)} \quad (i \geq 2).$$

$M^{(m,q)}$ are defined as the coefficients of the expansion of function,

$$\sum_{q=n(1-g)}^{0} M^{(n,q)} x^{-q} = \frac{1}{2} \left( -\frac{1}{2} \sum_{l=1}^{\tau_1} \tau_{\iota x^{gl}} - \right)^{n}$$

Such that

$$(\lambda_1, \cdots, \lambda_g, \mu_1, \cdots, \mu_g, H_1, \cdots, H_g, t_1, \cdots, t_g)$$

is the Hamiltonian Structure of the equation of the $A_g$ type.

We suppose throughout this paper that $2\alpha + 1$ is not an integer. This equation has an irregular singularity at $x = \infty$ of the Poincaré rank $g+2$ and $g$ regular singular points $x = \lambda_k$ ($k = 1, \cdots, g$). We also make the following assumption:

(A) none of $x = \lambda_k$ ($k = 1, \cdots, g$) is logarithmic singularity.

Since exponents at each regular singular point, $x = \lambda_k$, are 0 and 2, we deduce from the assumption (A) that $H_i$ ($i = 1, \cdots, g$) are rational functions of $t = (t_1, \cdots, t_g)$, $\lambda = (\lambda_1, \cdots, \lambda_g)$ and $\mu = (\mu_1, \cdots, \mu_g)$. The explicit forms of $H_i$ will be given in Section 2, they play important roles in our studies.

Now we state the Main Theorem:

**Main Theorem.** The holonomic deformation of the linear ordinary differential equation (0.1)-(0.2) is governed by the completely integrable Hamiltonian system:

$$(\overline{H}) \quad \frac{\partial \lambda_k}{\partial t_j} = \frac{\partial H_j}{\partial \mu_k}, \quad \frac{\partial \mu_k}{\partial t_j} = -\frac{\partial H_j}{\partial \lambda_k} \quad (k, j = 1, \cdots, g),$$

with the Hamiltonian functions $\overline{H}_j$ defined by (0.6).

Since the completely integrable Hamiltonian system $\overline{H}$ determines the holonomic deformation of linear equations of the $A_g$ type, we call $\overline{H}$ the $A_g$-system.
1 Holonomic deformation of linear equation of the second order.

In this section, we recall the theory of the holonomic deformation of linear differential equation of the form:

\[
\frac{d^2 y}{dx^2} + p_1(x,t) \frac{dy}{dx} + p_2(x,t)y = 0, \tag{1.1}
\]

We make in the following of this section a review of known results, which are available for us to study the holonomic deformation of linear equation of the \( A_g \)-type.

**Proposition 1.1.** The equation (1.1) has a fundamental system of solutions whose monodromy and Stokes multiplier are independent of \( t \), if and only if there exist rational functions of \( x \), \( A_j(x) \), \( B_j(x) \), such that the following system of partial differential equations is completely integrable:

\[
\begin{align*}
\frac{\partial^2 y}{\partial x^2} + p_1(x,t) \frac{\partial y}{\partial x} + p_2(x,t)y &= 0, \\
\frac{\partial y}{\partial t_j} &= B_j(x)y + A_j(x) \frac{\partial y}{\partial x}, \quad (j = 1, \ldots, g).
\end{align*}
\]

**Proposition 1.2.** The conditions of the complete integrability of (1.2) are given by:

\[
\begin{align*}
\frac{\partial A_j}{\partial t_i} + A_j \frac{\partial A_i}{\partial x} &= \frac{\partial A_i}{\partial t_j} + A_i \frac{\partial A_j}{\partial x}, \\
\frac{\partial^3}{\partial x^3} A_j - 4P \frac{\partial}{\partial x} A_j - 2A_j \frac{\partial}{\partial x} P + 2 \frac{\partial}{\partial t_j} P &= 0, \quad (j = 1, \ldots, g),
\end{align*}
\]

where

\[
P(x,t) = -p_2(x,t) + \frac{1}{4} p_1^2(x,t) + \frac{1}{2} \frac{\partial}{\partial x} p_1(x,t).
\]

If we make the change of the unknown function:

\[
y = \Phi(x)z, \quad \Phi(x) = \exp\left( -\frac{1}{2} \int^x p_1(x,t) \, dx \right), \tag{1.6}
\]
then (1.1) is transformed into an equation of the form:

\[
\frac{d^2 z}{dx^2} = P(x,t)z,
\]

where \( P(x,t) \) is the function given by (1.5). It follows that:

**Proposition 1.3.** The holonomic deformation of (1.1) is reduced to that of (1.7) Finally, the holonomic deformation of (1.1) is reduced to the existence of rational functions \( A_j(x) (j = 1, \cdots, g) \), satisfying the system (1.3)-(1.4) of partial differential equations. We will call (1.2) the extended system of (1.1) and the functions \( A_j(x) \) the deformation functions.

## 2 Deformation functions \( A_j(x) \).

In the following of this paper, we consider the holonomic deformation of linear equations of the form:

\[
\frac{d^2 y}{dx^2} + p_1(x,t) \frac{dy}{dx} + p_2(x,t)y = 0,
\]

where

\[
p_1(x,t) = -2x^{g+1} - \sum_{j} x^j + 1 - \sum_{k} \frac{1}{x - \lambda_k},
\]

\[
p_2(x,t) = -(2\alpha + 1)x^g - 2 \sum_{j} H_j x^{g-j} + \sum_{k} \frac{\mu_k}{x - \lambda_k},
\]

For the limiting pages, here we only can give the results, omit their proofs.

Firstly we determine the deformation functions.

**Proposition 2.1.** For \( j = 1, \cdots, g \), the deformation functions \( A_j(x) \) are given as follows:

\[
A_j(x) = \frac{\overline{Q}_j(x)}{\Lambda(x)},
\]

where \( \Lambda(x) = \prod_{j=1}^{g}(x - \lambda_j) \), and \( \overline{Q}_j(x) \) is a polynomial of degree \( j - 1 \).

The explicit form of \( \overline{Q}_j(x) \) will be given by proposition 3.2. In order to prove this proposition, we need following lemmata.

**Lemma 2.1.** As function of \( x \), \( A_j(x) \) is holomorphic on \( \mathbb{C} \setminus \{\lambda_1, \cdots, \lambda_g\} \).
Lemma 2.2. For $k = 1, \cdots, g$; $x = \lambda_k$ is a pole of the first order of $A_j(x)$.

Lemma 2.3. $A_j(x)$ admits a zero of order $g + 1 - j$ at $x = \infty$.

Proposition 2.1 is an immediate consequence of lemmata 2.1, 2.2 and 2.3.

To give the explicit form of $\overline{Q}_j(x)$, we prove following lemma:

Lemma 2.4. For $i \geq 3$, $a_i(t)$ defined by (0.8) satisfies

$$(2.4) \quad a_1(t) = \frac{1}{2}, \quad a_2(t) = 0, \quad a_i(t) = -\frac{1}{2} \sum_{m=1}^{i-2} T_{g+m+2-i} a_m \quad (i \geq 3).$$

Proposition 2.2. If differential equation (0.1)-(0.2) admits the holonomic deformation, then the deformation functions $A_j(x) = \overline{Q}_j(x) \overline{\Lambda}(x)$ ($j = 1, \cdots, g$) are determined as

$$(2.5) \quad \overline{Q}_j(x) = 2 \sum_{i=0}^{j-1} a_{i+1}(t) Q_{j-i}(x), \quad (j = 1, \cdots, g)$$

where

$$Q_j(x) = -\frac{1}{2} \sum_{n=0}^{j-1} \sigma_n x^{j-1-n}.$$  

$$(2.6) \quad \sigma_n = (-1)^{n+1} e_n, \quad (n = 0, 1, \cdots, g),$$

and $e_n$ denotes the $n$-th elementary polynomial of the $g$ variables, $\lambda_1, \cdots, \lambda_g$, in particular we define $e_0 = 1$.

Remark 2.1. From the proof of proposition 2.2, we know when $\overline{Q}_j(x)$ is of the form of (2.5), then the degree of $\Delta_j(x)$ is at most $g - 1$.

Proposition 2.3. $Q_j(x)$ and $\overline{Q}_j(x)$ have the following properties:

$$(2.7) \quad Q_j(\lambda_k) = \frac{1}{2} N^{j,k}, \quad \overline{Q}_j(\lambda_k) = \frac{1}{2} \overline{N}^{j,k}.$$ 

3 Equation of the SL-type.

In this section, we will investigate the equation of the SL-type:

$$\frac{d^2 z}{dx^2} = P(x, t) z,$$

$$(3.1) \quad P(x, t) = -p_2(x, t) + \frac{1}{4} p_1^2(x, t) + \frac{1}{2} \frac{\partial}{\partial x} p_1(x, t).$$
By using (0.2), we see that $P(x, t)$ can be written in the following form:

$$P(x, t) = x^{2g+2} + \sum_{i=0}^{g} F_i x^{g+i} + 2 \sum_{j=0}^{g} K_j x^{g-j}$$

where we denote by $\Sigma_{(k)}$ the sum for $k = 1, \cdots, g$. And we have:

$$F_j = \frac{1}{4} \sum_{i=2+j}^{g} T_i T_{g+2+j-i} + T_j + 2\alpha \delta_{j0} \quad (0 \leq j \leq g)$$

$$K_j = H_j + T_j^* + \frac{1}{2} \sum_{(k)} \lambda_k^2 + \frac{1}{4} \sum_{(k)} \sum_{m=1}^{j-2} T_{m+g+2-j} \lambda_k^m \quad (1 \leq j \leq g)$$

$$\nu_k = \mu_k - \frac{1}{2} \left( \sum_{(k)} \frac{1}{\lambda_k - \lambda_l} + \sum_{(i)} T_i \lambda_k^{-i+1} + 2\lambda_k^{g+1} \right) \quad (1 \leq k \leq g)$$

Here we denote by $\Sigma_{(l)}$ the sum for $l = 1, \cdots, g$ except for $l = k$. Let $e_j^{(k)}$ be the $j$-th elementary symmetric polynomial of $g-1$ variables, $\lambda_l$ ($l = 1, \cdots, g$, $\neq k)$, in particular, we put $e_0^{(k)} = 1$. Moreover, we define $\sigma_j^{(k)} = (-1)^{j+1} e_j^{(k)}$. For the simplicity of presentation, we put:

$$N_k = \frac{1}{\Lambda'(\lambda_k)}, \quad N_{j,K} = -\sigma_{j-1}^{(k)}, \quad k, j = 1, \cdots, g,$$

where $\Lambda(x) = \prod_{i=1}^{g} (x - \lambda_i)$, and $\Lambda'(x) = \frac{d}{dx} \Lambda(x)$.

We have following two propositions. Since the proofs are almost same, we omit them (see [5]).

**Proposition 3.1.** In the linear equation (0.1)-(0.2) $H_j$ ($j = 1, \cdots, g$) are given by:

$$H_j = \frac{1}{2} \sum_{(k)} [N_k N_{j,K} \mu_k^2 - U_{jk} \mu_k - N_k N_{j,K}(2\alpha + 1)]\lambda_k^g]$$
where

\[ U_{jk} = N_k N^{j,k}(2\lambda_k^{g+1} + \sum_{(i)} T_i \lambda_{i}^{-1}) - \sum_{(i)}^{(k)} \frac{N_i N^{j,i} + N_l N^{j,l}}{\lambda_i - \lambda_k}. \]

**Proposition 3.2.** In the linear equation (3.1)-(3.2), \( K_j \) \((j = 1, \cdots, g)\) are written as follows:

\[(3.8) \quad K_j = \frac{1}{2} \sum_{(k)} \left( N_k N^{j,k} \nu_k^2 - \sum_{(i)}^{(k)} \frac{N_i N^{j,i} \nu_k}{\lambda_k - \lambda_i} - N_k N^{j,k} V_k \right), \]

where \( V_k = \lambda_k^{2g+2} + \lambda_k^g \sum_{i=0}^g F_i \lambda^i_k + \frac{3}{4} \sum_{(i)}^{(k)} \frac{1}{(\lambda_i - \lambda_0)^2} \).

For the \( M^{(m,q)} \) given by (0.9), we have following lemma.

**Lemma 3.1.** For arbitrary natural numbers \( m \) and \( n; \) nonnegative integer \( q \) satisfying \( 1 - g \leq -q \leq 0 \), we have:

\[(3.9) \quad M^{(m+n,q)} = \sum_{r=q}^{m+n} M^{(m,r)} M^{(n,q-r)}. \]

### 4 the canonical transformation.

Using \( H_j \) and \( K_j \) given by (2.7) and (2.8) respectively, we define \( \overline{H}_j \) and \( \overline{K}_j \) \((j = 1, \cdots, g)\) as follows:

\[(4.1) \quad \overline{H}_j = 2 \sum_{i=0}^{j-1} a_{i+1}(t) \left( H_{j-i} + T_{j-i}^* \right) \quad (j = 1, \cdots, g), \]

\[(4.2) \quad \overline{K}_j = 2 \sum_{i=0}^{j-1} a_{i+1}(t) K_{j-i} \quad (j = 1, \cdots, g), \]

Note that (4.1) is nothing but (0.6). Combining (2.7) with (4.1) and (2.8) with (4.2), we obtain

\[(4.3) \quad \overline{H}_j = \frac{1}{2} \sum_{(k)} \left[ N_k \overline{N}^{j,k} \mu_2^k - \overline{U}_{jk} \mu_k - N_k \overline{N}^{j,k}(2\alpha + 1) \lambda_k^2 + \overline{T}_j \right], \]
\[
\overline{IC}_j = \frac{1}{2} \sum_{(k)} \left( N_k \overline{N}^{j,k} - \sum_{(l)} \frac{N_l \overline{N}^{j,l}}{\lambda_k - \lambda_l} \nu_k - N_k \overline{N}^{j,k} \nu_k \right),
\]

where

\[
\overline{N}^{j,k} = 2 \sum_{i=0}^{j-1} a_{i+1}(t) N^{j-i,k},
\]

\[
\overline{U}_{j,k} = 2 \sum_{i=0}^{j-1} a_{i+1}(t) U_{j-i,k},
\]

\[
\overline{T}_j^* = 2 \sum_{0i=}^{j-1} a_{i+1}(t) T^{*j,i}.
\]

We have

**Lemma 4.1.** \( \overline{K}_j \) and \( \overline{H}_j \) have the following relation:

\[
(4.6) \quad \overline{K}_j = \overline{H}_j + \frac{1}{2} \sum_{k} \lambda^j_k.
\]

**Proposition 4.1.** The transformation defined by (2.5) and (4.6)

\[
(\lambda, \mu, \overline{H}, t) \rightarrow (\lambda, \nu, \overline{I}, t)
\]

is canonical, where \( \lambda = (\lambda_1, \cdots, \lambda_g) \), \( \mu = (\mu_1, \cdots, \mu_g) \), \( \overline{H} = (\overline{H}_1, \cdots, \overline{H}_g) \), \( \nu = (\nu_1, \cdots, \nu_g) \), \( \overline{K} = (\overline{K}_1, \cdots, \overline{K}_g) \) and \( t = (t_1, \cdots, t_g) \).

## 5 the \( A_g \)-system.

In this section, we will prove Main Theorem. By means of propositions 1.2, 1.3 and 2.3, it suffices to establish the following theorem:

**Theorem 5.1.** The conditions (1.3), (1.4) of the complete integrability are equivalent to the following completely integrable Hamiltonian system:

\[
\frac{\partial \overline{K}_j}{\partial \nu_k} = \frac{\partial \lambda_k}{\partial t_j}, \quad \frac{\partial \overline{K}_j}{\partial \nu_k} = \frac{\partial \nu_k}{\partial t_j}, \quad (j, k = 1, \cdots, g).
\]

**Lemma 5.1.** The equation (1.4) induces the system (\( \overline{K} \)).

**Lemma 5.2.** The equations (1.4) is derived from the system (\( \overline{K} \)).

**Lemma 5.3.** The equation (1.3) is derived from the system (\( \overline{K} \)).

**Lemma 5.4.** System (\( \overline{K} \)) is complete integrable.

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