1. Introduction

We consider time-global existence and blow-up of solutions of the following system related to chemotaxis

\[
\begin{aligned}
    b_t &= \nabla \cdot (\nabla b - \chi b \nabla \phi(s)) \quad \text{in} \quad \Omega \times (0, \infty), \\
    0 &= \Delta s - s + b \quad \text{in} \quad \Omega \times (0, \infty),
\end{aligned}
\]

(1.1)

under the conditions

\[
\begin{aligned}
    \frac{\partial b}{\partial n} = \frac{\partial s}{\partial n} &= 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \\
    b(\cdot, 0) &= b_0 \quad \text{in} \quad \Omega,
\end{aligned}
\]

(1.2)

where \( \chi \) is a positive constant and \( \phi \) is a smooth function on \((0, \infty)\) with \( \phi' > 0 \). The system is a simplified Keller-Segel model. Keller-Segel model was introduced by Keller and Segel [11] to describe the initiation of chemotactic aggregation of cellular slime molds. On Keller-Segel model and simplified Keller-Segel models, time-local existence of the solutions has been studied by [19] and blow-up of the solutions has been studied by [4, 10, 9, 14, 18].

The domain \( \Omega \) and the non-trivial initial function \( b_0 \) are only confined to the following case:

(A1) \( \Omega \) is the open ball of radius \( L \) with center at the origin in \( \mathbb{R}^N \).

(A2) \( b_0 \) is smooth and nonnegative on \( \bar{\Omega} \), and is radially symmetric when \( N \geq 2 \).

Under these assumptions, there exists a unique solution \((b(x, t), s(x, t))\) to (1.1) and (1.2) defined maximal interval of existence \([0, T_{\text{max}}]\), which is radially symmetric in \( x \) when \( N \geq 2 \), smooth in \( \bar{\Omega} \times (0, T_{\text{max}}) \) and \( b(x, t) > 0, s(x, t) > 0 \) for \((x, t) \in \Omega \times (0, T_{\text{max}})\). If \( T_{\text{max}} < \infty \),

\[
\limsup_{t \to T_{\text{max}}} (\|b(\cdot, t)\|_{L^\infty} + \|s(\cdot, t)\|_{L^\infty}) = \infty,
\]

by which we mean that \((b(x,t), s(x,t))\) blows up in finite time.

**Theorem 1** Let \(N = 1\) and \(\phi\) be smooth on \((0,\infty)\). Then the solution \((b,s)\) to (1.1), (1.2) is globally bounded, that is, \(T_{\text{max}} = \infty\) and \((b,s)\) satisfies
\[
\sup_{t \geq 0} (\|b(\cdot, t)\|_{L^\infty} + \|s(\cdot, t)\|_{L^\infty}) < \infty.
\]

We put
\[
M_a(t) = \int_{\Omega} b(x,t)|x|^a \, dx \quad \text{for} \ 0 \leq t < T_{\text{max}},
\]
where \(a\) is a positive constant. That is called the moment of order \(a\), of \(b(\cdot,t)\).

**Theorem 2** Assume \(\phi(s) = s^p\ (p > 0)\), (A1) and (A2).

(1) \(N = 2\):
   (a) If \(0 < p < 1\), then the solution is globally bounded in time.
   (b) \(p = 1\):
       (i) If \(\|b_0\|_{L^1} < 8\pi/\chi\), then the solution is globally bounded in time.
       (ii) If \(\|b_0\|_{L^1} > 8\pi/\chi\) and \(M_2(0)\) is sufficiently small, then the solution blows up in finite time.
   (c) If \(p > 1\) and \(M_2(0)\) is sufficiently small, then the solution blows up in finite time.

(2) If \(N \geq 3\) and \(M_{(N-2)p+2}(0)\) is sufficiently small, then the solution blows up in finite time.

**Theorem 3** Assume \(\phi(s) = \log s\), (A1) and (A2).

(1) If \(N = 2\), then the solution is globally bounded in time.

(2) If \(N \geq 3\):
   (a) If \(\chi < 2/(N-2)\), then the solution is globally bounded in time.
   (b) If \(\chi > 2N/(N-2)\) and \(M_2(0)\) is sufficiently small, then the solution blows up in finite time.

### 2. Time-global existence and boundedness

The purpose in this section is to sketch the proofs of Theorem 1 and (i) in Theorems 2 and 3.

Let \(G\) be the Green function of \(-\Delta + 1\) in \(\Omega\) with homogeneous Neumann boundary conditions. For \(N \geq 2\) we put
\[
E(r) = (2\pi)^{-N/2} r^{(2-N)/2} \kappa_{(N-3)/2}(r) \quad \text{for} \ r > 0,
\]
where \(\kappa_{\nu}\) is the modified Bessel function of the second kind of order \(\nu\) (see [13]). \(E\) is a fundamental solution of \(-\Delta + 1\).

For the solution \((b,s)\) to (1.1), (1.2) define the functions \(S\) and \(B\) by
\[ S(r, t) = \int_{|x| \leq r} s(x, t) dx, \quad B(r, t) = \int_{|x| \leq r} b(x, t) dx \]  
(2.1)

for \(0 \leq r \leq L\) and \(0 \leq t < T_{\text{max}}\), respectively. \(B\) and \(S\) satisfy

\[ \frac{\partial B}{\partial t} = r^{N-1} \frac{\partial}{\partial r} \left( r^{1-N} \frac{\partial B}{\partial r} \right) + \frac{\chi}{\omega_N} (B - S) \phi'(s) r^{1-N} \frac{\partial B}{\partial r}, \]  
(2.2)

\[ 0 = r^{N-1} \frac{\partial}{\partial r} \left( r^{1-N} \frac{\partial S}{\partial r} \right) - S + B, \]  
(2.3)

for \(0 < r < L\) and \(0 < t < T_{\text{max}}\), and

\[ B(0, t) = S(0, t) = 0, \quad B(L, t) = S(L, t) = \|b_0\|_{L^1}, \]

where \(\omega_N\) is the surface area of the unit sphere \(S^{N-1}\) in \(\mathbb{R}^N\).

In order to show the boundedness and time-global existence of solutions \((b, s)\) to (1.1), (1.2), we begin with the following lemmas. These lemmas are shown by the arguments similar to those in [14] and [18], respectively, so we omit the proofs. In what follows, \(C\) denotes a generic positive constant depending on \(L\) and \(N\).

**Lemma 2.1** Let \(N \geq 2\). Then

\[ s(x, t) \geq C\|b_0\|_{L^1} \quad \text{for} \quad x \in \bar{\Omega} \quad \text{and} \quad t \in (0, T_{\text{max}}). \]

**Lemma 2.2** If the following condition

\[ \sup_{0 \leq t < T_{\text{max}}} \|s(\cdot, t)\|_{L^\infty} < \infty, \quad \sup_{0 \leq t < T_{\text{max}}} \|\nabla \phi(s(\cdot, t))\|_{L^\infty} < \infty, \]

holds, then \(T_{\text{max}} = \infty\) and

\[ \sup_{t > 0} \|b(\cdot, t)\|_{L^\infty} < \infty. \]

For the following lemma, see [18].

**Lemma 2.3** Let \(N \geq 2\). Then the following holds:

\[ B(|x|, t) E(|x|) \leq s(x, t) \leq C\|b_0\|_{L^1} E(|x|) \quad \text{in} \quad \Omega \setminus \{0\} \times (0, T_{\text{max}}). \]

**Sketch of proofs of Theorem 1 and (i) in Theorems 2 and 3.** By Lemmas 2.1, 2.2 and Appendixes in [14] and [18], it suffices to show that

\[ \sup_{0 \leq t < T_{\text{max}}} \|s(\cdot, t)\|_{L^\infty} < \infty, \quad \sup_{0 \leq t < T_{\text{max}}} \|\nabla s(\cdot, t)\|_{L^\infty} < \infty, \]  
(2.4)

In the case of \(N = 1\), (2.4) is shown by the arguments similar to those in [14]. Hence we will prove (2.4) in the case of \(N \geq 2\).

We put

\[ \Phi(u) = \begin{cases} \frac{p\|b_0\|_{L^1}^p}{u^{p-1}} & \text{in the case of Theorem 2,} \\ u^{-1} & \text{in the case of Theorem 3} \end{cases} \]
for \( u > 0 \). It follows from Lemma 2.3, (2.2) and \( \partial B / \partial r \geq 0 \) that \( B \) satisfies
\[
\frac{\partial B}{\partial t} \leq r^{N-1} \frac{\partial}{\partial r} \left( r^{1-N} \frac{\partial B}{\partial r} \right) + \frac{\chi}{\omega_N} \Phi(E) r^{1-N} \frac{\partial B}{\partial r}.
\]

We can construct the function \( W(r) \) such that
\[
W(r) \sim r^N \text{ as } r \to 0,
\]
\[
\|b_0\|_{L^1} < W(L) \quad \text{and} \quad B(r, 0) \leq W(r) \quad \text{for } 0 \leq r \leq L,
\]
and that
\[
0 \geq r^{N-1} \frac{d}{dr} \left( r^{1-N} \frac{dW}{dr} \right) + \frac{\chi}{\omega_N} B \phi'(s) r^{1-N} \frac{dW}{dr} \quad \text{for } 0 < r < L.
\]

Hence, the comparison theorem yields that
\[
B(r, t) \leq W(r) \quad \text{for } 0 \leq r \leq L, 0 \leq t < T_{\text{max}},
\]
which implies \( B(r, t) \leq C r^N \).

Since \( B(r, t) \leq C r^N \) for \( 0 \leq r \leq L \) and \( 0 \leq t < T_{\text{max}} \), it follows from (2.3) that
\[
B(r, t) \leq C r^N \quad \text{for } 0 \leq r \leq L, 0 \leq t < T_{\text{max}}.
\]

Then we have that
\[
|\nabla s(x, t)| = \frac{|S(|x|, t) - B(|x|, t)|}{\omega_N |x|^{N-1}} \leq C
\]
for \( x \in \Omega \) and \( 0 \leq t < T_{\text{max}} \). The boundedness of \( \|s(\cdot, t)\|_{L^\infty} \) with respect to \( t \in [0, T_{\text{max}}] \) follows from the estimate above of \( |\nabla s| \) and
\[
\min_{x \in \Omega} s(x, t) \leq \frac{\|b_0\|_{L^1}}{|\Omega|} \quad \text{for } 0 \leq t < T_{\text{max}},
\]
where \( |\Omega| \) is the volume of \( \Omega \). Thus the proofs of (i) of Theorems 2 and 3 are complete.

3. Blow-up of solutions

The purpose in this section is to show the blow-up of solutions for the system (1.1), (1.2) in the case of \( N \geq 2 \).

In order to show the blow-up of solutions \((b, s)\) to (1.1), (1.2) in [14] and [18], a differential inequality on a moment \( M_k(t) \) of \( b \) is constructed by use of some estimates of \( s \), and under some conditions on \( b_0 \) it is shown that the moment of \( b \) converges to \( 0 \) as \( t \) tends some \( T_0 \in (0, \infty) \) by use of the differential inequality.

The following lemma is an immediate consequence of Hölder’s inequality.

**Lemma 3.1** Let \( f \) be an integrable function on \( \Omega \), and \( p_1, p_2 \) and \( p_3 \) be numbers satisfying \( 0 \leq p_1 < p_2 < p_3 \). Then
\[
\int_{\Omega} |f||x|^{p_2} dx \leq \left\{ \int_{\Omega} |f||x|^{p_1} dx \right\}^{(p_3-p_2)/(p_3-p_1)} \left\{ \int_{\Omega} |f||x|^{p_3} dx \right\}^{(p_2-p_1)/(p_3-p_1)}.
\]
Let $S$ and $B$ be the same functions as in (2.1). The following lemmas are stated in [15] and [18].

**Lemma 3.2** The inequality holds:
\[
\frac{d}{dt}M_k(t) \leq k(k + N - 2) \int_{\Omega} b(x, t)|x|^{k-2}dx \\
+ \frac{k\chi}{\omega_N} \int_{\Omega} \phi'(s(x, t))b(x, t) \{S(|x|, t) - B(|x|, t)\} |x|^{k-N}dx
\]
on $(0, T_{\text{max}})$, where $k \geq 2$.

**Lemma 3.3** Let $N \geq 3$. There exists a positive constant $\delta$ such that
\[
\frac{\partial}{\partial r}(r^{N-1}s(x, t)) \geq 0
\]
in $\{x \in \mathbb{R}^N : |x| \leq \delta\} \times (0, T_{\text{max}})$, where $r = |x|$.

**Lemma 3.4** Let $N \geq 2$. Then the following holds:
\[
s(x, t) \leq \frac{1}{\omega_N|x|^{N-1}} \int_{|y|=|x|} E(|x-y|)d\sigma||b_0||_{L^1} + \int_{\Omega} K(x, y)b(y, t)dy
\]
in $\Omega \setminus \{0\} \times (0, T_{\text{max}})$.

**Sketch of proof of (ii) of Theorem 2.** Let $k = (N-2)p+2$. In order to prove the theorem, it suffices to show the following inequality
\[
\frac{d}{dt}M_k(t) \leq k(k + N - 2)||b_0||_{L^1}^{2/k} M_k(t)^{(k-2)/k} \\
+C||b_0||_{L^1}^{p+1}(k-2)/k M_k(t)^{2/k} - C||b_0||_{L^1}^{p+1}
\]
(3.1) for $t \in (0, T_{\text{max}})$. In fact, if $M_k(0)$ is sufficiently small so that the right-hand side of (3.1) is negative at $t = 0$, there exists $T_0 \in (0, \infty)$ such that
\[
M_k(t) \to 0 \quad \text{as} \quad t \to T_0.
\]

Hence, $T_{\text{max}}$ must be finite and $T_{\text{max}} \leq T_0$. By Appendixes in [14] and [18], we have
\[
\limsup_{t \to T_{\text{max}}} \|b(\cdot, t)\|_{L^\infty} = \infty.
\]

Let us first show (3.1) in the case of $p \geq 1$. Using Lemmas 2.3 and the properties of the fundamental solution, we obtain that
\[
\int_{\Omega} s^{p-1}(x, t)b(x, t)B(|x|, t)|x|^{k-N}dx \geq C||b_0||_{L^1}^{p+1}
\]
and that
\[
S(|x|, t) \leq C||b_0||_{L^1}|x|^2.
\]
It follows from Lemma 2.3 and (3.3) and the properties of the fundamental solution that
\[
\int_{\Omega} s^{p-1}(x, t)b(x, t)S(|x|, t)|x|^{k-N}dx \leq C\|b_0\|_{L^1}^p M_2(t). \tag{3.4}
\]

Lemma 3.2 together with (3.2), (3.4) and Lemma 3.1 yields (3.1).

Let us consider the case \(0 < p < 1\). By Lemmas 2.3 and the properties of the fundamental solution, we have
\[
\int_{\Omega} s^{p-1}(x, t)b(x, t)B(|x|, t)|x|^{k-N}dx \\
\geq C\|b_0\|_{L^1}^{p-1} \int_{\Omega} b(x, t)B(|x|, t)dx = \frac{C}{2}\|b_0\|_{L^1}^{p+1}. \tag{3.5}
\]

It follows from Lemmas 2.3 and 3.3 and the properties of the fundamental solution that
\[
\int_{|x| \leq \delta} s^{p-1}(x, t)b(x, t)S(|x|, t)|x|^{k-N}dx \leq C\|b_0\|_{L^1}^p \int_{|x| \leq \delta} b(x, t)|x|^2dx. \tag{3.6}
\]

By Lemma 2.1 and (3.3), we have
\[
\int_{\delta \leq |x| \leq L} s^{p-1}(x, t)b(x, t)S(|x|, t)|x|^{k-N}dx \\
\leq C\|b_0\|_{L^1}^p \int_{\delta \leq |x| \leq L} b(x, t)|x|^{k-N+2}dx \\
\leq C\|b_0\|_{L^1}^p \delta^{k-N} \int_{\delta \leq |x| \leq L} b(x, t)|x|^2dx. \tag{3.7}
\]

Combining (3.6) with (3.7) yields that
\[
\int_{\Omega} s^{p-1}(x, t)b(x, t)S(|x|, t)|x|^{k-N}dx \leq C\|b_0\|_{L^1}^p M_2(t). \tag{3.8}
\]

By (3.5) and (3.8), the similar argument to that in the case of \(p \geq 1\) gives us (3.1). Thus the proof is complete.

**Sketch of proof of (ii) in Theorem 3.** Observe that it follows from Lemma 3.4 and the properties of the fundamental solution that for \(0 \leq t < T_{\text{max}}\) and \(0 < |x| \leq L/2\),
\[
s(x, t) \leq \left\{ \frac{1}{\omega_N(N-2)|x|^{N-2}} + C(|x|^{3-N} + 1) \right\}\|b_0\|_{L^1}.
\]

For \(0 \leq t < T_{\text{max}}\) and \(0 < \delta \leq L/2\), we then have that
\[
\int_{\Omega} b(x, t)B(|x|, t) \frac{1}{|x|^{N-2}s(x, t)}dx \\
\geq \frac{1}{\|b_0\|_{L^1}} \left\{ \frac{1}{(N-2)\omega_N} + C\delta \right\}^{-1} \int_{|x| \leq \delta} b(x, t)B(|x|, t)dx
\]
\[
\geq \frac{(N-2)\omega_N}{2(1+C\delta)}\|b_0\|_{L^1}B(\delta,t)^2
\]
\[
\geq \frac{(N-2)\omega_N}{2(1+C\delta)}\left(\|b_0\|_{L^1} - \frac{1}{\delta^2}M_2(t)\right)^2
\]
\[
\geq \frac{(N-2)\omega_N}{2(1+C\delta)}\left(\|b_0\|_{L^1} - \frac{2}{\delta^2}M_2(t)\right)
\]
where \((\cdot)_+ = \max\{\cdot, 0\}\). It follows from Lemma 3.3 that
\[
\int_{\Omega} b(x,t)S(|x|,t) \frac{1}{|x|^{N-2}s(x,t)} dx
\]
\[
= \omega_N \int_{|x|\leq \delta} b(x,t)dx + \int_{\delta\leq |x| \leq L} b(x,t)S(|x|,t) \frac{1}{|x|^{N-2}s(x,t)} dx
\]
\[
\leq CM_2(t)
\]
in \((0,T_{\text{max}})\). Hence, combining Lemma 3.2 with (3.9) and (3.10) concludes that
\[
\frac{d}{dt}M_2(t) \leq \left\{2N - \frac{(N-2)\chi}{1+C\delta}\right\}\|b_0\|_{L^1} + C\chi(1+\delta^{-2})M_2(t)
\]
on \((0,T_{\text{max}})\). Suppose that \(\delta\) is sufficiently small so that \(2N(1+C\delta) - (N-2)\chi < 0\). Using the argument similar to that in the sketch of proof of Theorem 2, then we have the proof.

References


