数理解析研究所講究録
975 巻 1996 年 163-175

A Theory of Function Lattices on Finite Topological Spaces for Image Processing

Shinshu University, Yatsuka Nakamura (中村　八束)
Shinshu University, Yasunari Shidama (師玉　康成)

ABSTRACT. The main object of this paper is to formulate image processing for gray scaled and colored images mathematically. Bounded real functions on a finite topological space compose an abstraction of gray scaled images on a plane. We can introduce some operators for such functions using the nature of the underlying finite topological space, the concept of which was introduced previously by the authors. These operators correspond to some actual image processing for gray scaled images, which are a sort of neighborhood processing. The boundary, closure and interior of gray scaled images are defined naturally. Some other smoothing filters can also be introduced.

1. INTRODUCTION AND NOTATION

The concept of finite topological spaces was introduced by the authors ([1][4]). Such notions as closure, boundary, etc., which

---

1991 *Mathematics Subject Classification.* 06A99,54H99,68R99.
Key words and phrases. function lattice, image processing, finite topology, closure of function, boundary of function, interior of function, neighborhood, digitized space.

relate to those of general topological spaces, were introduced on this space. Subsets of the space can be viewed as monochromatic images and such topological notions then correspond to processing operations for monochromatic images.

Here we consider some functions on a finite topological space consisting of a function lattice and represent gray scaled images. Neighborhood operations for underlying subsets are generalized for such functions. Such generalized operations correspond to various image processing procedures for gray or colored images.

Other operations which were adopted by engineers are also formulated abstractly in such a framework.

Let X be a set. $(X, U(.))$ is a *finite topological space* (*FTS*) if for all $x \in X$, $U(x) \subseteq X$, i.e., $U(.)$ is a set valued function taking a value in 2^X. An FTS $(X, U(.))$ is *filled* if for all $x \in X$, $x \in U(x)$.

In the following, we assume the FTS treated is filled. For a subset A of X, we write

$$A^i = \{x : U(x) \subseteq A\},$$

and call it the *interior of* A. Similarly, we write

$$A^b = \{x : U(x) \cap A \neq \phi\},$$

and call it the *closure of* A. The *boundary of* A is defined by

$$A^\partial = A^b \cap (A^c)^b,$$

where A^c is the complement of A.

2. General Operations for Graphical Images

Given an FTS $\zeta = (X, U(.))$, two sets Y and Z, $K(.)$ denotes a mapping from X to $\bigcup\{Y^{U(x)} : x \in X\}$ such that for all $x \in X$, $K(x) \subseteq Y^{U(x)}$. Let f be an element of Y^X, i.e., f is a mapping from X to Y. For any $x \in X$, a restriction of f to $U(x)$, written as $f|U(x)$, is clearly an element of $Y^{U(x)}$. When $f|U(x) \in K(x)$, we say *f is under $K(.)$*. We write

$$\mathrm{GImage}(K, \zeta, Y) = \{g : g \in Y^X \text{ and } g \text{ is under } K\}.$$

For such GImage, the following hold:

(1) Let k be a mapping such that $k \in Y^X$. If $K(x) = \{\ k|U(x)\ \}$, then GImage$(K, \zeta, Y) = \{k\}$, where $k|U(x)$ is a restriction of a mapping k to $U(x)$,

(2) if $K_1(x) \subseteq K_2(x)$ for all $x \in X$, then GImage$(K_1, \zeta, Y) \subseteq$ GImage(K_2, ζ, Y),

(3) if $K(x) = Y^{U(x)}$ for all $x \in X$, then GImage$(K, \zeta, Y) = Y^X$, and

(4) if $K(x) = \phi$ for all $x \in X$, then GImage$(K, \zeta, Y) = \phi$.

Let a be an element of $\prod_{x \in X} Z^{K(x)}$. Then, it is clear that

$$a_x \in Z^{K(x)} \text{ for all } x \in X.$$

For an element f of GImage(K, ζ, Y), we introduce a transformation

$$f^{<a>}(x) = a_x(f|U(x)),$$

and it is clear that

$$f^{<a>} \in Z^X.$$

This means that an operation $< a >$ is an operator from GImage(K, ζ, Y) to Z^X. For this operation, we get the following:

(5) Let k be a mapping such that $k \in X^Z$. If $a_x(s) = k(x)$ for all $s \in Y^{U(x)} (x \in X)$, then $f^{<a>} = k$,

(6) let h be a mapping such that $h \in Y^Z$. If $a_x(s) = h(s(x))$ for all $s \in Y^{U(x)}$, then $f^{<a>} = h \circ f$, and

(7) if $Y = Z$ and $a_x(s) = s(x)$ for all $x \in X$ and all $s \in Y^{U(x)}$, then $f^{<a>} = f$.

3. Processing for Gray Images

In this section, the notations are almost the same as §2, but we assume here that $Y = Z = R^1$ and for every $x \in X$ and every $k \in K(x)$ there exists a real number r_x and for all $y \in U(x)$, holds $|k(y)| \leq r_x$. We write such K as K_∞. It means that we treat only locally bounded functions on X.

A set GImage(K_∞, ζ, Y) and X^Z (in this case, GImage$(K_\infty, \zeta, Y) \subseteq Z^X$) become vector lattices (see Kelly and Namioka [2]) by an order defined by

$$f_1 \leq f_2 \Leftrightarrow f_1(x) \leq f_2(x) \text{ for all } x \in X.$$

Then,

(1) $0 \le f$ and $t \ge 0$(t is real) implies $0 \le tf$,

(2) $0 \le f_1$ and $0 \le f_2$ implies $0 \le f_1 + f_2$, and

(3) $(f_1 \wedge f_2) + f_3 = (f_1 + f_3) \wedge (s_2 + f_3)$ and $(f_1 \vee f_2) + f_3 = (f_1 + f_3) \vee (s_2 + f_3)$.

Let us put $f^+ = f \vee 0$ and $f^- = -f \wedge 0$ for $f \in \mathrm{GImage}(K_\infty, \zeta, Y)$. It is obvious that $f^+ \in \mathrm{GImage}(K_\infty, \zeta, Y)$ and $f^- \in \mathrm{GImage}(K_\infty, \zeta, Y)$. Then,

(4) $f \in \mathrm{GImage}(K_\infty, \zeta, Y)$ holds $0 \le f^+$, $0 \le f^-$, $f \le f^+$, $-f \le f^-$ and $f = f^+ - f^-$.

For $h \in K_\infty(x)$, we put $i_x(h) = \inf_{y \in U(x)} h(y)$ and $b_x(h) = \sup_{y \in U(x)} h(y)$. Then, $i \in \prod_{x \in X} Z^{K_\infty(x)}$ and $b \in \prod_{x \in X} Z^{K_\infty(x)}$ so that we can consider operators $f^{<i>}$ and $f^{<b>}$ for any $f \in \mathrm{GImage}(K_\infty, \zeta, Y)$. We call $f^{<i>}$ the *interior of f* and $f^{<b>}$ the *closure of f*. It is clear that

(5) $f \in \mathrm{GImage}(K_\infty, \zeta, Y)$ implies $f^{<i>} \in Z^X$ and $f^{<b>} \in Z^X$,

(6) $f^{<i>} \le f \le f^{<b>}$,

(7) $(f_1 \wedge f_2)^{<i>} = f_1^{<i>} \wedge f_2^{<i>}$,

(8) $(f_1 \vee f_2)^{<b>} = f_1^{<b>} \vee f_2^{<b>}$,

(9) $f_1^{<i>} \vee f_2^{<i>} \le (f_1 \vee f_2)^{<i>}$,

(10) $f_1^{<b>} \wedge f_2^{<b>} \ge (f_1 \wedge f_2)^{<b>}$,

(11) $f_1 \le f_2$ implies $f_1^{<i>} \le f_2^{<i>}$ and $f_1^{<b>} \le f_2^{<b>}$,

(12) $(-f)^{<i>} = -(f^{<b>}), (-f)^{<b>} = -(f^{<i>})$,

(13) $0 \le t$(t is real) implies $(tf)^{<i>} = t(f^{<i>})$ and $(tf)^{<b>} =$

$t(f^{<b>})$,

(14) $f_1^{<i>} + f_2^{<i>} \leq (f_1 + f_2)^{<i>}$, and

(15) $f_1^{<b>} + f_2^{<b>} \geq (f_1 + f_2)^{<b>}$.

The following is an interesting result about the + operation and closure.

Lemma 1. *For* $f \in \mathrm{GImage}(K, \zeta, Y)$, $(f^{<b>})^+ = (f^+)^{<b>}$.

Proof. As $f \leq f^+$, it is clear that $f^{<b>} \leq (f^+)^{<b>}$ by (11). Moreover, $0 \leq (f^+)^{<b>}$ holds. Thus, $f^{<b>} \vee 0 \leq (f^+)^{<b>} \vee 0 = (f^+)^{<b>}$, i.e., $(f^{<b>})^+ \leq (f^+)^{<b>}$. Next, we see $f \leq f^{<b>}$. Thus, $f^+ \leq (f^{<b>})^+$. Here we divide the cases: If $f(x) \leq 0$, then $0 < (f^+)^{<b>}$ or $(f^+)^{<b>} = 0$, the former of which derives $(f^+)^{<b>} = (f^{<b>})^+$, i.e., $(f^+)^{<b>} \leq (f^{<b>})^+$, and the latter of which derives $(f^+)^{<b>} = 0 \leq (f^{<b>})^+$. If $f(x) > 0$, then $f^+(x) = f(x) > 0$. Thus $(f^+)^{<b>}(x) = (f^{<b>})^+$. $\square$

Similarly we have:

Lemma 2. *For* $f \in \mathrm{GImage}(K_\infty, \zeta, Y)$, $(f^{<i>})^- = (f^-)^{<b>}$.

Let us continue to list equations about the interior and the closure.

(16) $1 \in \mathrm{GImage}(K_\infty, \zeta, Y), 1^{<i>} = 1$ and $1^{<b>} = 1$.

If we put $f^c = 1 - f$, then

(17) $(f^c)^{<i>} = (f^{<b>})^c$ and $(f^c)^{<b>} = (f^{<i>})^c$.

The *boundary* of f is defined by

$$f^{\partial} = f^{<b>} \wedge (f^c)^{<b>} (= f^{<b>} \wedge (f^{<i>})^c).$$

Then,

(18) $0 \leq f \leq 1$ holds $0 \leq f^{\partial} \leq 1$,

(19) $f^{\partial} \leq f^{<b>}$,

(20) for $B \subseteq X$, $\chi_B^{<i>} = \chi_{B^i}$ where χ_B is a characteristic function of a set B,

(21) for $B \subseteq X$, $\chi_B^{<b>} = \chi_{B^b}$, and

(22) for $B \subseteq X$, $\chi_B^{<\partial>} = \chi_{B^\partial}$.

4. Local Mean of Functions

Notations used here are the same as §3. Let us assume that for all $x \in X$, there exists a real signed measure space $(U(x), \mathcal{B}_x, m_x)$, where $\mathcal{B}_x$ is a σ-field in $U(x)$ and a total variation of m_x is finite. We choose $K_m(x)$ as

$$K_m(x) = \{h_x : h_x \text{ is } \mathcal{B}_x\text{-measurable on } U(x) \text{ and } m_x\text{-integrable}\}.$$

For $g_x \in K_m(x)$, we define $\widetilde{m}_x$ as

$$\widetilde{m}_x(g_x) = \int_{U(x)} g_x \; m_x(dx).$$

Then, for $f \in \mathrm{GImage}(K_m, \zeta, Y)$, we can define

$$f^{<\widetilde{m}>} = \int (f|U(x)) m_x(dx),$$

which is called a *local mean* of f. Most filtering operations or differential operations like Sobel's operator (e.g., see [7]), can be represented by these mean operations. If we put $U(x) = X$ for each point x, then the Fourier cosine transformation (e.g., see [6]) is also formalized in such a frame. For mean operations, we get the following:

(1) if $f \in$GImage(K_∞, ζ, Y), then $f \in$GImage(K_m, ζ, Y),

(2) for any real number t_1 and t_2, $(a_1 f_1 + a_2 f_2)^{<\widetilde{m}>} = a_1 f_1^{<\widetilde{m}>} + a_2 f_2^{<\widetilde{m}>}$, and

(3) for two (finite)signed measures m_1 and m_2, $f^{<\widetilde{m1+m2}>} = f^{<\widetilde{m_1}>} + f^{<\widetilde{m_2}>}$.

If p_x is a probability measure for all $x \in X$, then

(4) $0 \leq f$ implies $0 \leq f^{<\widetilde{p}>}$,

(5) $f_1 \leq f_2$ implies $f_1^{<\widetilde{p}>} \leq f_2^{<\widetilde{p}>}$,

(6) for $f \in$ GImage(K, ζ, Y), $f \in$ GImage(K_p, ζ, Y) and $f^{<i>} \leq f^{<\widetilde{p}>} \leq f^{<b>}$,

(7) $(f^c)^{<\widetilde{p}>} = (f^{<\widetilde{p}>})^c$,

(8) for $B \subseteq X$, $\chi_B^{<p>} = p_x(B)$, and

(9) if $f_n \uparrow f$ pointwise, then $f_n^{<p>} \uparrow f^{<p>}$.

We can add some natural conditions to the definition of means. If $\mathcal{X}$ is a σ-field of X, then $\mathcal{B}_x = \mathcal{X} \cap U(x) (= \{C \cap U(x) : C \in \mathcal{X}\})$, and $m_.(B)$ is $\mathcal{X}$-measurable for all $B \in \mathcal{X}$. In this case, m is called a *channel* on $(X, U(.))$, and it is the same as the concept of an information channel (Khinchin[3], Umegaki[8]).

A channel is called *T-stationary* if $m_{Tx}(B) = m_x(T^{-1}B)$ for all $x \in X$ and $B \in \mathcal{X}$, where T is a measurable transformation on $(X, \mathcal{B})$. A stationary channel corresponds to the situation in which a filter of image processing is invariant under some geometric transformation.

For information theoretic relation

(10) $0 \leq f \leq 1$ holds $(-f \log f)^{<p>} \leq -(f^{<p>} \log f^{<p>})$, where $0 \log 0 = 0$,

which means that the entropy increases by a filtering operation.

Now we define a *convolution* of two information channels m_1, m_2 as

$$(m_1 \otimes m_2)_x(E) = \int_{U_2(x)} m_{1y}(E \cap U(y)) m_{2x}(dy),$$

where $U_2(x) = \{z : z \in U(y), \text{for some } y \in U(x)\}$. Clearly $m_1 \otimes m_2$ is again a channel, but it is on $(X, U_2(.))$.

For a convolution, we get the following theorem.

Theorem 1. *If* $f \in$GImage(K, ζ, Y), *then* $(f^{<m_1>})^{<m_2>} = f^{<m_1 \otimes m_2>}$.

This theorem shows that two or more successive operations of filtering are substituted by a one time filtering operation.

5. Processing for Color Images

In this section, we shall formulate color image processings. Let us put $Y = Z = R^3$ and assume a channel m and a σ-field on X

as in §3. As a set valued function K, we put:

$$K^1_m(x) = \{h :< h, e_i > \text{is}$$
$$\mathcal{B}_x\text{−measurable and } m_x\text{−integrable}, i = 1, 2, 3\},$$

where e_i is a unit vector in $Y = R^3$, e.g., $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Then, for $f \in$GImage(K^1_m, ζ, Y) we define

$$f^{<m>} = \sum_{i=1}^{3} \int_{U(x)} < f(y), e_i > dm(y)e_i,$$

where $< u_1, u_2 >$ is an inner product of vectors u_1 and u_2. A space GImage(K^1_m, ζ, Y) is not a lattice anymore, but we can introduce the concepts of *interior* and *closure* relative to h as

$$f^{<i,h>} = \inf_{y \in U(x)} < f(y), h_1 > h_1 + < f(x), h_2 > h_2 + < f(x), h_3 > h_3,$$

where h_1, h_2, h_3 are an orthogonal basis of R^3. Similarly,

$$f^{<b,h>} = \sup_{y \in U(x)} < f(y), h_1 > h_1 + < f(x), h_2 > h_2 + < f(x), h_3 > h_3.$$

The *complement* of f is defined as

$$f^c = 1 - f,$$

where 1 is a function taking a constant value of (1,1,1) in $Y = R^3$.

The *boundary* of a color image f relative to h can be defined as

$$f^{<\partial,h>} = f^{<b,h>} \wedge (f^c)^{<b,h>}.$$

We can introduce an order related to h as

$$f_1 \leq_h f_2 \Leftrightarrow 0 \leq < f_2 - f_1, h_1 > .$$

Then we see

(1) $f^{<i,h>} \leq_h f$ and $f \leq_h f^{<b,h>}$, and
(2) $f^{<i,h>} \leq_h f^{<|m|>}$ and $f^{<|m|>} \leq_h f^{<b,h>}$.

We consider the following set

$$K_m^2(x) = \{h :< h, e_i > \text{ is } \mathcal{B}_x\text{-measurable and } m_x\text{-square-integrable}, i = 1, 2, 3\}.$$

We can introduce a norm and an inner product in the set $K_m^2(x)$ as:

$$||f||_{m,x} = \sqrt{\int_{U(x)} < f(y), f(y) > |m_x|(dy)}$$

and

$$< f_1, f_2 >_{m,x} = \int_{(U(x)} < f_1(y), f_2(y) > |m_x|(dy).$$

Then, the space becomes a Hilbert space and for a projection P_x to a 3-dimensional subspace (we call this a 3-dimensional projection) in this Hilbert space,

$$f^{<P>}(x) = P_x(f|U(x))$$

defines an operation from GImage(K_m^2, ζ, Y) to Z^X. This operation is called a *projective operation* and is a formalization of color conversions, which depends on the neighborhood colors of each pixel.

Let us give the following theorem about the projective operation.

Theorem 2. *For a projective operation P, there exist 3 vector valued functions h_{1x}, h_{2x}, h_{3x} taking values in $Y \times U(x)$ for any $x \in X$ and*

$$f_i^{<P>} = \int_{U(x)} < f(y), h_{ix}(y) > |m|(dy)(i = 1,2,3)$$

where i in the left term means the i-th coordinate in R^3.

The proof of the above theorem is easy by the following form of expansion by Schatten [5]:

$$P_x = h_{1x} \otimes \overline{h_{1x}} + h_{2x} \otimes \overline{h_{2x}} + h_{3x} \otimes \overline{h_{3x}}.$$

References

1. H. Imura and M. Eguchi, *Finite topological spaces*, Formalized Mathematics **2** (1992), 180-194.
2. J.L. Kelly, I. Namioka et al., *Linear topological spaces*, Van Nostrand (1961).
3. A. Khinchin, *Mathematical foundations of information theory*, Dover Publ., (1957).
4. Y. Nakamura, *Finite topology concept for discrete spaces*, Proceeding of the Eleventh Symposium on Applied Functional Analysis (edited by H.Umegaki) (1988), 111-116.
5. R. Schatten, *Norm ideals of completely continuous operators*, Springer-Verlag (1960).
6. Gary S. Stoudt, *Fourier series via Sturm-Liouville equations*, Indiana University of PA (1996).
7. Sundaramoorthy et al., *A system for automatic 3D medical image analysis*, Computerized Medical Imaging and Graphics, **19** (1995), 131-143.

8. H. Umegaki, *Representations and extremal properties of averaging operators and their applications to information channels*, J. Math. Anal. Appl. **25** (1969), 41-73.

Department of Information Engineering, Faculty of Engineering, Shinshu University, 500 Wakasato, Nagano 380, Japan
E-mail address: `ynakamur@cs.shinshu-u.ac.jp`

Department of Information Engineering, Faculty of Engineering, Shinshu University, 500 Wakasato, Nagano 380, Japan
E-mail address: `shidama@cs.shinshu-u.ac.jp`