数理解析研究所講究録
973巻 1996年 177-191

Finite element approximation for some quasilinear elliptic problems

Yuki Matsuzawa*

November 3, 1996

1 Introduction

Our purpose is to study the finite element approximation for some simple quasilinear elliptic problems.

Let $\Omega \subset \mathbf{R}^N$ be an N-dimensional polyhedral domain and $A: \mathbf{R} \to \mathbf{R}$ a Lipschitz continuous function satisfying

$$A(s) \geq C_a \qquad \left({}^\forall s \in \mathbf{R}\right)$$

with a constant $C_a > 0$. We are interested in the boundary value problem

$$-\nabla \cdot (A(u)\nabla u) = f \quad \text{in } \Omega \tag{1}$$
$$u = 0 \quad \text{on } \partial\Omega \tag{2}$$

and its numerical computations, where

$$f = f_0 + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} f_i.$$

Based on our previous work concerning the L^∞ estimate for the Ritz operator associated with the second order elliptic operator of irregular coefficients ([5]), we can extend some results by [1].

*Department of Mathematics, Osaka University, Machikaneyamacho 1-1, Toyonakashi, 560, JAPAN

Namely we can show the existence of the approximate solution u_h as well as the order estimates for $\|u_h - u\|_{H^1}$ and $\|u_h - u\|_{L^\infty}$, provided that f is small in some sense. Furtherermore, even for the general f we can show the convergence in those norms.

The problem (1) with (2) is formulated variationally. First, V denotes $H_0^1(\Omega)$ and

$$a(w : u, v) = \int_\Omega A(w)\nabla u \cdot \nabla v \qquad (u, v \in V),$$

where $w \in L^\infty(\Omega)$. Next,

$$F(v) = \int_\Omega \left(f_0 v - \sum_{i=1}^{N} f_i \frac{\partial v}{\partial x_i} \right) \qquad (v \in V). \tag{3}$$

Then $u \in V \cap L^\infty(\Omega)$ satisfying

$$a(u : u, v) = F(v) \qquad \left({}^\forall v \in V\right) \tag{4}$$

is regarded as a weak solution for (1) with (2).

We suppose $f_i \in L^p(\Omega)$ $(0 \le i \le N)$ for $p > \max\{N, 2\}$ and hence

$$|F(v)| \le C\beta \|v\|_{W^{1,p'}} \qquad (v \in V),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $C > 0$ being a constant, and $\beta = \sum_{i=0}^{N} \|f_i\|_{L^p}$.

The problem (4) is discretized as follows. Let $\{\tau_h\}_{0<h\le h_0}$ be a family of regular triangulations of Ω and

$$\begin{aligned} W_h &= \left\{\chi_h \in C\left(\overline{\Omega}\right) \mid \chi_h|_T : \text{ linear} \qquad \left({}^\forall T \in \tau_h\right)\right\}, \\ V_h &= W_h \cap V, \end{aligned}$$

$h > 0$ being a size parameter.

Then, we take $u_h \in V_h$ satisfying

$$a(u_h : u_h, v_h) = F(v_h) \qquad \left({}^\forall v_h \in V_h\right). \tag{5}$$

The existence of such u_h will be assured by Brouwer's fixed point theorem, where some a priori estimates of the solution $w_h = T_h u_h$ for

$$a(u_h : w_h, v_h) = F(v_h) \qquad \left({}^\forall v_h \in V_h\right)$$

are necessary.

We make use of the previous argument ([5]) for this part and the next section is devoted to it. Henceforth, $u \in V \cap L^\infty(\Omega)$ denotes a weak solution for (1) with (2), which is supposed to exist.

2 A priori estimate for linear problems

We take coefficients $a_{ij} = \delta_{ij}a(x) \in L^\infty(\Omega)$ satisfying

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \qquad \left(\xi = (\xi_1, \cdots, \xi_N) \in \mathbf{R}^N, x \in \Omega\right), \tag{6}$$

$\lambda > 0$ being a constant.

Introducing

$$a(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} \qquad (u, v \in V),$$

we consider the problem

$$a(u_h, v_h) = F(v_h) \qquad \left({}^\forall v_h \in V_h\right), \tag{7}$$

where $F(v)$ is defined by (3).

Unique existence of such $u_h \in V_h$ is assured by Riesz' representation theorem and Poincaré's inequality

$$\|v\|_{L^2} \leq C_p \|\nabla v\|_{L^2} \qquad (v \in V). \tag{8}$$

Then, we can claim the following theorem.

Theorem 1 *Let $N \leq 3$ and $P_0(T) \in \overline{T}$ for any $T \in \tau_h$, where $P_0(T)$ denotes the center of the circumscibing ball of T. Then, there exists a constant $C > 0$ determined only by $p > \max\{N, 2\}$, N, and C_p such that*

$$\|u_h\|_{L^\infty} \leq C\lambda^{-1} \sum_{i=0}^{N} \|f_i\|_{L^p}. \tag{9}$$

Proof : We introduce the non-linear operator $J_h : W_h \to W_h$ by

$$J_h\chi_h|_a = \max\{\chi_h|_a, 0\},$$

where $a \in T$ denotes a vertex and $T \in \tau_h$. For a constant $k \geq 0$, let

$$\begin{aligned} \chi &= \chi_k = u_h - k \in W_h \\ \eta &= \eta_k = J_h\chi \in V_h. \end{aligned}$$

Then

$$\begin{aligned} \lambda \|\nabla\eta\|_{L^2}^2 &\le a(\eta,\eta) \\ &= -a(u_h-\eta,\eta)+a(u_h,\eta). \end{aligned}$$

Here, Lemma 1 of [5] implies

$$\begin{aligned} a(u_h-\eta,\eta) &= a(u_h-k-\eta,\eta) \\ &= a(\chi-J_h\chi, J_h\chi) \\ &\ge 0 \end{aligned}$$

so that

$$\begin{aligned} \lambda \|\nabla\eta\|_{L^2}^2 &\le a(u_h,\eta) \\ &= F(\eta) \\ &\le \sum_{i=0}^{N} \|f_i\|_{L^2(\omega)} \|\eta\|_{H^1} \\ &\le (C_p+1)\|\nabla\eta\|_{L^2} \sum_{i=0}^{N} \|f_i\|_{L^2(\omega)}, \end{aligned}$$

where $\omega=\omega_k=\mathrm{supp}\ \eta$. In other words

$$\|\nabla\eta\|_{L^2} \le C\lambda^{-1}\sum_{i=0}^{N}\|f_i\|_{L^2(\omega)}.$$

For $1\le q\le 2$ we have

$$\|\nabla\eta\|_{L^q} \le |\omega|^{\frac{1}{q}-\frac{1}{2}}\|\nabla\eta\|_{L^2}$$

and

$$\|f_i\|_{L^2(\omega)} \le |\omega|^{\frac{1}{2}-\frac{1}{p}}\|f_i\|_{L^p(\Omega)}.$$

We note the relation $\eta|_{\partial\Omega}=0$ to deduce

$$\|\eta\|_{L^{q^*}} \le C\|\nabla\eta\|_{L^q},$$

where $\frac{1}{q^*}=\frac{1}{q}-\frac{1}{N}$. Futhermore,

$$\begin{aligned} \|\eta\|_{L^1} &= \|\eta\|_{L^1(\omega)} \\ &\le |\omega|^{1-\frac{1}{q^*}}\|\eta\|_{L^{q^*}}. \end{aligned}$$

Combining those inequalities, we get

$$\begin{aligned} \|\eta_k\|_{L^1} &= \|\eta\|_{L^1} \\ &\le C\lambda^{-1} |\omega|^{1-\frac{1}{q^*}+\frac{1}{q}-\frac{1}{2}} \sum_{i=0}^{N} \|f_i\|_{L^2(\omega)} \\ &\le C\lambda^{-1} |\omega|^{\gamma} \sum_{i=0}^{N} \|f_i\|_{L^p(\Omega)} \\ &= C\lambda^{-1} |\omega_k|^{\gamma} \sum_{i=0}^{N} \|f_i\|_{L^p(\Omega)} . \end{aligned}$$

Here

$$\begin{aligned} \gamma &= 1 - \frac{1}{q^*} + \frac{1}{q} - \frac{1}{2} + \frac{1}{2} - \frac{1}{p} \\ &= 1 + \frac{1}{N} - \frac{1}{p} > 1. \end{aligned}$$

We recall Lemma 2 of [5]. Namely,

$$|T| \, \|\eta\|_{L^\infty(T)} \le (N+1) \, \|\eta\|_{L^1(T)} ,$$

where $T \in \tau_h$ and $0 \le \eta \in V_h$.

Let

$$\begin{aligned} \rho(t) &= |\omega_t| = |\text{supp } \eta_t| \\ &= |\text{supp } J_h(u_h - t)| \end{aligned}$$

for $t \ge 0$. Because of the definition of J_h, it holds that

$$\int_k^\infty \rho(t)dt = \sum_{T\in\tau_h} |T| \, \|\eta_k\|_{L^\infty(T)} \qquad (k \ge 0) . \tag{10}$$

The right-hand side of (10) is dominated from above by

$$\begin{aligned} (N+1) \sum_{T\in\tau_h} \|\eta_k\|_{L^1(T)} &= (N+1) \, \|\eta_k\|_{L^1(\Omega)} \\ &\le (N+1)C\lambda^{-1} |\omega_k|^{\gamma} \sum_{i=0}^{N} \|f_i\|_{L^p(\Omega)} \\ &= (N+1)C\lambda^{-1} \rho(k)^{\gamma} \sum_{i=0}^{N} \|f_i\|_{L^p(\Omega)} . \end{aligned}$$

Similarly to [4] (*c.f.* [5]), the integral inequality

$$\int_k^\infty \rho(t)dt \leq (N+1)C\lambda^{-1}\rho(k)^\gamma \sum_{i=0}^N \|f_i\|_{L^p(\Omega)} \qquad (k \geq 0)$$

implies $\rho(k) = 0 \quad (k \geq k^*)$ for

$$k^* = \frac{\gamma}{\gamma-1} |\Omega|^{\gamma-1} (N+1)C\lambda^{-1} \sum_{i=0}^N \|f_i\|_{L^p(\Omega)}$$

or equivalently, $u_h(x) \leq k^* \quad \left(x \in \overline{\Omega}\right)$. The inequality $-u_h(x) \leq k^* \quad \left(x \in \overline{\Omega}\right)$ follows similarly. We get the conclusion (9). □

3 Solvability of the discrete problem

We recall the non-linear operator $T_h: \ V_h \to V_h$ defined by

$$a\left(u_h : T_h u_h, v_h\right) = F(v_h) \qquad \left({}^\forall v_h \in V_h\right).$$

We can apply Theorem 1 for $a_{ij}(x) = A\left(u_h(x)\right)\delta_{ij}$. For $\lambda = C_a > 0$ (6) holds. There is a constant $C > 0$ determined by N, $p > \max\{N, 2\}$, and the Poincaré constant C_p satisfying

$$\|T_h u_h\|_{L^\infty} \leq C C_a^{-1} \sum_{i=0}^N \|f_i\|_{L^p(\Omega)}$$

for any $u_h \in V_h$.

In other words,

$$T_h\left(V_h\right) \subset B = \left\{v_h \in V_h \mid \|v_h\|_{L^\infty} \leq K\right\},$$

where $K = CC_a^{-1}\sum_{i=0}^N \|f_i\|_{L^p(\Omega)}$. Therefore, Brouwer's fixed point theorem assures the following.

Theorem 2 *The non-linear operator T_h has a fixed point in B so that the discretized problem (5) has a solution.*

We note that [1] derived the same conclusion for $N = 2$ based on the Rannacher-Scott type estimate

$$\|R_h u\|_{W^{1,p}} \leq C \|u\|_{W^{1,p}}, \tag{11}$$

where $2 = N \leq p \leq \infty$ and $R_h : V \rightarrow V_h$ denotes the Ritz operator corresponding to elliptic operator satisfing some condition. For A (11) need the smoothess of coeficent. Using the duality argument, Theorem 2 is proven without smoothness of $A(s)$.

4 Error estimates for small data

Following the argument [1], we can derive the H^1 and L^∞ error estimates for the case of $\gamma < 1$, where $\gamma = C_a^{-1} L \|\nabla u\|_{L^p}$ with $p > \max\{N, 2\}$ and L being the Lipschitz constant of A on $I = [-l, l]$, $l = \max\{K, \|u\|_{L^\infty}\}$.

Acutually, the relations (4) and (5) imply for $v_h \in V_h$ that

$$\begin{aligned} a(u_h : u - u_h, v_h) &= a(u_h : u, v_h) - a(u_h : u_h, v_h) \\ &= a(u_h : u, v_h) - F(v_h) \\ &= a(u_h : u, v_h) - a(u : u, v_h) \\ &= \int_\Omega (A(u_h) - A(u)) \nabla u \cdot \nabla v_h. \end{aligned}$$

Therefore,

$$\begin{aligned} a(u_h : u - u_h, u - u_h) &= a(u_h : u - u_h, u - v_h) + a(u_h : u - u_h, v_h - u_h) \\ &= \int_\Omega A(u_h) \nabla (u - u_h) \cdot \nabla (u - v_h) \\ &\quad + \int_\Omega (A(u_h) - A(u)) \nabla u \cdot \nabla (v_h - u_h). \end{aligned} \tag{12}$$

The solution $u_h \in V_h$ of (5) satisfies $T_h u_h = u_h \in B$ and hence $\|u_h\|_{L^\infty} \leq K$. There exists a constant $M > 0$ such that

$$\|A(u_h)\|_{L^\infty} \leq M.$$

The first term of the right-hand side of (12) is dominated from above by

$$M \|\nabla(u - u_h)\|_{L^2} \|\nabla(u - v_h)\|_{L^2}.$$

On the other hand, the second term is estimated as

$$L\int_{\Omega}|u-u_h|\,|\nabla u|\,|\nabla(v_h-u_h)|\leq L\,\|u-u_h\|_{L^{\frac{2p}{p-2}}}\,\|\nabla u\|_{L^p}\,\|\nabla(v_h-u_h)\|_{L^2}\,.$$

In use of Sobolev's imbedding

$$H_0^1(\Omega)\hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$$

we have

$$\|u-u_h\|_{L^{\frac{2p}{p-2}}}\leq C\,\|\nabla(u-u_h)\|_{L^2}$$

because $p>\max\{N,2\}$.

Combining those estimates, we get

$$\begin{aligned}C_a\,\|\nabla(u-u_h)\|_{L^2}^2 &\leq a(u_h:u-u_h,u-u_h)\\ &\leq M\,\|\nabla(u-u_h)\|_{L^2}\,\|\nabla(u-v_h)\|_{L^2}\\ &\quad+L\,\|\nabla(u-u_h)\|_{L^2}\,\|\nabla u\|_{L^p}\,\|\nabla(v_h-u_h)\|_{L^2}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}C_a\,\|\nabla(u-u_h)\|_{L^2} &\leq M\,\|\nabla(u-v_h)\|_{L^2}\\ &\quad+L\,\|\nabla u\|_{L^p}\,\{\|\nabla(v_h-u)\|_{L^2}+\|\nabla(u-u_h)\|_{L^2}\}\end{aligned}$$

and hence

$$(1-\gamma)\,\|\nabla(u-u_h)\|_{L^2}\leq C_a^{-1}M\,\|\nabla(u-v_h)\|_{L^2}+\gamma\,\|\nabla(v_h-u)\|_{L^2}\,.$$

We have proven the following.

Theorem 3 *In the case of $\gamma<1$,*

$$\|\nabla(u-u_h)\|_{L^2}\leq\frac{C_a^{-1}M+\gamma}{1-\gamma}\inf_{v_h\in V_h}\|\nabla(u-v_h)\|_{L^2}\,.$$

In particular, $u_h\to u$ in $H_0^1(\Omega)$.

Now, we want to estimate $\|u_h-u\|_{L^\infty}$, supposing $u\in W^{1,p}(\Omega)$ for $p>\max\{N,2\}$.

Let $\hat{u}_h \in V_h$ be the solution of

$$a(u : \hat{u}_h, v_h) = F(v_h) \qquad (v_h \in V_h)\,. \tag{13}$$

Denote the Ritz operator associated with the bilinear form

$$a(u : v, w) = \int_\Omega A(u)\nabla v \cdot \nabla w \qquad (v, w \in V)$$

by $R_h : V \to V_h$. We have for $p > \max\{N, 2\}$ that

$$\|R_h v\|_{L^\infty} \le C C_a^{-1} M \|v\|_{W^{1,p}} \qquad \left(v \in V \cap W^{1,p}\right)$$

([5]).

Therefore, $\hat{u}_h = R_h u$ satisfies

$$\begin{aligned} \|\hat{u}_h - u\|_{L^\infty} &= \|(R_h - 1)(u - \chi_h)\|_{L^\infty} \\ &\le \|u - \chi\|_{L^\infty} + C C_a^{-1} M \|u - \chi_h\|_{W^{1,p}}\,, \end{aligned}$$

where $\chi_h \in V_h$. For any $v_h \in V_h$ we have

$$\begin{aligned} a(u_h : u_h - \hat{u}_h, v_h) &= a(u_h : u_h, v_h) - a(u_h : \hat{u}_h, v_h) \\ &= F(v_h) - a(u_h : \hat{u}_h, v_h) \\ &= a(u : \hat{u}_h, v_h) - a(u_h : \hat{u}_h, v_h) \\ &= \int_\Omega (A(u) - A(u_h))\, \nabla \hat{u}_h \cdot \nabla v_h. \end{aligned}$$

The right-hand side is equal to

$$\int_\Omega \sum_{j=1}^N \left(-f_j \frac{\partial v_h}{\partial x_j}\right),$$

where $f_j = -\left(A(u) - A(u_h)\right) \frac{\partial \hat{u}_h}{\partial x_j}$.

We have

$$a(u_h : u_h - \hat{u}_h, v_h) = \int_\Omega \sum_{j=1}^N \left(-f_j \frac{\partial v_j}{\partial x_j}\right) \qquad \left({}^\forall v_h \in V_h\right).$$

In use of Theorem 1 of §2 we obtain

$$\begin{aligned}\|u_h - \hat{u}_h\|_{L^\infty} &\le CC_a^{-1}\sum_{j=1}^{N}\|f_j\|_{L^p} \\ &\le CC_a^{-1}M\,\|A'\|_{L^\infty(I)}\,\|u-u_h\|_{L^\infty}\,\|\hat{u}_h\|_{W^{1,p}}.\end{aligned}$$

We recall that $A(u) \in W^{1,p}$ by $u \in W^{1,p} \subset L^\infty$ and that the estimate (11) holds if Ω is convex. Under this assumption we have

$$\|u_h - \hat{u}_h\|_{L^\infty} \le CC_a^{-1}M\,\|A'\|_{L^\infty}\,\|u\|_{W^{1,p}}\,\|u-u_h\|_{L^\infty}.$$

Putting $\gamma = CC_a^{-1}M\,\|A'\|_{L^\infty}\,\|u\|_{W^{1,p}}$, we have

$$\begin{aligned}\|u-u_h\|_{L^\infty} &\le \|u-\hat{u}_h\|_{L^\infty} + \|\hat{u}_h - u_h\|_{L^\infty} \\ &\le \|u-\chi_h\|_{L^\infty} + CC_a^{-1}M\,\|u-\chi_h\|_{W^{1,p}} + \gamma\,\|u-u_h\|_{L^\infty}.\end{aligned}$$

This implies the following theorem.

Theorem 4 *Under the above assumptions, furthermore, let Ω is convex and $\gamma < 1$.*

Then we have the estimate

$$\|u-u_h\|_{L^\infty} \le \frac{C}{1-\gamma}\left(1 + C_a^{-1}M\right)\inf_{\chi_h\in V_h}\|u-\chi_h\|_{W^{1,p}},$$

where C depend only on $p > \max\{N,2\}$, N, the Poincaré constant, and the constant C in (11).

In particular, $u_h \to u$ in L^∞.

5 Convergence for large data

Even in the case of $\gamma \ge 1$, when $u \in W^{1,p}(\Omega) \cap H_0^1(\Omega)$ with $p > \max\{N,2\}$, and the weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (1) with (2) is unique, the convergence

$$u_h \to u \qquad \text{in } H_0^1(\Omega)$$

holds as $h \to 0$. Those assumptions are actually hold when Ω and f_i are regular.

Define the weak solution $u \in H_0^1(\Omega) \cap L^\infty$ for (1) with (2) by

$$\int_\Omega A(u)Du \cdot Dv = \int_\Omega \left(f_0 v - \sum_{i=1}^N f_i \frac{\partial n}{\partial x_i} \right).$$

When Ω, f_i $(0 \le i \le N)$, and A is smooth, the weak solution is classical solution.

From the theorem of Giorgi-Stampacchia, $u \in C^\alpha(\overline{\Omega})$ $(0 < \alpha < 1)$ follows so that we get the linear elliptic regularity of L^∞ coefficient. Furthermore, from $A(u) \in C^\alpha(\overline{\Omega})$ and the theorem of Morrey, $u \in W^{1,p}(\Omega)$ and $A(u) \in W^{1,p}(\Omega)$ $(1 <^\forall p < \infty)$.

Since

$$\nabla \cdot (A(u)\nabla u) = \nabla A(u) \cdot \nabla u + A(u) \cdot \Delta u,$$

we have the problem

$$-\Delta u = \frac{1}{A(u)}\{\nabla A(u) \cdot \nabla u + f\} \quad \text{in } \Omega \tag{14}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{15}$$

From $\nabla A(u) \in L^p$ and $\nabla u \in L^p$, the right-hand side of (14) belong to $L^{\frac{p}{2}}(\Omega)$ $(2 < p < \infty)$. L^p estimate implies $u \in W^{2,q}(\Omega)$ $(q > N)$ and hence $u \in C^{1+\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$ from the theorem of Morrey.

Therefore, the right-hand side of (14) belong to $C^\alpha(\overline{\Omega})$ and hence $u \in C^{2+\alpha}(\overline{\Omega})$. From the result of Douglas-Dupont-Serrin ([3]: the uniqueness of classical solution), we get also the uniqueness of weak solution.

Furthermore, for Ritz operator $\hat{R}_h : V \to V_h$ associated with the elliptic operator

$$\hat{A}v = -\nabla \cdot (A(u)\nabla v)$$

when the estimate of Rannacher-Scott [6] type

$$\left\|\hat{R}_h v\right\|_{W^{1,q}} \le C \|v\|_{W^{1,q}}$$

holds for

$$q > \begin{cases} 1 & (N = 1) \\ 2 & (N = 2) \\ 6 & (N = 3), \end{cases}$$

(therefore, always when $N = 1$,) we can show $u_h \to u$ in $L^\infty(\Omega)$.

Let $u \in W^{1,p}(\Omega) \cap H_0^1(\Omega)$ and $p > \max\{N, 2\}$. The relation (4) and (5) imply for fixed $v_h \in V_h$ and $\lambda = C_a > 0$ that

$$
\begin{aligned}
\lambda \|\nabla(u_h - v_h)\|_{L^2}^2 &\le a(u_h : u_h - v_h, u_h - v_h) \\
&= a(u_h : u_h, u_h - v_h) - a(u_h : v_h, u_h - v_h) \\
&= F(u_h - v_h) - a(u_h : v_h, u_h - v_h) \\
&= a(u : u, u_h - v_h) - a(u_h : v_h, u_h - v_h) \\
&= \int_\Omega (A(u) - A(u_h))\nabla u \cdot \nabla(u_h - v_h) \\
&\quad + \int_\Omega A(u_h)\nabla(u - v_h) \cdot \nabla(u_h - v_h)
\end{aligned}
$$

Here, we remark

$$
\begin{aligned}
\|u_h\|_{L^\infty} &\le K, \quad M = \max_{|s| \le K} |A(s)|, \\
L &= \sup_{s,s'} \left| \frac{A(s) - A(s')}{s - s'} \right| \qquad (s, s' \in [-l, l]),
\end{aligned}
$$

and $l = \max\ K, \|u\|_{L^\infty}$. Then

$$
\begin{aligned}
\int_\Omega A(u_h)\nabla(u - v_h) \cdot \nabla(u_h - v_h) &= a(u_h : u - v_h, u_h - v_h) \\
&\le M \|\nabla(u - v_h)\|_{L^2} \|\nabla(u_h - v_h)\|_{L^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \int_\Omega (A(u) - A(u_h))\nabla u \cdot \nabla(u_h - v_h) \right| &\le \|A(u) - A(u_h)\|_{L^q} \|\nabla u\|_{L^p} \|\nabla(u_h - v_h)\|_{L^2} \\
&\le L \|u - u_h\|_{L^q} \|\nabla u\|_{L^p} \|\nabla(u_h - v_h)\|_{L^2},
\end{aligned}
$$

where

$$
\frac{1}{q} + \frac{1}{p} + \frac{1}{2} = 1.
$$

Therefore,

$$
\lambda \|\nabla(u_h - v_h)\|_{L^2} \le M \|\nabla(u - v_h)\|_{L^2} + L \|u - u_h\|_{L^q} \|\nabla u\|_{L^p}.
$$

and hence

$$\begin{aligned}
\|\nabla(u_h - u)\|_{L^2} &\leq \|\nabla(u_h - v_h)\|_{L^2} + \|\nabla(v_h - u)\|_{L^2} \\
&\leq \left(\frac{M}{\lambda} + 1\right) \|\nabla(u - v_h)\|_{L^2} \\
&\quad + \frac{L}{\lambda} \|u - u_h\|_{L^q} \|\nabla u\|_{L^p} \\
&\leq \left(\frac{M}{\lambda} + 1\right) \|\nabla(u - v_h)\|_{L^2} \\
&\quad + \frac{L}{2\lambda} \|\nabla(u - u_h)\|_{L^2} + C \|u - u_h\|_{L^2} \\
&\leq 2\left(\frac{M}{\lambda} + 1\right) \|\nabla(u - v_h)\|_{L^2} + C \|u - u_h\|_{L^2}.
\end{aligned}$$

From $u \in H_0^1(\Omega)$, $\inf_{v_h \in V_h} \|\nabla(u - v_h)\|_{L^2} \to 0 \quad (h \downarrow 0)$ follows. We shall show $u \to u_h$ in $L^2(\Omega)$.

The problem (1) implies

$$\begin{aligned}
\lambda \|\nabla u_h\|_{L^2}^2 &\leq a(u_h : u_h, u_h) \\
&= F(u_h) \\
&\leq \sum_{i=0}^{N} \|f_i\|_{L^2} \|\nabla u_h\|_{L^2}
\end{aligned}$$

and hence

$$\|\nabla u_h\|_{L^2} \leq \lambda^{-1} \sum_{i=0}^{N} \|f_i\|_{L^2}.$$

On the other hand, Theorem 1 implies that there exists a constant K such that

$$\|u_h\|_{L^\infty} \leq K.$$

Taking subsequences,

$$\begin{aligned}
u_h &\to w \quad && w - H_0^1(\Omega),\ w^* - L^\infty(\Omega) = \left(L^1(\Omega)\right)^* \\
u_h &\to w \quad && \text{in } L^2(\Omega).
\end{aligned}$$

We shall show that $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a weak solution for (1) with (2). Then the uniqueness of the weak solution ([3]) implies $w = u$ and we can complete the proof.

For any $v \in C_0^\infty(\Omega)$ there exists $\{v_h\}$ $(v_h \in V_h)$ such that

$$\|\nabla(v_h - v)\|_{L^p} \to 0 \qquad (p > \max\{N, 2\}).$$

therefore,

$$\begin{aligned}|F(v_h) - F(v)| &= \left|\int_\Omega f_0(v_h - v) - \sum_{i=1}^{N} f_i \frac{\partial}{\partial x_i}(v_h - v)\right| \\ &\le C\|\nabla(v_h - v)\|_{L^{p'}} \\ &\le C\|\nabla(v_h - v)\|_{L^p} \to 0 \qquad (p' < 2 < p).\end{aligned}$$

On the other hand,

$$\begin{aligned}a(u_h : u_h, v_h) &= \int_\Omega (A(u_h) - A(w))\nabla u_h \cdot \nabla v_h \\ &\quad + a(w : u_h, v_h - v) + a(w : u_h, v).\end{aligned}$$

Since $u_h \to w$ in $H_0^1(\Omega)$, we have

$$a(w : u_h, v) \to a(w : w, v).$$

Furthermore,

$$\begin{aligned}&\left|\int_\Omega (A(u_h) - A(w))\nabla u_h \cdot \nabla v_h\right| \\ &\le L\|u_h - w\|_{L^q}\|\nabla u_h\|_{L^2}\|\nabla v_h\|_{L^p}.\end{aligned} \qquad (16)$$

For $q < \frac{2N}{N-2}$, we have $u_h \to w$ in $L^q(\Omega)$ and hence the right-hand side of (16) converge to zero.

Finally,

$$|a(w : u_h, v_h)| \le M\|\nabla u_h\|_{L^2}\|\nabla(v_h - v)\|_{L^2} \to 0$$

and hence

$$a(w : w, v) = F(v) \qquad \left(^\forall v \in C_0^\infty(\Omega)\right).$$

Therefore,

$$a(w : w, v) = F(v) \qquad \left(^\forall v \in H_0^1(\Omega)\right).$$

This completes the proof in the case of $H_0^1(\Omega)$ convergence.

Next, we prove about the case of L^∞ convergence.

Let $\hat{u}_h \in V_h$ be the solution of (13). Since $\|\hat{u}_h - u\|_{L^\infty} \to 0$, we have

$$\begin{aligned}\|u_h - \hat{u}_h\|_{L^\infty} &\leq C\lambda^{-2}\sum_{j=1}^{N}\left\|(A(u) - A(u_h))\frac{\partial \hat{u}_h}{\partial x_j}\right\|_{L^\infty} \\ &\leq C\lambda^{-2}ML\,\|u - u_h\|_{L^{pr'}}\,\|\nabla \hat{u}_h\|_{L^{pr}} \qquad (p > N, p \geq 2),\end{aligned}$$

where

$$pr' = \frac{2N}{N-2}, \qquad pr = \frac{2N}{\frac{2N}{p} - (N-2)}.$$

Therefore, there exist $q > \max\{N, \frac{2N}{4-N}\}$ such that

$$\left\|\nabla \hat{R}_h u\right\|_{L^q} \leq C\,\|u\|_{L^q}$$

and hence

$$\|u_h - u\|_{L^\infty} \leq \|u_h - \hat{u}_h\|_{L^\infty} + \|\hat{u}_h - u\|_{L^\infty} \to 0.$$

References

[1] Brenner, S.C., Scott, L.R., *The Mathematical Theory of Finite Element Methods*, Springer, New York, 1994.

[2] J., Douglas, Jr., T., Dupont, *A Galerkin Method for a Nonlinear Diriclet Problem*, Math. Comp. **29** (1975) 689-696.

[3] J., Douglas, Jr., T., Dupont, J., Serrin, *Uniqueness and Comparison Teorems for Nonlinear Elliptic Equations in Divergence Form*, Arch. Rational Mech. Anal. **42** (1971) 157-168

[4] Hartman, P., Stampacchia, G., *On some non-linear elliptic differential-functional equations*, Acta Math. **115** (1966) 271-310.

[5] Matsuzawa, Y., Suzuki, T., *An L^∞ bound of the finite element approximation of piecewise linear trial function*, to appear in Adv. Math. Sci. Appl..

[6] Rannacher, R., Scott, R., *Some optimal error estimates for piecewise linear finite element approximation*, Math. Comp. **38** (1982) 437-445.