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A Riemannian Geometrical Analysis of the Filament Equation

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abstract

The motion of an isolated thin vortex filament with very small curvature is governed by the filament equation. We describe this equation as a geodesic equation on a group manifold $C^\infty(L, SO(3))$ in view of the fact that T_t'' plays the role of the *angular velocity* in the equation of motion for the tangent vector T_t.

We investigate the instability of the filament equation by the calculation of the sectional curvature. We show that the spiral vortex filament is not unstable for the *short wave length* perturbation. We also investigate the instability of the Hasimoto soliton to find it is not unstable for particular perturbation which may have some relation to the integrability.

1. Introduction

The dynamics of an isolated thin vortex filament with very small curvature, embedded in a perfect (incompressible and inviscid) fluid is well approximated by the *local induction equation* (or the *filament equation*) [1]:

$$\frac{\partial \gamma_t}{\partial t} = \frac{\partial \gamma_t}{\partial s} \times \frac{\partial^2 \gamma_t}{\partial s^2} \tag{1}$$

with a suitable normalization of the time $t \in \mathbf{R}$, where $s \in L$ is the arc length parameter and $\gamma_t : L \to M$ is the C^∞ embedding of a 1-dimensional manifold L into a 3-dimensional oriented manifold M. The manifold L, which denotes the space of the arclength parameter s, is S^1 if the vortex filament is spatially periodic or a closed, though it is the real numbers $\mathbf{R}$ in general case. In a conventional term in physics, $\gamma_t(s)$ corresponds to the position vector of a point on the filament denoted by the arc length parameter s at the time t. This equation was originally derived and studied by Da Rios [2] and Levi Civita [3], and later re-derived and compared with experiments ([4], [5], [6]).

A well known local coordinate system at each point $\gamma_t(s) \in M$ on the filament is spanned by orthonormal (by a metric on M) basis belong to $T_{\gamma_t(s)}M$. They are, the unit tangent vector $T_t(s) \equiv \partial\gamma_t/\partial s$, the principal normal vector $N_t(s)$ and the binormal vector $B_t(s)$. These vectors satisfy the Frenet equation:

$$\frac{d}{ds}\begin{pmatrix} T_t \\ N_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0 & \kappa_t & 0 \\ -\kappa_t & 0 & \tau_t \\ 0 & -\tau_t & 0 \end{pmatrix}\begin{pmatrix} T_t \\ N_t \\ B_t \end{pmatrix},$$

where $\kappa_t(s) \in R$ is the curvature and $\tau_t(s) \in R$ the torsion of the filament. Then one can easily obtain the equation of motion for the tangent vector $T_t(s)$ by differentiation of Eq.(1) with respect to s:

$$\frac{\partial T_t}{\partial t} = T_t \times T_t'', \quad \left(T_t''(s) \equiv \frac{\partial^2 T_t(s)}{\partial s^2}\right), \tag{2}$$

where a prime denotes a partial differentiation with respect to s. Eq.(2) can be transformed to the cubic-nonlinear Schrödinger equation (NLS) for the function $\psi(s,t)$, by the Hasimoto transformation $\psi(s,t) = \kappa \exp\left(\int^s \tau \, ds\right)$[7]. According to this fact, the filament equation is said to be integrable because NLS is integrable.

2. Hamiltonian Description

Now, we determine the correspondence $\eta : T_{\gamma_t(s)}M \to \mathcal{G}$, where $\mathcal{G}$ is the Lie algebra $C^\infty(L, so(3))$. In a faithful 3×3 matrix representation ρ^{I}, this correspondence maps the unit orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of a Cartesian coordinate system on $T_{\gamma_t(s)}M$ to the basis of the Lie algebra $\mathcal{G}$ as follows:

$$\lambda^{\mathrm{I}}(\mathbf{e}_1) = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^{\mathrm{I}}(\mathbf{e}_2) = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \ \text{and} \ \lambda^{\mathrm{I}}(\mathbf{e}_3) = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $\lambda^{\mathrm{I}} = \rho^{\mathrm{I}} \circ \eta : T_{\gamma_t(s)}M \to \mathcal{M}(3,3)_{\mathcal{F}(M)}$. ($\mathcal{M}(3,3)_{\mathcal{F}(M)}$ is the 3×3 matrix $\mathcal{F}(M)$−module for the functions $\mathcal{F}(M)$ on M.) In this representation of $\mathcal{G}$, the Lie bracket $[\ *\ ,\ *\ ]^{\mathrm{I}}$: $\rho^{\mathrm{I}}(\mathcal{G}) \times \rho^{\mathrm{I}}(\mathcal{G}) \to \rho^{\mathrm{I}}(\mathcal{G})$ is given by the local commutator of the 3×3 matrices:

$$[\Omega_1^{\mathrm{I}},\ \Omega_2^{\mathrm{I}}]^{\mathrm{I}}(s) = \frac{1}{\sqrt{2}\, i}\{\Omega_1^{\mathrm{I}}(s)\Omega_2^{\mathrm{I}}(s) - \Omega_2^{\mathrm{I}}(s)\Omega_1^{\mathrm{I}}(s)\}, \tag{3}$$

where $\Omega_i^{\mathrm{I}} = \rho^{\mathrm{I}}(\Omega_i)$ for $\tilde{\Omega}_i^{\mathrm{I}} \in \mathcal{G}$ $(i = 1, 2)$.

Eq.(2) implies that T_t is a solution of the following equation for $X_t \in \rho^{\mathrm{I}}(C^\infty(L, SO(3)))$:

$$\frac{\partial X_t}{\partial t} = [X_t,\ \Omega_t^{\mathrm{I}}]^{\mathrm{I}}, \quad \left(\Omega_t^{\mathrm{I}} = \lambda^{\mathrm{I}}(T_t'') \in \mathcal{G}\right), \tag{4}$$

where $C^\infty(L, SO(3))$ is the sections of the trivial fiber bundle $L \times SO(3)$.

In other words, the group $G = C^\infty(L, SO(3))$ acts on itself from the right:

$$\Omega_t^{\mathrm{I}} = \rho^{\mathrm{I}}\left(\Phi_t^{-1}\frac{\partial \Phi_t}{\partial t}\right), \quad (\Phi_t \in G = C^\infty(L, SO(3)))\,. \tag{5}$$

On the analogy of the rotation of a rigid body in the classical mechanics, the equations from (3) to (5) imply that $\eta(T_t'') \in \mathcal{G}$ corresponds to the *angular velocity* of the motion of the vectors X_t in the body-fixed coordinates.

It is well known that Eq. (2) is closely related to the evolution equation for the continuum limit of the one-dimensional classical Heisenberg chain (see Faddeev and Takhtajan [14], etc.). But there is a slight difference because the configuration space for the motion of a vortex filament is the (infinite-dimensional) manifold $C^\infty(L, SO(3))$ while that for the continuum classical Heisenberg model is $C^\infty(L, SU(2))$.

The dual space $\mathcal{G}^*$ of the algebra $\mathcal{G}$ is introduced by the natural pairing $\langle\, * \,,\, * \,\rangle : \mathcal{G}^* \times \mathcal{G} \to \mathbf{R}$. It is also represented by the elements of $\mathcal{M}(3,3)_{\mathcal{F}(M)}$. One obtains Eq. (2) as equation of motion on $\mathcal{G}^*$ because it is found to be a Hamilton's equation. In fact, if one introduce the *quadratic left-invariant* Hamiltonian $H^{\mathrm{I}} : \rho^{\mathrm{I}}(\mathcal{G}^*) \to \mathbf{R}$ and the Lie-Poisson bracket $\{\, * \,,\, * \,\}^{\mathrm{I}} : C^\infty(\rho^{\mathrm{I}}(\mathcal{G}^*)) \times C^\infty(\rho^{\mathrm{I}}(\mathcal{G}^*)) \to C^\infty(\rho^{\mathrm{I}}(\mathcal{G}^*))$ [17] as

$$H^{\mathrm{I}}(\omega_t^{\mathrm{I}}) = -\frac{1}{2}\int_L ds\; tr(\omega_t^{\mathrm{I}}(s),\, \partial_s^2\omega_t^{\mathrm{I}}(s)), \quad (\,\omega_t^{\mathrm{I}} = \lambda^{\mathrm{I}}(T_t) \in \rho^{\mathrm{I}}(\mathcal{G}^*)\,), \tag{6}$$

and

$$\{F,\, G\}^{\mathrm{I}}(\omega_t^{\mathrm{I}}) = \left\langle \omega_t^{\mathrm{I}},\, \left[\frac{\delta F}{\delta\omega^{\mathrm{I}}},\, \frac{\delta G}{\delta\omega^{\mathrm{I}}}\right]^{\mathrm{I}} \right\rangle, \quad \left(\frac{\delta F}{\delta\omega^{\mathrm{I}}},\, \frac{\delta G}{\delta\omega^{\mathrm{I}}} \in \rho^{\mathrm{I}}(\mathcal{G}) \text{ and } \omega_t^{\mathrm{I}} \in \rho^{\mathrm{I}}(\mathcal{G}^*)\right), \tag{7}$$

respectively, where $\delta F/\delta\omega^{\mathrm{I}} \in \rho^{\mathrm{I}}(\mathcal{G})$ is defined by

$$\left.\frac{d}{d\tau}F(\omega^{\mathrm{I}} + \tau\sigma)\right|_{\tau=0} = \left\langle \sigma,\, \frac{\delta F}{\delta\omega^{\mathrm{I}}} \right\rangle,$$

then one can immediately obtain Eq.(2) from the Hamilton's equation of motion:

$$\frac{d}{dt}F(\omega_t^{\mathrm{I}}) = \{F,\, H\}^{\mathrm{I}}(\omega_t^{\mathrm{I}}). \tag{8}$$

On the analogy of the rotation of a rigid body, $\eta(T_t) \in \mathcal{G}^*$ has the sense of the *angular momentum* (see [17], etc.).

3. Riemannian Geometrical Description

In 1966, Arnold [8] deduced *right-* (or *left-*) *invariant* metric on a group manifold from a *quadratic right-* (or *left-*) *invariant* Hamiltonian, which makes the geodesic equation equivalent to the corresponding Hamilton's equation.

He applied this method to the instability analysis of the Euler equation which describes the motion of a perfect fluid (see also [10]). In the rest of this report, first, we describe Eq.(2) as the geodesic equation on the manifold $G = C^\infty(L, SO(3))$ following Arnold's method. Next, we investigate the instability of the motion of a vortex filament by the calculation of the sectional curvature.

The Hamiltonian of (6) induces a *left-invariant* Riemannian metric $\langle\langle\, * \,,\, * \,\rangle\rangle^{\mathrm{I}}_\phi : T_\phi G \times T_\phi G \to \mathbf{R}$ at $\phi \in G$:

$$\langle\langle \Omega_{1\_\phi},\, \Omega_{2\_\phi}\rangle\rangle^{\mathrm{I}}_\phi = \langle g^{\mathrm{I}}(\Omega^{\mathrm{I}}_1), \Omega^{\mathrm{I}}_2\rangle = -\int_L ds\ tr(\partial_s^{-2}\Omega^{\mathrm{I}}_1(s),\, \Omega^{\mathrm{I}}_2(s)). \tag{9}$$

In Eq.(9), the one-to-one correspondence $g^{\mathrm{I}} : \rho^{\mathrm{I}}(\mathcal{G}) \to \rho^{\mathrm{I}}(\mathcal{G}^*)$ and left invariant vectors $\Omega_{j\_\Phi} \in T_\Phi G$ (j=1, 2) are defined as

$$g^{\mathrm{I}}(\Omega^{\mathrm{I}}(s)) \equiv -\partial_s^{-2}\Omega^{\mathrm{I}}(s) = -\int^s\int^{s_2} \Omega^{\mathrm{I}}(s_1)\, ds_1 ds_2 \quad \text{and} \quad \Omega_{j\_\Phi} f(\Phi) \equiv \frac{d}{d\epsilon} f\left(\Phi\ \exp(\epsilon\Omega_j)\right)\Big|_{\epsilon=0},$$

respectively, where $\Omega^{\mathrm{I}} \in \rho^{\mathrm{I}}(\mathcal{G})$, $g^{\mathrm{I}}(\Omega^{\mathrm{I}}) \in \rho^{\mathrm{I}}(\mathcal{G}^*)$ and $f \in C^\infty(G)$.

The Riemannian manifold defined above admits a unique Riemannian connection ∇^{I} [16]:

$$\nabla^{\mathrm{I}}_{\Omega_{1\_\Phi}}\Omega_{2\_\Phi} = \frac{1}{2}\left([\Omega^{\mathrm{I}}_1, \Omega^{\mathrm{I}}_2] + \partial_s^2[\Omega^{\mathrm{I}}_1, \partial_s^{-2}\Omega^{\mathrm{I}}_2] + \partial_s^2[\Omega^{\mathrm{I}}_2, \partial_s^{-2}\Omega^{\mathrm{I}}_1]\right)\Big|_\Phi . \tag{10}$$

Then the geodesic equation, $\nabla^{\mathrm{I}}_{\Omega_{t\_\Phi_t}}\Omega_{t\_\Phi_t} = 0$, leads to the equation of the motion for T_t'':

$$\frac{\partial T_t''}{\partial t} = \left(T_t \times T_t''\right)'' . \tag{11}$$

4. Modification of the Representation

As we will show in next section, we can investigate the instability of the geodesic equation by considering the variation of the geodesic. But what we can investigate from the formulation in previous section is the motion of T'' not of T itself. To obtain Eq.(1) as a geodesic equation, we change the representation of the algebra $\mathcal{G}$.

We adopt following local commutator $[\, * \,,\, * \,]^{\mathrm{II}} : \rho^{\mathrm{II}}(\mathcal{G}) \times \rho^{\mathrm{II}}(\mathcal{G}) \to \rho^{\mathrm{II}}(\mathcal{G})$ instead of Eq.(3):

$$[\Omega_1,\, \Omega_2]^{\mathrm{II}}(s) = \frac{1}{\sqrt{2}\, i}\partial_s^{-2}\{\ \partial_s^2\Omega^{\mathrm{II}}_1(s)\ \partial_s^2\Omega^{\mathrm{II}}_2(s) - \partial_s^2\Omega^{\mathrm{II}}_2(s)\ \partial_s^2\Omega^{\mathrm{II}}_1(s)\ \},\quad (\Omega^{\mathrm{II}}_1, \Omega^{\mathrm{II}}_2 \in \rho^{\mathrm{II}}(\mathcal{G})). \tag{12}$$

In this case, we correspond $T(s)$ and the elements of $\mathcal{G}$, i.e., $\Omega^{\mathrm{II}}_t(s) = \lambda^{\mathrm{II}}(T_t(s)) = \rho^{\mathrm{II}}(\eta(T_t(s))) \in \rho^{\mathrm{II}}(\mathcal{G})$. We also modify the Hamiltonian and Riemannian metric as follows:

$$H^{\mathrm{II}}(\omega^{\mathrm{II}}_t) = -\frac{1}{2}\int_L ds\ tr(\partial_s^{-2}\omega^{\mathrm{II}}_t(s),\, \omega^{\mathrm{II}}_t(s)),\quad \left(\omega^{\mathrm{II}}_t = \lambda^{\mathrm{II}}(T_t'') \in \rho^{\mathrm{II}}(\mathcal{G}^*)\right), \tag{13}$$

and

$$\langle\langle \Omega_{1\_\phi},\ \Omega_{2\_\phi}\rangle\rangle^{\rm II}_{\phi} = \langle g^{\rm II}(\Omega^{\rm II}_1), \Omega^{\rm II}_2\rangle = -\int_L ds\ tr(\partial_s^2\Omega^{\rm II}_1(s),\ \Omega^{\rm II}_2(s)), \tag{14}$$

where modified one-to-one correspondence $g^{\rm II} : \rho^{\rm II}(\mathcal{G}) \to \rho^{\rm II}(\mathcal{G}^*)$ is defined as $g^{\rm II}(\Omega^{\rm II}(s)) = -\partial_s^2\Omega^{\rm II}(s)$. In this representation from Eq.(12) to Eq.(14), though the correspondences between vectors $(T,\ T'')$ and the spaces $(\mathcal{G},\ \mathcal{G}^*)$ are exchanged, the Hamiltonian and the Riemannian metric are also modified to give same quantity as in previous representation. In fact, once the shape of a vortex filament is determined, one can easily confirm $H^{\rm I} = H^{\rm II} = \kappa^2/2$. According to this modification, we obtain Eq.(11) (the equation of motion for T'') as Hamilton's equation and Eq.(2) (the equation of motion for T) as the geodesic on the manifold $G = C^\infty(L,\ SO(3))$.

5. Instability Analysis

We will consider how the instability of the motion of a vortex filament can be predicted by calculating the sectional curvature on G. The infinitesimal variational field $\xi_{t\_\Phi} \in T_\Phi G$ between two neighboring geodesic curves is governed by the following Jacobi equation:

$$\nabla_{\Omega_{t\_\Phi_t}}\nabla_{\Omega_{t\_\Phi_t}}\xi_{t\_\Phi_t} = -R(\xi_{t\_\Phi_t}, \Omega_{t\_\Phi_t})\Omega_{t\_\Phi_t}, \tag{15}$$

where $R(\Omega_{1\_\Phi},\ \Omega_{2\_\Phi}) = [\nabla_{\Omega_{1\_\Phi}},\ \nabla_{\Omega_{2\_\Phi}}] - \nabla_{[\Omega_{1\_\Phi},\ \Omega_{2\_\Phi}]}$ is the Riemannian curvature tensor. Then the norm of $\xi_{t\_\Phi} \in T_\Phi G$ evolves as,

$$\frac{d^2}{dt^2}\langle\langle \xi_{t\_\Phi_t},\ \xi_{t\_\Phi_t}\rangle\rangle_{\Phi_t} = 2\left\{\langle\langle \nabla_{\Omega_{t\_\Phi_t}}\xi_{t\_\Phi_t},\ \nabla_{\Omega_{t\_\Phi_t}}\xi_{t\_\Phi_t}\rangle\rangle_{\Phi_t} - \langle\langle \xi_{t\_\Phi_t}, R(\xi_{t\_\Phi_t},\ \Omega_{t\_\Phi_t})\Omega_{t\_\Phi_t}\rangle\rangle_{\Phi_t}\right\}. \tag{16}$$

Eq.(16) indicates if the sectional curvature $\langle\langle \xi_{t\_\Phi_t}, R(\xi_{t\_\Phi_t}, \Omega_{t\_\Phi_t})\Omega_{t\_\Phi_t}\rangle\rangle_{\Phi_t}$ is negative, then the right hand side of Eq.(16) must be positive, that is, the system is unstable in this sense. In our formulation (by the representation $\rho^{\rm II}$, the sectional curvature on G is calculated as

$$\langle\langle\ \Omega_{1\_\Phi_t},\ R(\Omega_{1\_\Phi_t}, \Omega_{2\_\Phi_t})\,\Omega_{2\_\Phi_t}\ \rangle\rangle_{\Phi_t} = \int_L ds\ f_{\Omega_1,\Omega_2}(s)\ ds, \tag{17}$$

for the following function $f_{\Omega_1,\Omega_2}(s) : L \to \mathbf{R}$:

$$\begin{aligned} f_{\Omega_1,\Omega_2}(s) \ &= \ tr\left([\Omega_1,\ \Omega_1''][\Omega_2,\ \Omega_2'']\right) + \frac{3}{4}tr\left([\Omega_1,\ \Omega_2'']\,\partial_s^{-2}[\Omega_1,\ \Omega_2'']\right) \\ &- \ \frac{1}{4}tr\left(\left\{[\Omega_1,\ \Omega_2''] + [\Omega_2,\ \Omega_1'']\right\}\left\{[\Omega_1,\ \Omega_2''] + [\Omega_2,\ \Omega_1'']\right\}\right) \qquad (18) \\ &- \ \frac{1}{2}\left\{tr\left(\Omega_1''^2\right)\ tr\left(\Omega_2\Omega_2''\right) + tr\left(\Omega_2''^2\right)\ tr\left(\Omega_1\Omega_1''\right)\right\} \\ &- \ \frac{1}{2}tr\left(\Omega_1''\Omega_2''\right)\left\{\ tr\left(\Omega_1\Omega_2''\right) + tr\left(\Omega_1''\Omega_2\right)\right\}, \end{aligned} \tag{19}$$

where we removed the symbol $^{\mathrm{II}}$ for simplicity. If $L = S^1$ which means that the vortex filament is spatially periodic, the third and fourth terms in the right hand side of Eq.(18) are positive while the second term is negative. The balance between these terms determines the instability. One of the Ω_j's ($j = 1, 2$) corresponds to the tangent vector of the filament and the other perturbation. In the following of this report, we assume $\Omega_1 = \eta(T)$.

In the simplest example of the straight vortex filament, it is not unstable because all terms but the third in the right hand side of $f_{\Omega_1,\Omega_2}(s)$ vanishes. In fact, it can be easily confirmed because $(\Omega_1)' = \eta(\partial_s T) = 0$.

For the second example, let us consider the instability of the motion of a *spiral* vortex filament. The tangent vector $T_t \in C^\infty(S^1, SO(3))$ is described for real constants a, h and $c = 1/\sqrt{(a^2+h^2)}$ as

$$T_t(s) = (-ac\sin(cs),\ ac\cos(cs),\ hc)\,, \tag{20}$$

which is a special solution of Eq.(2). For the validity of the *local induction approximation*, we assume $a/c \approx a/h \approx \epsilon << 1$ because the curvature of the filament, $\kappa = a/c$, has to be very small. We consider the perturbation $\Omega_2 = \eta(\xi)$ which satisfies following two conditions: (i) $\langle\langle \Omega_{1\_\phi_t}, \Omega_{2\_\phi_t}\rangle\rangle_{\phi_t} = 0$ and (ii) $tr(\Omega_{1\_\phi_t},\ \Omega_{2\_\phi_t}) = 0$. The condition (i) means the orthogonal relation between the tangent vector and the perturbation vector by the metric on $C^\infty(S^1,\ SO(3))$, while (ii) is the same relation by the metric on M. The latter is introduced by a physical requirement that ξ doesn't stretch or shorten the filament. Then the perturbation for the spiral filament can be generally written as:

$$\xi_{t_0} = (f(s),\ g(s),\ a\{f(s)\cos(s/c) - g(s)\sin(s/c)\}/h),$$

where $f(s), g(s) : S^1 \to \mathbf{R}$.

After the calculation of the sectional curvature, we found that the positive third term is of $O(1)$ order while others are higher than $O(\epsilon)$ order, if the functions $f(s)$ and $g(s)$ have Fourier modes the wavelength of which is shorter than c^{-1}. Thus, if the perturbation has 'short wave length', the spiral vortex filament is not unstable. On the contrast, if the perturbation has 'long enough wavelength' $\lambda >> c^{-1}$, this system is unstable because the negative second term becomes dominant. This result is consistent with the Helmholtz's classical work.

For third example, we consider the stability of the Hasimoto soliton. Hasimoto found that Eq.(2) can be transformed to the cubic-nonlinear Schrödinger equation (NLS) for the function $\psi(s,t)$, by the transformation $\psi(s,t) = \kappa\exp\left(\int^s \tau\, ds\right)$[7]. He constructed one soliton solution for the infinitely long vortex filament as follows:

$$T_t(s)_x = 1 - 2\mu\,\mathrm{sech}^2(\eta),\ T_t(s)_y + iT_t(s)_z = -2\mu\mathrm{sech}(\eta)\left(\tanh(\eta) - i\frac{\nu}{\tau_0}\right)e^{\left(\tau_0 s + (\nu^2 - \tau^2)t\right)}, \tag{21}$$

where μ, ν and τ_0 are constants and $\eta = \nu(s - 2\tau_0 t)$. To describe the vortex motion as a geodesic in this case, we adopt Riemannian submanifold of $C^\infty(\mathbf{R}, SO(3))$ the elements of which have finite norms. The calculation of the sectional curvature of this system is more difficult than the *spiral* vortex filament because $L = \mathbf{R}$ means the existence of infinite continuous Fourier modes. For first trial, we choose $\xi_0 = \kappa_0 N_0$ as the perturbation to investigate the instability. Though the calculation cannot be completed analytically, we found that the sectional curvature is positive by numerical integration. Thus we can say that the one soliton solution is not unstable for this perturbation. We are now investigating the case of more general perturbation [15].

Discussion

It is plausible that the property of positive curvature has some relation to the integrability of the filament equation. In the case of the Hasimoto soliton, we suppose the non-negativeness of the curvature, in other words, the nature of *anti-instability*, will also appear for more general perturbation. The sectional curvature will nearly vanishes for many cases because the soliton solution (21) is spatially localized. We will also investigate the instability of the vortex filaments of more general shapes and apply this formalism for the other equations including the soliton equations in the future.

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