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AUTHOR(S):

NAGAI, TOSHITAKA; SENBA, TAKASI

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BEHAVIOR OF RADially SYMMETRIC SOLUTIONS OF A SYSTEM RELATED TO CHEMOTAXIS

TOSHITAKA NAGAI 永井 敏隆

Department of Mathematics, Kyushu Institute of Technology, Tobata, Kitakyushu 804
JAPAN

TAKASI SENBA 仙葉 隆

Department of Applied Mathematics, Miyazaki University, Kibana, Miyazaki 889-21
JAPAN

1. Introduction

We consider time-global existence and blow-up of solutions of the following system related to chemotaxis

$$\begin{cases} b_t = \nabla \cdot (\nabla b - \chi b \nabla \phi(s)) & \text{in } \Omega \times (0, \infty), \\ 0 = \Delta s - s + b & \text{in } \Omega \times (0, \infty), \end{cases} \quad (1.1)$$

under the conditions

$$\begin{cases} \frac{\partial b}{\partial n} = \frac{\partial s}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ b(\cdot, 0) = b_0 & \text{in } \Omega, \end{cases} \quad (1.2)$$

where χ is a positive constant and ϕ is a smooth function on $(0, \infty)$ with $\phi' > 0$. The system is a simplified Keller-Segel model. Keller-Segel model was introduced by Keller and Segel [11] to describe the initiation of chemotactic aggregation of cellular slime molds. On Keller-Segel model and simplified Keller-Segel models, time-local existence of the solutions has been studied by [19] and blow-up of the solutions has been studied by [4, 10, 9, 14, 18].

The domain Ω and the non-trivial initial function b_0 are only confined to the following case:

(A1) Ω is the open ball of radius L with center at the origin in $\mathbf{R}^N$.

(A2) b_0 is smooth and nonnegative on $\bar{\Omega}$, and is radially symmetric when $N \geq 2$.

Under these assumptions, there exists a unique solution $(b(x, t), s(x, t))$ to (1.1) and (1.2) defined maximal interval of existence $[0, T_{max})$, which is radially symmetric in x when $N \geq 2$, smooth in $\bar{\Omega} \times (0, T_{max})$ and $b(x, t) > 0$, $s(x, t) > 0$ for $(x, t) \in \Omega \times (0, T_{max})$. If $T_{max} < \infty$,

$$\limsup_{t \rightarrow T_{max}} (\|b(\cdot, t)\|_{L^\infty} + \|s(\cdot, t)\|_{L^\infty}) = \infty,$$

by which we mean that $(b(x, t), s(x, t))$ blows up in finite time.

Theorem 1 *Let $N = 1$ and ϕ be smooth on $(0, \infty)$. Then the solution (b, s) to (1.1), (1.2) is globally bounded, that is, $T_{max} = \infty$ and (b, s) satisfies*

$$\sup_{t \geq 0} (\|b(\cdot, t)\|_{L^\infty} + \|s(\cdot, t)\|_{L^\infty}) < \infty.$$

We put

$$M_a(t) = \int_{\Omega} b(x, t) |x|^a dx \quad \text{for } 0 \leq t < T_{max},$$

where a is a positive constant. That is called the moment of order a , of $b(\cdot, t)$.

Theorem 2 *Assume $\phi(s) = s^p$ ($p > 0$), (A1) and (A2).*

(1) $N = 2$:

(a) *If $0 < p < 1$, then the solution is globally bounded in time.*

(b) $p = 1$:

(i) *If $\|b_0\|_{L^1} < 8\pi/\chi$, then the solution is globally bounded in time.*

(ii) *If $\|b_0\|_{L^1} > 8\pi/\chi$ and $M_2(0)$ is sufficiently small, then the solution blows up in finite time.*

(c) *If $p > 1$ and $M_2(0)$ is sufficiently small, then the solution blows up in finite time.*

(2) *If $N \geq 3$ and $M_{(N-2)p+2}(0)$ is sufficiently small, then the solution blows up in finite time.*

Theorem 3 *Assume $\phi(s) = \log s$, (A1) and (A2).*

(1) *If $N = 2$, then the solution is globally bounded in time.*

(2) $N \geq 3$:

(a) *If $\chi < 2/(N-2)$, then the solution is globally bounded in time.*

(b) *If $\chi > 2N/(N-2)$ and $M_2(0)$ is sufficiently small, then the solution blows up in finite time.*

2. Time-global existence and boundedness

The purpose in this section is to sketch the proofs of Theorem 1 and (i) in Theorems 2 and 3.

Let G be the Green function of $-\Delta + 1$ in Ω with homogeneous Neumann boundary conditions. For $N \geq 2$ we put

$$E(r) = (2\pi)^{-N/2} r^{(2-N)/2} \kappa_{(N-2)/2}(r) \quad \text{for } r > 0,$$

where κ_ν is the modified Bessel function of the second kind of order ν (see [13]). E is a fundamental solution of $-\Delta + 1$.

For the solution (b, s) to (1.1), (1.2) define the functions S and B by

$$S(r, t) = \int_{|x| \leq r} s(x, t) dx, \quad B(r, t) = \int_{|x| \leq r} b(x, t) dx \quad (2.1)$$

for $0 \leq r \leq L$ and $0 \leq t < T_{max}$, respectively. B and S satisfy

$$\frac{\partial B}{\partial t} = r^{N-1} \frac{\partial}{\partial r} \left(r^{1-N} \frac{\partial B}{\partial r} \right) + \frac{\chi}{\omega_N} (B - S) \phi'(s) r^{1-N} \frac{\partial B}{\partial r}, \quad (2.2)$$

$$0 = r^{N-1} \frac{\partial}{\partial r} \left(r^{1-N} \frac{\partial S}{\partial r} \right) - S + B, \quad (2.3)$$

for $0 < r < L$ and $0 < t < T_{max}$, and

$$B(0, t) = S(0, t) = 0, \quad B(L, t) = S(L, t) = \|b_0\|_{L^1},$$

where ω_N is the surface area of the unit sphere S^{N-1} in $\mathbf{R}^N$.

In order to show the boundedness and time-global existence of solutions (b, s) to (1.1), (1.2), we begin with the following lemmas. These lemmas are shown by the arguments similar to those in [14] and [18], respectively, so we omit the proofs. In what follows, C denotes a generic positive constant depending on L and N .

Lemma 2.1 *Let $N \geq 2$. Then*

$$s(x, t) \geq C \|b_0\|_{L^1} \quad \text{for } x \in \bar{\Omega} \text{ and } t \in (0, T_{max}).$$

Lemma 2.2 *If the following condition*

$$\sup_{0 \leq t < T_{max}} \|s(\cdot, t)\|_{L^\infty} < \infty, \quad \sup_{0 \leq t < T_{max}} \|\nabla \phi(s(\cdot, t))\|_{L^\infty} < \infty,$$

holds, then $T_{max} = \infty$ and

$$\sup_{t > 0} \|b(\cdot, t)\|_{L^\infty} < \infty.$$

For the following lemma, see [18].

Lemma 2.3 *Let $N \geq 2$. Then the following holds :*

$$B(|x|, t) E(|x|) \leq s(x, t) \leq C \|b_0\|_{L^1} E(|x|) \quad \text{in } \Omega \setminus \{0\} \times (0, T_{max}).$$

Sketch of proofs of Theorem 1 and (i) in Theorems 2 and 3. By Lemmas 2.1, 2.2 and Appendixes in [14] and [18], it suffices to show that

$$\sup_{0 \leq t < T_{max}} \|s(\cdot, t)\|_{L^\infty} < \infty, \quad \sup_{0 \leq t < T_{max}} \|\nabla s(\cdot, t)\|_{L^\infty} < \infty. \quad (2.4)$$

In the case of $N = 1$, (2.4) is shown by the arguments similar to those in [14]. Hence we will prove (2.4) in the case of $N \geq 2$.

We put

$$\Phi(u) = \begin{cases} p \|b_0\|_{L^1}^p u^{p-1} & \text{in the case of Theorem 2,} \\ u^{-1} & \text{in the case of Theorem 3} \end{cases}$$

for $u > 0$. It follows from Lemma 2.3, (2.2) and $\partial B / \partial r \geq 0$ that B satisfies

$$\frac{\partial B}{\partial t} \leq r^{N-1} \frac{\partial}{\partial r} \left(r^{1-N} \frac{\partial B}{\partial r} \right) + \frac{\chi}{\omega_N} \Phi(E) r^{1-N} \frac{\partial B}{\partial r}.$$

We can construct the function $W(r)$ such that

$$W(r) \sim r^N \text{ as } r \rightarrow 0,$$

$$\|b_0\|_{L^1} < W(L) \quad \text{and} \quad B(r, 0) \leq W(r) \quad \text{for } 0 \leq r \leq L,$$

and that

$$0 \geq r^{N-1} \frac{d}{dr} \left(r^{1-N} \frac{dW}{dr} \right) + \frac{\chi}{\omega_N} B \phi'(s) r^{1-N} \frac{dW}{dr} \text{ for } 0 < r < L.$$

Hence, the comparison theorem yields that

$$B(r, t) \leq W(r) \quad \text{for } 0 \leq r \leq L, 0 \leq t < T_{\max},$$

which implies $B(r, t) \leq Cr^N$.

Since $B(r, t) \leq Cr^N$ for $0 \leq r \leq L$ and $0 \leq t < T_{\max}$, it follows from (2.3) that

$$S(r, t) \leq Cr^N \quad \text{for } 0 \leq r \leq L, 0 \leq t < T_{\max}.$$

Then we have that

$$|\nabla s(x, t)| = \frac{|S(|x|, t) - B(|x|, t)|}{\omega_N |x|^{N-1}} \leq C$$

for $x \in \Omega$ and $0 \leq t < T_{\max}$. The boundedness of $\|s(\cdot, t)\|_{L^\infty}$ with respect to $t \in [0, T_{\max})$ follows from the estimate above of $|\nabla s|$ and

$$\min_{x \in \Omega} s(x, t) \leq \frac{\|b_0\|_{L^1}}{|\Omega|} \quad \text{for } 0 \leq t < T_{\max},$$

where $|\Omega|$ is the volume of Ω . Thus the proofs of (i) of Theorems 2 and 3 are complete.

3. Blow-up of solutions

The purpose in this section is to show the blow-up of solutions for the system (1.1), (1.2) in the case of $N \geq 2$.

In order to show the blow-up of solutions (b, s) to (1.1), (1.2) in [14] and [18], a differential inequality on a moment $M_k(t)$ of b is constructed by use of some estimates of s , and under some conditions on b_0 it is shown that the moment of b converges to 0 as t tends some $T_0 \in (0, \infty)$ by use of the differential inequality.

The following lemma is an immediate consequence of Hölder's inequality.

Lemma 3.1 *Let f be an integrable function on Ω , and p_1, p_2 and p_3 be numbers satisfying $0 \leq p_1 < p_2 < p_3$. Then*

$$\int_{\Omega} |f| |x|^{p_2} dx \leq \left\{ \int_{\Omega} |f| |x|^{p_1} dx \right\}^{(p_3-p_2)/(p_3-p_1)} \left\{ \int_{\Omega} |f| |x|^{p_3} dx \right\}^{(p_2-p_1)/(p_3-p_1)}.$$

Let S and B be the same functions as in (2.1). The following lemmas are stated in [15] and [18].

Lemma 3.2 *The inequality holds :*

$$\begin{aligned} \frac{d}{dt} M_k(t) &\leq k(k+N-2) \int_{\Omega} b(x,t) |x|^{k-2} dx \\ &\quad + \frac{k\chi}{\omega_N} \int_{\Omega} \phi'(s(x,t)) b(x,t) \{S(|x|,t) - B(|x|,t)\} |x|^{k-N} dx \end{aligned}$$

on $(0, T_{max})$, where $k \geq 2$.

Lemma 3.3 *Let $N \geq 3$. There exists a positive constant δ such that*

$$\frac{\partial}{\partial r} (r^{N-1} s(x,t)) \geq 0 \quad \text{in } \{x \in R^N : |x| \leq \delta\} \times (0, T_{max}),$$

where $r = |x|$.

Lemma 3.4 *Let $N \geq 2$. Then the following holds :*

$$s(x,t) \leq \frac{1}{\omega_N |x|^{N-1}} \int_{|y|=|x|} E(|x-y|) d\sigma \|b_0\|_{L^1} + \int_{\Omega} K(x,y) b(y,t) dy$$

in $\Omega \setminus \{0\} \times (0, T_{max})$.

Sketch of proof of (ii) of Theorem 2. Let $k = (N-2)p + 2$. In order to prove the theorem, it suffices to show the following inequality

$$\begin{aligned} \frac{d}{dt} M_k(t) &\leq k(k+N-2) \|b_0\|_{L^1}^{2/k} M_k(t)^{(k-2)/k} \\ &\quad + C \|b_0\|_{L^1}^{p+(k-2)/k} M_k(t)^{2/k} - C \|b_0\|_{L^1}^{p+1} \end{aligned} \quad (3.1)$$

for $t \in (0, T_{max})$. In fact, if $M_k(0)$ is sufficiently small so that the right-hand side of (3.1) is negative at $t = 0$, there exists $T_0 \in (0, \infty)$ such that

$$M_k(t) \rightarrow 0 \quad \text{as } t \rightarrow T_0.$$

Hence, T_{max} must be finite and $T_{max} \leq T_0$. By Appendixes in [14] and [18], we have

$$\limsup_{t \rightarrow T_{max}} \|b(\cdot, t)\|_{L^\infty} = \infty.$$

Let us first show (3.1) in the case of $p \geq 1$. Using Lemmas 2.3 and the properties of the fundamental solution, we obtain that

$$\int_{\Omega} s^{p-1}(x,t) b(x,t) B(|x|,t) |x|^{k-N} dx \geq C \|b_0\|_{L^1}^{p+1}. \quad (3.2)$$

and that

$$S(|x|,t) \leq C \|b_0\|_{L^1} |x|^2. \quad (3.3)$$

It follows from Lemma 2.3 and (3.3) and the properties of the fundamental solution that

$$\int_{\Omega} s^{p-1}(x, t) b(x, t) S(|x|, t) |x|^{k-N} dx \leq C \|b_0\|_{L^1}^p M_2(t). \quad (3.4)$$

Lemma 3.2 together with (3.2), (3.4) and Lemma 3.1 yields (3.1).

Let us consider the case $0 < p < 1$. By Lemmas 2.3 and the properties of the fundamental solution, we have

$$\begin{aligned} & \int_{\Omega} s^{p-1}(x, t) b(x, t) B(|x|, t) |x|^{k-N} dx \\ & \geq C \|b_0\|_{L^1}^{p-1} \int_{\Omega} b(x, t) B(|x|, t) dx = \frac{C}{2} \|b_0\|_{L^1}^{p+1}. \end{aligned} \quad (3.5)$$

It follows from Lemmas 2.3 and 3.3 and the properties of the fundamental solution that

$$\int_{|x| \leq \delta} s^{p-1}(x, t) b(x, t) S(|x|, t) |x|^{k-N} dx \leq C \|b_0\|_{L^1}^p \int_{|x| \leq \delta} b(x, t) |x|^2 dx. \quad (3.6)$$

By Lemma 2.1 and (3.3), we have

$$\begin{aligned} & \int_{\delta \leq |x| \leq L} s^{p-1}(x, t) b(x, t) S(|x|, t) |x|^{k-N} dx \\ & \leq C \|b_0\|_{L^1}^p \int_{\delta \leq |x| \leq L} b(x, t) |x|^{k-N+2} dx \\ & \leq C \|b_0\|_{L^1}^p \delta^{k-N} \int_{\delta \leq |x| \leq L} b(x, t) |x|^2 dx. \end{aligned} \quad (3.7)$$

Combining (3.6) with (3.7) yields that

$$\int_{\Omega} s^{p-1}(x, t) b(x, t) S(|x|, t) |x|^{k-N} dx \leq C \|b_0\|_{L^1}^p M_2(t). \quad (3.8)$$

By (3.5) and (3.8), the similar argument to that in the case of $p \geq 1$ gives us (3.1). Thus the proof is complete.

Sketch of proof of (ii) in Theorem 3. Observe that it follows from Lemma 3.4 and the properties of the fundamental solution that for $0 \leq t < T_{max}$ and $0 < |x| \leq L/2$,

$$s(x, t) \leq \left\{ \frac{1}{\omega_N(N-2)|x|^{N-2}} + C(|x|^{3-N} + 1) \right\} \|b_0\|_{L^1}.$$

For $0 \leq t < T_{max}$ and $0 < \delta \leq L/2$, we then have that

$$\begin{aligned} & \int_{\Omega} b(x, t) B(|x|, t) \frac{1}{|x|^{N-2} s(x, t)} dx \\ & \geq \frac{1}{\|b_0\|_{L^1}} \left\{ \frac{1}{(N-2)\omega_N} + C\delta \right\}^{-1} \int_{|x| \leq \delta} b(x, t) B(|x|, t) dx \end{aligned}$$

$$\begin{aligned}
&\geq \frac{(N-2)\omega_N}{2(1+C\delta)\|b_0\|_{L^1}} B(\delta, t)^2 \\
&\geq \frac{(N-2)\omega_N}{2(1+C\delta)\|b_0\|_{L^1}} \left(\|b_0\|_{L^1} - \frac{1}{\delta^2} M_2(t) \right)_+^2 \\
&\geq \frac{(N-2)\omega_N}{2(1+C\delta)} \left(\|b_0\|_{L^1} - \frac{2}{\delta^2} M_2(t) \right),
\end{aligned} \tag{3.9}$$

where $(\cdot)_+ = \max\{\cdot, 0\}$. It follows from Lemma 3.3 that

$$\begin{aligned}
&\int_{\Omega} b(x, t) S(|x|, t) \frac{1}{|x|^{N-2} s(x, t)} dx \\
&= \omega_N \int_{|x| \leq \delta} b(x, t) |x|^2 dx + \int_{\delta \leq |x| \leq L} b(x, t) S(|x|, t) \frac{1}{|x|^{N-2} s(x, t)} dx \\
&\leq C M_2(t)
\end{aligned} \tag{3.10}$$

in $(0, T_{max})$. Hence, combining Lemma 3.2 with (3.9) and (3.10) concludes that

$$\frac{d}{dt} M_2(t) \leq \left\{ 2N - \frac{(N-2)\chi}{1+C\delta} \right\} \|b_0\|_{L^1} + C\chi(1+\delta^{-2}) M_2(t)$$

on $(0, T_{max})$. Suppose that δ is sufficiently small so that $2N(1+C\delta) - (N-2)\chi < 0$. Using the argument similar to that in the sketch of proof of Theorem 2, then we have the proof.

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