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NONLINEAR MULTIPARAMETER EIGENVALUE PROBLEMS

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1 Introduction

This paper is concerned with the nonlinear multiparameter problem

$$u''(r) + \frac{N-1}{r}u'(r) + \sum_{k=1}^{n}\mu_k f_k(u(r)) = \lambda g(u(r)), \quad 0 < r < 1, \tag{1.1}$$

$$u(r) > 0, \quad 0 \le r < 1, \tag{1.2}$$

$$u'(0) = 0, \quad u(1) = 0, \tag{1.3}$$

where $N \ge 3$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_n) \in R_+^n$ $(n \ge 1)$ and $\lambda \in R_+$ are parameters.

The aim of this paper is to establish asymptotic formulas of the variational eigenvalues $\lambda = \lambda(\mu, \alpha)$ obtained by Ljusternik-Schnirelman (LS) theory on general level set

$$N_{\mu,\alpha} := \{u \in X : \Lambda(\mu, u) := \frac{1}{2}\int_B |\nabla u|^2 dx - \sum_{k=1}^{n}\mu_k \int_B F_k(u(x))dx = -\alpha\omega\},$$

where $X := W_0^{1,2}(B)$ is the usual real Sobolev space, $\alpha > 0$ is a parameter, ω is the measure of the unit sphere in R^N and $F_k(u) := \int_0^u f_k(s)ds$.
We assume the following conditions (A.1) - (A.3):

(A.1) f_k, g are locally Lipschitz continuous, odd in u and $f_k(u) \ge 0$, $g(u) > 0$ for $u > 0$.
(A.2) There exists a constant $0 < \epsilon_1$ such that for $1 \le k \le n$

$$f_k(u)u - (2 + \epsilon_1)F_k(u) \ge 0, \quad u \in R. \tag{1.4}$$

(A.3) There exist constants $1 < p_1 < p_2 < \cdots < p_n < 1+4/N$ and $K_k > 0$ $(k = 0, 1, \cdots, n)$ such that as $u \to \infty$

$$\frac{g(u)}{u} \to K_0, \frac{f_k(u)}{u^{p_k}} \to K_k. \tag{1.5}$$

Furthermore, there exist constants $1 < q_1 < q_2 < \cdots < q_n < 1 + 4/N$ with $q_k \le p_k$ and $J_k > 0$ $(k = 0, 1, \cdots, n)$ such that as $u \downarrow 0$

$$\frac{g(u)}{u} \to J_0, \frac{f_k(u)}{u^{q_k}} \to J_k. \tag{1.6}$$

Our motivation comes from the problem of determining asymptotic direction of eigenvalues (the limit of the ratio of two eigenvalues) of linear multiparameter ordinary differential equations, which are sometimes studied by using Prüfer transformation. For linear theory, we refer to Faierman [3] and Turin [9] and the references therein.

Recently, in Shibata [7], the asymptotic direction for nonlinear two-parameter problems was studied for the simplest case of equation (1.1)-(1.3):

$$u''(x) + \mu u(x)^p = \lambda u(x)^q, \quad 0 < x < 1, \tag{1.7}$$

$$u(x) > 0, \quad 0 < x < 1, \tag{1.8}$$

$$u(0) = u(1) = 0. \tag{1.9}$$

Here $\mu, \lambda > 0$ are parameters and $1 \le q < p < q+2$ are constants. In [7], by LS-theory on general level set $N_{\mu,\alpha}$, the following asymptotic formula of λ as $\mu \to \infty$ for a fixed $\alpha > 0$ was given:

$$\lambda(\mu, \alpha) = C_1 \mu^{\frac{q+3}{p+3}} + o(\mu^{\frac{q+3}{p+3}}), \tag{1.10}$$

where

$$C_1 = \left\{ \left(\frac{q+1}{p+1}\right)^{\frac{q+3}{2(p-q)}} \frac{(p+3)(q+1)(p-q)\alpha}{2(2q-p+3)} \sqrt{\frac{2}{\pi(q+1)}} \frac{\Gamma\left(\frac{p+3}{2(p-q)}\right)}{\Gamma\left(\frac{q+3}{2(p-q)}\right)} \right\}^{\frac{2(p-q)}{p+3}}. \tag{1.11}$$

We shall extend this formula to our general situation.

2 Main Results

Let

$$\|u\|_X^2 := \frac{1}{\omega}\int_B |\nabla u|^2 dx, \quad \|u\|_p^p = \frac{1}{\omega}\int_B |u(x)|^p dx, \quad (u,v) := \frac{1}{\omega}\int_B u(x)v(x)dx, \tag{2.1}$$

$$\|u\|_\infty := \sup_{x\in B} |u(x)|, \quad \Phi_k(u) := \frac{1}{\omega}\int_B F_k(u(x))dx, \quad \Psi(u) := \frac{1}{\omega}\int_B G(u(x))dx. \tag{2.2}$$

For a given $(\mu, \alpha) \in R_+^{n+1}$, $\lambda = \lambda(\mu, \alpha)$ is called the variational eigenvalue if the following conditions (B.1) - (B.2) are satisfied:

(B.1) $(\mu, \alpha, \lambda(\mu,\alpha), u_{\mu,\alpha}(x)) \in R_+^{n+2} \times N_{\mu,\alpha}$ satisfies (1.1) - (1.3).

(B.2)

$$\Psi(u_{\mu,\alpha}) = \beta(\mu,\alpha) := \inf_{u\in N_{\mu,\alpha}} \Psi(u). \tag{2.3}$$

$\lambda(\mu, \alpha)$ is explicitly represented as

$$\lambda(\mu,\alpha) = \frac{2\alpha + \sum_{k=1}^n \mu_k((f_k(u_{\mu,\alpha}), u_{\mu,\alpha}) - 2\Phi_k(u_{\mu,\alpha}))}{(g(u_{\mu,\alpha}), u_{\mu,\alpha})}. \tag{2.4}$$

Indeed, multiplying (1.1) by $u_{\mu,\alpha}$ and integration by parts, we obtain

$$- \|u_{\mu,\alpha}\|_X^2 + \sum_{k=1}^{n} \mu_k (f_k(u_{\mu,\alpha}), u_{\mu,\alpha}) = \lambda(\mu,\alpha)(g(u_{\mu,\alpha}), u_{\mu,\alpha}). \tag{2.5}$$

This along with the fact that $u_{\mu,\alpha} \in N_{\mu,\alpha}$ yields (2.4).

Now we introduce (C-i) and (D-i)-condition for a sequence $\{(\mu,\alpha)\} \subset R_+^{n+1}$:

(C-i) Let $1 \le i \le n$ be fixed. Then

$$\alpha \mu_i^{\frac{2}{p_i-1}} \to \infty, \tag{2.6}$$

$$\alpha^2 \mu_i^{N-2} \to \infty. \tag{2.7}$$

Furthermore, for $k \neq i$

$$\mu_k \alpha^{\frac{2(p_k-p_i)}{N+2-p_i(N-2)}} \mu_i^{-\frac{N+2-p_k(N-2)}{N+2-p_i(N-2)}} \to 0. \tag{2.8}$$

(D-i) Let $1 \le i \le n$ be fixed. Then

$$\alpha \mu_i^{\frac{2}{q_i-1}} \to \infty, \tag{2.9}$$

$$\alpha^2 \mu_i^{N-2} \to 0. \tag{2.10}$$

Furthermore, for $k \neq i$

$$\mu_k \alpha^{\frac{2(q_k-q_i)}{N+2-q_i(N-2)}} \mu_i^{-\frac{N+2-q_k(N-2)}{N+2-q_i(N-2)}} \to 0. \tag{2.11}$$

Finally, let w denote the ground state solution of the following nonlinear scalar field equation, which uniquely exists:

$$-\Delta w = w^{p_i} - w \quad in\ R^N, \tag{2.12}$$

$$w > 0 \quad in\ R^N, \tag{2.13}$$

$$\lim_{|x|\to\infty} w(x) = 0. \tag{2.14}$$

Moreover, let W be the ground state of (2.12)-(2.14) with p_i replaced by q_i.

Now we state our results.

Theorem 2.1 *Assume (A.1) - (A.3). Then the following asymptotic formula holds for* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfying* (C-i):

$$\lambda(\mu,\alpha) = C_2 \left(\alpha \mu_i^{\frac{2}{p_i-1}}\right)^{\frac{2(p_i-1)}{N+2-(N-2)p_i}} + o\left(\left(\alpha \mu_i^{\frac{2}{p_i-1}}\right)^{\frac{2(p_i-1)}{N+2-(N-2)p_i}}\right) \tag{2.15}$$

where

$$C_2 = K_0^{-1} K_i^{\frac{4}{N+2-(N-2)p_i}} \left\{ \frac{N+2-(N-2)p_i}{(4+N-Np_i)\|w\|_{L^2(R^N)}^2} \right\}^{\frac{2(p_i-1)}{N+2-(N-2)p_i}}. \tag{2.16}$$

Theorem 2.2 *Assume (A.1) - (A.3). Then the following asymptotic formula holds for* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfying* (D-i):

$$\lambda(\mu,\alpha) = C_3(\alpha\mu_i^{\frac{2}{q_i-1}})^{\frac{2(q_i-1)}{N+2-(N-2)q_i}} + o\Big((\alpha\mu_i^{\frac{2}{q_i-1}})^{\frac{2(q_i-1)}{N+2-(N-2)q_i}}\Big) \tag{2.17}$$

where

$$C_3 = J_0^{-1} J_i^{\frac{4}{N+2-(N-2)q_i}} \left\{ \frac{N+2-(N-2)q_i}{(4+N-Nq_i)\|W\|_{L^2(R^N)}^2} \right\}^{\frac{2(q_i-1)}{N+2-(N-2)q_i}}. \tag{2.18}$$

Typical examples of f_k, g are as follows.

Example 2.3 Let $1 < p_1 < \cdots < p_i < \cdots < p_n < 1 + 4/N$ and $1 < q_1 < \cdots < q_i < \cdots < q_n < 1 + 4/N$ with $q_k \le p_k$. Then

$$f_k(u) = |u|^{p_k-1}u, \quad g(u) = u, \tag{2.19}$$

$$f_k(u) = (1 + |u|^{p_k-q_k})|u|^{q_k-1}u, \quad g(u) = u. \tag{2.20}$$

satisfies (A.1) - (A.3).

3 Fundamental lemmas for Theorem 2.1

Since Theorem 2.2 can be obtained by the similar arguments as those used to prove Theorem 2.1, we shall give a proof of Theorem 2.1. The existence of variational eigenvalues can be proved by using the result of Zeidler [10]. We begin with the basic equality which will play important roles. Let $\sigma_{\mu,\alpha} = \max_{0\le r\le 1} u_{\mu,\alpha}(r)$ $(= u_{\mu,\alpha}(0))$.

Lemma 3.1 *The following equality holds for* $0 \le r \le 1$*:*

$$\begin{aligned} \frac{1}{2}(u'_{\mu,\alpha}(r))^2 &+ \int_0^r \frac{N-1}{s} u'_{\mu,\alpha}(s)^2 ds + R(\mu,\alpha,u_{\mu,\alpha}(r)) \\ &= R(\mu,\alpha,\sigma_{\mu,\alpha}) = \frac{1}{2}(u'_{\mu,\alpha}(1))^2 + \int_0^1 \frac{N-1}{s} u'_{\mu,\alpha}(s)^2 ds > 0, \end{aligned} \tag{3.1}$$

where

$$R(\mu,\alpha,u) = \sum_{k=1}^n \mu_k F_k(u) - \lambda(\mu,\alpha)G(u). \tag{3.2}$$

This lemma can be obtained by direct calculation. Hence, we omit the proof. Let

$$h_0(u) := g(u) - u, \quad h_k(u) := f_k(u) - u^{p_k}, \tag{3.3}$$

$$H_0(u) := \int_0^u h_0(s)ds, \quad H_k(u) := \int_0^u h_k(s)ds. \tag{3.4}$$

Then by (1.5) we see that as $u \to \infty$

$$\frac{h_0(u)}{u}, \ \frac{h_k(u)}{u^{p_k}}, \ \frac{H_0(u)}{u^2}, \ \frac{H_k(u)}{u^{p_k+1}} \to 0. \tag{3.5}$$

Lemma 3.2 *Assume that* $\{(\mu,\alpha)\}\subset R^{n+1}_{+}$ *satisfies (C-i). Then*

$$\lambda(\mu,\alpha)\geq C(\alpha\mu_i^{\frac{2}{p_i-1}})^{\frac{2(p_i-1)}{N+2-p_i(N-2)}}. \tag{3.6}$$

Proof. Let $\tau=\tau_{\mu,\alpha}:=(\alpha\mu_i^{\frac{2}{p_i-1}})^{\frac{(p_i-1)}{N+2-p_i(N-2)}}$. Furthermore, let $w_\tau(S)$ satisfy

$$\begin{aligned}
&(3.7)\qquad & w_\tau''(S) &+ \frac{N-1}{S}w_\tau'(S)+w_\tau(S)^{p_i}-w_\tau(S)=0,\quad 0<S<\tau_{\mu,\alpha},\\
&(3.8)\qquad & w_\tau(S) &> 0,\quad 0\leq S<\tau_{\mu,\alpha},\\
&(3.9)\qquad & w_\tau'(0) &= w_\tau(\tau_{\mu,\alpha})=0.
\end{aligned}$$

We write, for simplicity,

$$\|w_\tau\|_X^2:=\int_0^{\tau_{\mu,\alpha}}S^{N-1}w_\tau'(S)^2dS,\ \ \|w_\tau\|_p^p:=\int_0^{\tau_{\mu,\alpha}}S^{N-1}|w_\tau(S)|^pdS.$$

We know from Kwong [6] that there uniquely exists a solution w_τ, and since $\tau_{\mu,\alpha}\to\infty$ by (2.6), $\|w_\tau\|_2,\|w_\tau\|_\infty\leq C$ for $\{(\mu,\alpha)\}$. Put

$$d_{\mu,\alpha}:=(\alpha\mu_i^{\frac{N-2}{2}})^{\frac{2}{N+2-p_i(N-2)}},\ \ U_{\mu,\alpha}(s):=c_{\mu,\alpha}d_{\mu,\alpha}w_\tau(S),\ \ s:=\tau_{\mu,\alpha}^{-1}S, \tag{3.10}$$

where $c_{\mu,\alpha}:=\inf\{t>0: td_{\mu,\alpha}w_\tau(\tau_{\mu,\alpha}s)\in N_{\mu,\alpha}\}$. Then $\Xi(t):=\Lambda(\mu,td_{\mu,\alpha}w_\tau(\tau_{\mu,\alpha}s))\to\infty$ as $t\to\infty$ and $\Xi(0)=0$, we see that $c_{\mu,\alpha}$ exists. We first show that

$$C^{-1}\leq c_{\mu,\alpha}\leq C. \tag{3.11}$$

By (2.8), we obtain for $k\neq i$

$$\begin{aligned}
&(3.12)\qquad & \alpha^{-1}\mu_k d_{\mu,\alpha}^{p_k+1}\tau_{\mu,\alpha}^{-N} &= \mu_k\alpha^{\frac{2(p_k-p_i)}{N+2-p_i(N-2)}}\mu_i^{-\frac{N+2-p_k(N-2)}{N+2-p_i(N-2)}}\to 0,\\
&(3.13)\qquad & d_{\mu,\alpha}^2\tau_{\mu,\alpha}^{2-N} &= \mu_i d_{\mu,\alpha}^{p_i+1}\tau_{\mu,\alpha}^{-N}=\alpha.
\end{aligned}$$

We have by (3.12) and (3.13)

$$\begin{aligned}
(3.14)\ -\alpha &= \Lambda(\mu,U_{\mu,\alpha})=\frac{1}{2}c_{\mu,\alpha}^2d_{\mu,\alpha}^2\tau_{\mu,\alpha}^{2-N}\|w_\tau\|_X^2-\frac{1}{p_i+1}\mu_ic_{\mu,\alpha}^{p_i+1}d_{\mu,\alpha}^{p_i+1}\tau_{\mu,\alpha}^{-N}\|w_\tau\|_{p_i+1}^{p_i+1}\\
&- \sum_{k\neq i}^n\frac{1}{p_k+1}\mu_kc_{\mu,\alpha}^{p_k+1}d_{\mu,\alpha}^{p_k+1}\tau_{\mu,\alpha}^{-N}\|w_\tau\|_{p_k+1}^{p_k+1}-\sum_{k=1}^n\mu_k\int_0^1 s^{N-1}H_k(U_{\mu,\alpha}(s))ds\\
&= \frac{1}{2}c_{\mu,\alpha}^2\alpha\|w_\tau\|_X^2-\frac{1}{p_i+1}c_{\mu,\alpha}^{p_i+1}\alpha\|w_\tau\|_{p_i+1}^{p_i+1}\\
&- \sum_{k\neq i}^n\frac{o(1)}{p_k+1}c_{\mu,\alpha}^{p_k+1}\alpha\|w_\tau\|_{p_k+1}^{p_k+1}-\sum_{k=1}^n\mu_k\int_0^1 s^{N-1}H_k(U_{\mu,\alpha}(s))ds.
\end{aligned}$$

At first, assume that $c_{\mu,\alpha} \to \infty$. Since $c_{\mu,\alpha}d_{\mu,\alpha} \to \infty$, we obtain by (2.8) and (3.12) that

$$(3.15)\qquad \begin{aligned} &\mu_k \int_0^1 s^{N-1} H_k(U_{\mu,\alpha}(s))ds \\ &= \mu_k \tau_{\mu,\alpha}^{-N} \int_0^{\tau_{\mu,\alpha}} S^{N-1} \frac{H(c_{\mu,\alpha}d_{\mu,\alpha}w_\tau(S))}{(c_{\mu,\alpha}d_{\mu,\alpha}w_\tau(S))^{p_k+1}} (c_{\mu,\alpha}d_{\mu,\alpha}w_\tau(S))^{p_k+1} dS \\ &= o(1)\mu_k c_{\mu,\alpha}^{p_k+1} d_{\mu,\alpha}^{p_k+1} \tau_{\mu,\alpha}^{-N} \|w_\tau\|_{p_k+1}^{p_k+1}. \end{aligned}$$

Then we obtain by (3.14) and (3.15) that

$$(3.16)\qquad \begin{aligned} -\alpha &= \Lambda(\mu, U_{\mu,\alpha}) \\ &= \alpha c_{\mu,\alpha}^2 \left\{ \frac{1}{2}\|w_\tau\|_X^2 - \frac{1+o(1)}{p_i+1} c_{\mu,\alpha}^{p_i-1} \|w_\tau\|_{p_i+1}^{p_i+1} - o(1) \sum_{k\neq i}^{n} \frac{1+o(1)}{p_k+1} c_{\mu,\alpha}^{p_k-1} \|w_\tau\|_{p_k+1}^{p_k+1} \right\}. \end{aligned}$$

Since $c_{\mu,\alpha} \to \infty$, this is a contradiction. Thus, $c_{\mu,\alpha} \le C$.
Next, assume that $c_{\mu,\alpha} \to 0$. Then there are two cases to consider.
Case 1: If $c_{\mu,\alpha}d_{\mu,\alpha} \to \infty$, then by the same argument as that used above, we also obtain (3.16). Since $c_{\mu,\alpha} \to 0$, this is a contradiction.
Case 2: Assume that $c_{\mu,\alpha}d_{\mu,\alpha} \le C$. By (1.6) we have for $u \le C$

$$(3.17)\qquad f_k(u) \le Cu^{q_k}, \quad F_k(u) \le Cu^{q_k+1}.$$

Then since $q_k \le p_k$ and $d_{\mu,\alpha} \to \infty$, we have by (3.12) and (3.13) that for $1 \le k \le n$

$$(3.18)\qquad \begin{aligned} \Phi_k(U_{\mu,\alpha}) &= \mu_k \int_0^1 s^{N-1} F_k(c_{\mu,\alpha}d_{\mu,\alpha}w_\tau(\tau_{\mu,\alpha}s))ds \le C\mu_k c_{\mu,\alpha}^{q_k+1} d_{\mu,\alpha}^{q_k+1} \tau_{\mu,\alpha}^{-N} \|w_\tau\|_{q_k+1}^{q_k+1} \\ &\le Cc_{\mu,\alpha}^{q_k+1}(\mu_k d_{\mu,\alpha}^{p_k+1} \tau_{\mu,\alpha}^{-N} \alpha^{-1})\alpha = Cc_{\mu,\alpha}^{q_k+1}\alpha. \end{aligned}$$

Then

$$(3.19)\qquad -\alpha = \Lambda(\mu, U_{\mu,\alpha}) \ge \alpha c_{\mu,\alpha}^2 \left\{ \frac{1}{2}\|w_\tau\|_X^2 - C\sum_{k=1}^{n} c_{\mu,\alpha}^{q_k-1} \right\}.$$

Since $c_{\mu,\alpha} \to 0$, this is a contradiction. Hence, we obtain (3.11). Now we obtain

$$(3.20)\qquad \begin{aligned} &\frac{2\alpha}{\lambda(\mu,\alpha)} \le (g(u_{\mu,\alpha}), u_{\mu,\alpha}) \le C\|u_{\mu,\alpha}\|_2^2 \le \Psi(u_{\mu,\alpha}) \\ &\le \Psi(c_{\mu,\alpha}d_{\mu,\alpha}w_\tau(\tau_{\mu,\alpha}s)) = Cc_{\mu,\alpha}^2 d_{\mu,\alpha}^2 \tau_{\mu,\alpha}^{-N} \|w_\tau\|_2^2 \le C\alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}} \mu_i^{-\frac{4}{N+2-p_i(N-2)}}; \end{aligned}$$

this implies (3.6). Thus the proof is complete. □

Lemma 3.3 *Assume that $\{\mu,\alpha)\} \subset R_+^{n+1}$ satisfies (C-i). Then $\lambda(\mu,\alpha)/\mu_i \to \infty$.*

Proof. By Lemma 3.2 and (2.7)

$$(3.21)\qquad \frac{\lambda(\mu,\alpha)}{\mu_i} \ge C(\alpha^2\mu_i^{N-2})^{\frac{p_i-1}{N+2-p_i(N-2)}} \to \infty.$$

□

Lemma 3.4 *Assume that* $\{\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies (C-i). Then* $\sigma_{\mu,\alpha} \to \infty$.

Proof. At first, we assume that there exists a subsequence of $\{\sigma_{\mu,\alpha}\}$ such that $\sigma_{\mu,\alpha} \to 0$ and derive a contradiction. We have by (2.8) that for $k \neq i$

$$\frac{\mu_k}{\mu_i} = o(1)(\alpha^2\mu_i^{N-2})^{\frac{p_i-p_k}{N+2-p_i(N-2)}}. \tag{3.22}$$

Then we obtain by (1.6), (3.1) and Lemma 3.3 that

$$\begin{aligned} C(\alpha^2\mu_i^{N-2})^{\frac{p_i-1}{N+2-p_i(N-2)}} &\le \frac{\lambda(\mu,\alpha)}{\mu_i} < \sum_{k=1}^n \frac{\mu_k}{\mu_i}\frac{F_k(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})} \\ &\le C(\sigma_{\mu,\alpha}^{q_i-1} + o(1)\sum_{k\neq i}(\alpha^2\mu_i^{N-2})^{\frac{p_i-p_k}{N+2-p_i(N-2)}}\sigma_{\mu,\alpha}^{q_k-1}). \end{aligned} \tag{3.23}$$

This is a contradiction. Hence, $0 < \delta \le \sigma_{\mu,\alpha}$. If there exists a subsequence of $\{\sigma_{\mu,\alpha}\}$ such that $\sigma_{\mu,\alpha} \le C$, then by (3.21) and (3.22)

$$\begin{aligned} C(\alpha^2\mu_i^{N-2})^{\frac{p_i-1}{N+2-p_i(N-2)}} &\le \frac{\lambda(\mu,\alpha)}{\mu_i} < \sum_{k=1}^n \frac{\mu_k}{\mu_i}\frac{F_k(\sigma_{\mu,\alpha})}{G(\sigma_{\mu,\alpha})} \\ &\le C(1 + o(1)(\alpha^2\mu_i^{N-2})^{\frac{p_i-p_k}{N+2-p_i(N-2)}}). \end{aligned} \tag{3.24}$$

This contradicts (2.7). Thus the proof is complete. □

Lemma 3.5 *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies (C-i). Then for* $1 \le k \le n$

$$\mu_k\sigma_{\mu,\alpha}^{p_k+1} \le C\mu_i\sigma_{\mu,\alpha}^{p_i+1}. \tag{3.25}$$

Proof. We define $1 \le j(\mu,\alpha) \le n$ by the rule

$$\mu_{j(\mu,\alpha)}\sigma_{\mu,\alpha}^{p_{j(\mu,\alpha)}+1} = \max_{1\le k\le n}\mu_k\sigma_{\mu,\alpha}^{p_k+1}. \tag{3.26}$$

Then there exists a infinite subsequence of $\{(\mu,\alpha)\}$ and $1 \le j \le n$ such that $j = j(\mu,\alpha)$ for all elements which belong to the subsequence. we consider this subsequence. We have by (3.1), (3.26) and Lemma 3.4 that for $0 \le r \le 1$

$$\begin{aligned} \frac{1}{2}u'_{\mu,\alpha}(r)^2 &\le \sum_{k=1}^n \mu_k(F_k(\sigma_{\mu,\alpha}) - F_k(u_{\mu,\alpha}(r))) - \lambda(\mu,\alpha)(G(\sigma_{\mu,\alpha}) - G(u_{\mu,\alpha}(r))) \\ &\le \sum_{k=1}^n \mu_k F_k(\sigma_{\mu,\alpha}) \le C\sum_{k=1}^n \mu_k\sigma_{\mu,\alpha}^{p_k+1} \le C\mu_j\sigma_{\mu,\alpha}^{p_j+1}. \end{aligned} \tag{3.27}$$

Let $0 \le r_1 = r_{1,\mu,\alpha} \le 1$ satisfy $u_{\mu,\alpha}(r_1) = 1/2\sigma_{\mu,\alpha}$. By mean value theorem and (3.27)

$$\left|\frac{u_{\mu,\alpha}(0) - u_{\mu,\alpha}(r_1)}{r_1}\right| = \left|\frac{\sigma_{\mu,\alpha}}{2r_1}\right| \le C\sqrt{\mu_j\sigma_{\mu,\alpha}^{p_j+1}},$$

that is,

$$C\sigma_{\mu,\alpha}^{\frac{1-p_j}{2}}\mu_j^{-1/2} \le r_1. \tag{3.28}$$

Since $u_{\mu,\alpha}(r)$ is decreasing in $0 \le r \le 1$, it follows from (3.28) that

$$\begin{aligned} \|u_{\mu,\alpha}\|_2^2 &\ge \int_0^{r_1} r^{N-1}u_{\mu,\alpha}(r)^2dr \ge C\sigma_{\mu,\alpha}^2 r_1^N \ge C\sigma_{\mu,\alpha}^2\left(\sigma_{\mu,\alpha}^{\frac{1-p_j}{2}}\mu_j^{-1/2}\right)^N \\ &\ge C\sigma_{\mu,\alpha}^{\frac{4-N(p_j-1)}{2}}\mu_j^{-N/2}. \end{aligned} \tag{3.29}$$

This along with (3.20) implies that

$$\sigma_{\mu,\alpha} \le C\left(\mu_j^{N/2}\alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}}\mu_i^{-\frac{4}{N+2-p_i(N-2)}}\right)^{\frac{2}{4-N(p_j-1)}}. \tag{3.30}$$

It follows from (1.5), (3.1) and Lemma 3.4 that

$$\lambda(\mu,\alpha) \le C\sum_{k=1}^n \mu_k\sigma_{\mu,\alpha}^{p_k-1}. \tag{3.31}$$

Then by Lemma 3.3, (3.26) and (3.31)

$$\begin{aligned} C(\alpha^2\mu_i^{N-2})^{\frac{p_i-1}{N+2-p_i(N-2)}} &\le \frac{\lambda(\mu,\alpha)}{\mu_i} \le C\mu_i^{-1}\mu_j\sigma_{\mu,\alpha}^{p_j-1} \\ &\le C\mu_i^{-1}\mu_j\left(\mu_j^{N/2}\alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}}\mu_i^{-\frac{4}{N+2-p_i(N-2)}}\right)^{\frac{2(p_j-1)}{4-N(p_j-1)}} \\ &= C\mu_j^{\frac{4}{4-N(p_j-1)}}\alpha^{\frac{2(p_j-1)(4-N(p_i-1))}{(N+2-p_i(N-2))(4-N(p_j-1))}}\mu_i^{-\frac{8(p_j-1)}{(N+2-p_i(N-2))(4-N(p_j-1))}}\mu_i^{-1}; \end{aligned}$$

that is,

$$C \le \mu_j\alpha^{\frac{2(p_j-p_i)}{N+2-p_i(N-2)}}\mu_i^{-\frac{N+2-p_j(N-2)}{N+2-p_i(N-2)}}. \tag{3.32}$$

Since we assume (2.8), we find that there never exists an infinite subsequence of $\{(\mu,\alpha)\}$ satisfying (3.26) for $j \ne i$, namely, (3.26) holds for $j = i$ except finite elements of $\{(\mu,\alpha)\}$. Thus we obtain our conclusion. □.

Lemma 3.6 *Assume that $\{(\mu,\alpha)\} \subset R_+^{n+1}$ satisfies (C-i). Then*

$$\lambda(\mu,\alpha) \le C(\alpha\mu_i^{\frac{2}{p_i-1}})^{\frac{2(p_i-1)}{N+2-p_i(N-2)}}. \tag{3.33}$$

Proof. By Lemma 3.4, Lemma 3.5 and (3.31)

$$\lambda(\mu,\alpha) \le C\sum_{k=1}^n \mu_k\sigma_{\mu,\alpha}^{p_k-1} \le C\mu_i\sigma_{\mu,\alpha}^{p_i-1}. \tag{3.34}$$

Since (3.30) holds for $j = i$, we have

$$\sigma_{\mu,\alpha} \le C(\alpha^2 \mu_i^{N-2})^{1/(N+2-p_i(N-2))}. \tag{3.35}$$

Substituting (3.35) into (3.34), we obtain (3.33). □

We conclude this section by showing the following lemma. Let $\xi_{\mu,\alpha} := (\lambda(\mu,\alpha)/\mu_i)^{1/(p_i-1)}$.

Lemma 3.7 *Assume that* $\{(\mu,\alpha)\} \subset R_+^{n+1}$ *satisfies (C-i). Then*

$$C^{-1}\sigma_{\mu,\alpha} \le \xi_{\mu,\alpha} \le C\sigma_{\mu,\alpha}. \tag{3.36}$$

Proof. The second inequality follows from (3.34). By (3.21) and (3.35) we obtain

$$\sigma_{\mu,\alpha}^{p_i-1} \le C(\alpha^2\mu_i^{N-2})^{\frac{p_i-1}{N+2-p_i(N-2)}} \le C\frac{\lambda(\mu,\alpha)}{\mu_i}. \tag{3.37}$$

Thus the proof is complete. □

4 Proof of Theorem 2.1

We begin with recalling some fundamental properties of the ground state w of the equation (2.12) - (2.14). It is known that there uniquely exists the ground state w of (2.12)-(2.14) such that w is spherically symmetric, decreases with respect to $s = |x|, w \in C^2(R^N)$. Furthermore, for some constant $\delta > 0$ and for $|\gamma| \le 2$

$$|D^\gamma w(x)| \le Ce^{-\delta|x|}, \quad x \in R^N \tag{4.1}$$

For these properties, we refer to Berestycki and Lions [1] and Kwong [6]. Since w is spherically symmetric, (2.12) - (2.14) are equivalent to:

$$\begin{aligned} w''(s) &+ \frac{N-1}{s}w'(s) + w(s)^{p_i} - w(s) = 0, \quad s > 0, &(4.2)\\ w(s) &> 0, \quad s \ge 0, &(4.3)\\ \lim_{s\to\infty} w(s) &= 0. &(4.4) \end{aligned}$$

We put $w_{\mu,\alpha}(s) := \xi_{\mu,\alpha}^{-1}u_{\mu,\alpha}(r), \ s := \sqrt{\lambda(\mu,\alpha)}r$. By (1.1) - (1.3) we see that $w_{\mu,\alpha}(s)$ satisfies the following equation:

$$\begin{aligned} &w''_{\mu,\alpha}(s) + \frac{N-1}{s}w'_{\mu,\alpha}(s) + w_{\mu,\alpha}(s)^{p_i} - w_{\mu,\alpha}(s) \\ &+ \sum_{k=1,k\ne i}^{n} \lambda(\mu,\alpha)^{-1}\mu_k\xi_{\mu,\alpha}^{p_k-1}w_{\mu,\alpha}(s)^{p_k} \\ &+ \sum_{k=1}^{n}\lambda(\mu,\alpha)^{-1}\mu_k\xi_{\mu,\alpha}^{-1}h_k(\xi_{\mu,\alpha}w_{\mu,\alpha}(s)) \end{aligned} \tag{4.5}$$

$$-\xi_{\mu,\alpha}^{-1}h_0(\xi_{\mu,\alpha}w_{\mu,\alpha}(s)) = 0, \quad s \in I_{\mu,\alpha} := (0, \sqrt{\lambda(\mu,\alpha)}),$$

$$w_{\mu,\alpha}(s) > 0, \quad 0 \le s < \sqrt{\lambda(\mu,\alpha)}, \tag{4.6}$$

$$w'_{\mu,\alpha}(0) = 0, \quad w_{\mu,\alpha}(\sqrt{\lambda(\mu,\alpha)}) = 0. \tag{4.7}$$

By definition of $w_{\mu,\alpha}(s)$, we have:

$$\|w_{\mu,\alpha}\|^2_{X,\lambda} := \int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1} w'_{\mu,\alpha}(s)^2 ds = \lambda(\mu,\alpha)^{\frac{N-2}{2}} \xi_{\mu,\alpha}^{-2} \|u_{\mu,\alpha}\|^2_X, \tag{4.8}$$

$$\|w_{\mu,\alpha}\|^{p_k+1}_{p_k+1,\lambda} := \int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1} w_{\mu,\alpha}(s)^{p_k+1} ds = \lambda(\mu,\alpha)^{\frac{N}{2}} \xi_{\mu,\alpha}^{-(p_k+1)} \|u_{\mu,\alpha}\|^{p_k+1}_{p_k+1}, \tag{4.9}$$

$$\|w_{\mu,\alpha}\|^2_{2,\lambda} := \int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1} w_{\mu,\alpha}(s)^2 ds = \lambda(\mu,\alpha)^{\frac{N}{2}} \xi_{\mu,\alpha}^{-2} \|u_{\mu,\alpha}\|^2_2. \tag{4.10}$$

Furthermore, by (3.1), Lemma 3.5 and Lemma 3.7

$$\begin{aligned}
\frac{1}{2}\lambda(\mu,\alpha)\xi^2_{\mu,\alpha} w'_{\mu,\alpha}(s)^2 &= \frac{1}{2}u'_{\mu,\alpha}(r)^2 \le \sum_{k=1}^n \mu_k (F_k(\sigma_{\mu,\alpha}) - F_k(u_{\mu,\alpha}(r))) \\
&\le C\sum_{k=1}^n \mu_k \sigma_{\mu,\alpha}^{p_k+1} \le C\mu_i \sigma_{\mu,\alpha}^{p_i+1} \le C\mu_i \xi_{\mu,\alpha}^{p_i+1} = C\lambda(\mu,\alpha)\xi^2_{\mu,\alpha};
\end{aligned}$$

that is, for $s \in [0, \sqrt{\lambda(\mu,\alpha)}]$

$$w'_{\mu,\alpha}(s)^2 \le C. \tag{4.11}$$

Lemma 4.1 *Assume that* $\{(\mu,\alpha)\} \subset R^{n+1}_+$ *satisfies (C-i). Then*

$$\lambda(\mu,\alpha)^{-1}\mu_k \xi_{\mu,\alpha}^{p_k-1} w_{\mu,\alpha}(s)^{p_k} \to 0 \ (k \ne i), \tag{4.12}$$

$$\lambda(\mu,\alpha)^{-1}\mu_k \xi_{\mu,\alpha}^{-1} h_k(\xi_{\mu,\alpha}w_{\mu,\alpha}(s)) \to 0 \ (1 \le k \le n), \tag{4.13}$$

$$\xi_{\mu,\alpha}^{-1} h_0(\xi_{\mu,\alpha}w_{\mu,\alpha}(s)) \to 0. \tag{4.14}$$

for a fixed $s \in [0, \sqrt{\lambda(\mu,\alpha)})$.

This lemma can be obtained by direct calculation. Hence, we omit the proof.

Lemma 4.2 *Assume that* $\{(\mu,\alpha)\} \subset R^{n+1}_+$ *satisfies (C-i). Then* $\|w_{\mu,\alpha}\|_{2,\lambda}, \|w_{\mu,\alpha}\|_{X,\lambda} \le C$.

Proof. By (1.4) and (2.4)

$$\begin{aligned}
\epsilon_1 \sum_{k=1}^n \mu_k \Phi_k(u_{\mu,\alpha}) &\le 2\alpha + \epsilon_1 \sum_{k=1}^n \Phi_k(u_{\mu,\alpha}) \le C\lambda(\mu,\alpha)(g(u_{\mu,\alpha}), u_{\mu,\alpha}) \\
&\le C\lambda(\mu,\alpha)\|u_{\mu,\alpha}\|^2_2.
\end{aligned} \tag{4.15}$$

Since $u_{\mu,\alpha} \in N_{\mu,\alpha}$, we obtain by (3.20), (4.8) and (4.15) that

$$\begin{aligned}
\lambda(\mu,\alpha)^{\frac{2-N}{2}} \xi^2_{\mu,\alpha} \|w_{\mu,\alpha}\|^2_{X,\lambda} &= \|u_{\mu,\alpha}\|^2_X = 2(\sum_{k=1}^n \mu_k \Phi_k(u_{\mu,\alpha}) - \alpha) \le 2\sum_{k=1}^n \mu_k \Phi_k(u_{\mu,\alpha}) \\
&\le 2C\epsilon_1^{-1}\lambda(\mu,\alpha)\|u_{\mu,\alpha}\|^2_2 \le C\lambda(\mu,\alpha)\alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}} \mu_i^{-\frac{4}{N+2-p_i(N-2)}};
\end{aligned} \tag{4.16}$$

this along with Lemma 3.2 implies that

$$\|w_{\mu,\alpha}\|^2_{X,\lambda} \le C\lambda(\mu,\alpha)\xi^{-2}_{\mu,\alpha}\alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}}\mu_i^{-\frac{4}{N+2-p_i(N-2)}} = C\left\{\lambda(\mu,\alpha)^{-1}\alpha^{\frac{2(p_i-1)}{N+2-p_i(N-2)}}\mu_i^{\frac{4}{N+2-p_i(N-2)}}\right\}^{\frac{4-N(p_i-1)}{2(p_i-1)}} \le C. \tag{4.17}$$

Next, by Lemma 3.2, (3.20) and (4.10)

$$\|w_{\mu,\alpha}\|^2_{2,\lambda} = \lambda(\mu,\alpha)^{N/2}\xi^{-2}_{\mu,\alpha}\|u_{\mu,\alpha}\|^2_2 \le C\lambda(\mu,\alpha)^{N/2}\xi^{-2}_{\mu,\alpha}\alpha^{\frac{4-N(p_i-1)}{N+2-p_i(N-2)}}\mu_i^{-\frac{4}{N+2-p_i(N-2)}} = C\left\{\lambda(\mu,\alpha)^{-1}\alpha^{\frac{2(p_i-1)}{N+2-p_i(N-2)}}\mu_i^{\frac{4}{N+2-p_i(N-2)}}\right\}^{\frac{4-N(p_i-1)}{2(p_i-1)}} \le C.$$

Thus the proof is complete. □

The following two lemmas are variant of Shibata [7, Lemma 4.7]. Hence, the proofs are omitted.

Lemma 4.3 *Let $w = w(|x|)$ be the ground state of (5.2) - (5.4). Suppose that $\{(\mu,\alpha)\} \subset R^{n+1}_+$ satisfies (C-i). Then* $\lim w_{\mu,\alpha}(|x|) = w(|x|)$ *uniformly on any compact subsets on R^N.*

Lemma 4.4 *Assume that $\{(\mu,\alpha)\} \subset R^{n+1}_+$ satisfies (C-i). Then there exists $y_0(x) = y_0(|x|) \in L^2(R^N) \cap L^{p_k+1}(R^N)$ $(1 \le k \le n)$ such that $w_{\mu,\alpha}(|x|) \le y_0(|x|)$ for $x \in R^N$.*

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1: From Lemma 4.3, Lemma 4.4 and Lebesgue's convergence theorem

$$\|w_{\mu,\alpha}\|_{2,\lambda} \to \|w\|_{2,R^N}, \quad \|w_{\mu,\alpha}\|_{p_k+1,\lambda} \to \|w\|_{p_k+1,R^N} \quad (1 \le k \le n). \tag{4.18}$$

We know from Pohozaev's identity (cf. Strauss [8]) that

$$\|w\|^{p_i+1}_{p_i+1,R^N} = \frac{2(p_i+1)}{2+N-p_i(N-2)}\|w\|^2_{2,R^N}. \tag{4.19}$$

Furthermore, by Lemma 3.7, Lemma 4.2 and (4.10)

$$\begin{aligned}(g(u_{\mu,\alpha}),u_{\mu,\alpha}) &= \|u_{\mu,\alpha}\|^2_2 + \int_0^1 r^{N-1}h_0(u_{\mu,\alpha}(r))u_{\mu,\alpha}(r)dr \\ &= \lambda(\mu,\alpha)^{-N/2}\xi^2_{\mu,\alpha}\|w_{\mu,\alpha}\|^2_{2,\lambda} \\ &+\lambda(\mu,\alpha)^{-N/2}\xi^2_{\mu,\alpha}\int_0^{\sqrt{\lambda(\mu,\alpha)}} s^{N-1}\frac{h_0(\xi_{\mu,\alpha}w_{\mu,\alpha}(s))}{\xi_{\mu,\alpha}w_{\mu,\alpha}(s)}w_{\mu,\alpha}(s)^2ds \\ &= (1+o(1))\lambda(\mu,\alpha)^{-N/2}\xi^2_{\mu,\alpha}\|w_{\mu,\alpha}\|^2_{2,\lambda}.\end{aligned} \tag{4.20}$$

Similarly, we obtain

$$(4.21)\qquad \begin{aligned}(f_k(u_{\mu,\alpha}),u_{\mu,\alpha}) &= \|u_{\mu,\alpha}\|_{p_k+1}^{p_k+1}+\int_0^1 r^{N-1}h_k(u_{\mu,\alpha}(r))u_{\mu,\alpha}(r)dr\\ &= (1+o(1))\lambda(\mu,\alpha)^{-N/2}\xi_{\mu,\alpha}^{p_k+1}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1},\end{aligned}$$

$$(4.22)\qquad \Phi_k(u_{\mu,\alpha}) = \frac{1}{p_k+1}(1+o(1))\lambda(\mu,\alpha)^{-N/2}\xi_{\mu,\alpha}^{p_k+1}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1}.$$

By (2.4) and (4.20) - (4.22) we obtain

$$(1+o(1))\lambda(\mu,\alpha)^{(2-N)/2}\xi_{\mu,\alpha}^2\|w_{\mu,\alpha}\|_2^2 = 2\alpha+\sum_{k=1}^n\frac{p_k-1}{p_k+1}(1+o(1))\mu_k\lambda(\mu,\alpha)^{-N/2}\xi_{\mu,\alpha}^{p_k+1}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1};$$
(4.23)

this implies that

$$(4.24)\qquad \{A_1(\mu,\alpha)-A_2(\mu,\alpha)\}\lambda(\mu,\alpha)^{\frac{(N+2)-p_i(N-2)}{2(p_i-1)}}\mu_i^{-\frac{2}{p_i-1}} = 2\alpha,$$

where

$$(4.25)\qquad A_1(\mu,\alpha) = (1+o(1))\|w_{\mu,\alpha}\|_{2,\lambda}^2,$$

$$(4.26)\qquad \begin{aligned}A_2(\mu,\alpha) &= \frac{p_i-1}{p_i+1}(1+o(1))\|w_{\mu,\alpha}\|_{p_i+1}^{p_i+1}\\ &+ \sum_{k\neq i}^n\frac{p_k-1}{p_k+1}(1+o(1))\mu_k\lambda(\mu,\alpha)^{-1}\xi_{\mu,\alpha}^{p_k-1}\|w_{\mu,\alpha}\|_{p_k+1,\lambda}^{p_k+1}.\end{aligned}$$

Then by Lemma 4.1, (4.18), (4.19) and (4.24) we obtain

$$(4.27)\qquad \frac{\lambda(\mu,\alpha)}{\alpha^{\tau_1}\mu_i^{\tau_2}} \to \left(\frac{1}{\|w\|_{2,R^N}^2-\frac{p_i-1}{p_i+1}\|w\|_{p_i+1,R^N}^{p_i+1}}\right)^{\tau_1} = \left(\frac{(N+2)-p_i(N-2)}{(4+N-Np_i)\|w\|_{2,R^N}^2}\right)^{\tau_1},$$

where

$$\tau_1 = \frac{2(p_i-1)}{N+2-(N-2)p_i},\quad \tau_2 = \frac{4}{N+2-(N-2)p_i}.$$

Thus the proof is complete. □

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