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<tr>
<td>Author(s)</td>
<td>Hamano, Masahiro; Okada, Mitsuhiro</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1997), 976: 169-181</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60797">http://hdl.handle.net/2433/60797</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
A Hierarchy of the Fragments of the System of Inductive Definition (Preliminary Report)

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1 introduction

Gentzen [7] proved the consistency of PA (Peano Arithmetic) by using the transfinite induction up to the first epsilon number $\epsilon_0$. Here $\epsilon_0$ is $\lim_k \omega_k$, where $\omega_0 = 0$ and $\omega_{k+1} = \omega^\omega$. Later in [8] he proved that the accessibility (i.e., transfinite induction) proof up to any ordinal less than $\epsilon_0$, e.g., $\omega_k$ for any natural number $k$, is provable in $PA$.

In his [8] the nestedness complexity of implications used in the accessibility proof increases by one while the accessibility of one higher $\omega$-tower $\omega_{k+1}$ is proved from the accessibility of $\omega_k$. Hence by considering Gentzen's work [7, 8] a natural question arises: does the hierarchy of $\omega$-towers, $\{\omega_k\}_{k=1,2,...}$, correspond exactly to a certain hierarchy of fragments of $PA$?

Mints [10] answered this question by estimating the least upper bounds of accessibility ordinals for the fragments of $PA$, where the fragments are defined by means of the number of alternations of quantifiers, using one quantifier system developed in his former paper [9]. (Shirai [13] also gave a similar result by means of the number of quantifiers.)

The purpose of our paper is to investigate in a similar correspondence (between the hierarchy of critical ordinals and the hierarchy of fragment systems) for the system of $\xi$-iterated Inductive Definition $ID_{\xi}$ [6]. We first analyze in Section 2 Arai's optimal accessibility proof for $ID_{\xi}$ ([3]) to obtain a hierarchy of accessible ordinals for the fragments of intuitionistic $ID_{\xi}$, where the fragments are defined in terms of the nestedness complexity of implications. Then we show in Section 3 the least upper bounds of accessible ordinals (i.e., the critical ordinals) for those fragments, by analyzing Takeuti-Arai's consistency proofs of $ID_{\xi}$ ([3]). In fact, for the upper bounds proof we use the fragments of classical $ID_{\xi}$ in terms of the nestedness complexity of classical negations. Since the fragments of $ID_{\xi}$ obtained by means of the number of alternations of quantifiers (in a prenex normal form) are also characterized by the nestedness complexity of negations with the help of universal quantifiers (by representing an existential quantifier $\exists$ by means of $\neg \forall \neg$), our result for $ID_{\xi}$ corresponds to Mints' ([10]) for $PA$.
2 Provability of transfinite inductions on \( \omega(\xi, k, 0) \) in subsystems of \( S_\xi(ID_\xi(U_0)) \)

Let \((I, \prec)\) be the well ordered system whose order type is ordinal \( \xi + 1 \). Arai [1] proved the well ordering of Takeuti's system of ordinal diagram \( O(\xi + 1, 1) \) in the system \( ID_\xi \) (the intuitionistic system of \( \xi \)-times iterated inductive definition).

In this chapter we introduce a hierarchy of fragments \( S_\xi(ID_\xi(U_0)) \) of \( ID_\xi \) based on the nestedness complexity of implications, and observe Arai's well ordering proof of [1] on these fragments.

Now we recall the definitions of \( ID(U) \) and \( ID_\xi \) of Feferman [6].

**Definition 1** (System \( ID(U) \) and \( ID_\xi \), cf. Feferman [6])

For any positive operator form \( U \), \( ID(U) \) is obtained from \( PA \) by adding the following axiom schemata.

\[
\begin{align*}
(P_\xi 1) & \quad \forall x < \xi(A(P_{\xi a}, P_{\xi b}, x) \subseteq P_{\xi d}) \\
(P_\xi 2) & \quad \forall x < \xi(\forall y (P_{\xi a}, P_{\xi b}, x) \subseteq P_{\xi d} \subseteq \forall y V)
\end{align*}
\]

\((TI)_\xi \quad \text{Prog}[I, <, V] \rightarrow (I \subseteq V)\)

where \( P_{\xi a} := \{x, y | x < a \land P_{\xi d} y\} \)

\( ID^i := \bigcup \{ID_\xi(U) \mid U \text{ is a positive operator form}\} \)

The starting point of Arai's well ordering proof is to define the notion of accessibility with respect to \( \prec_i \) for \( i < \xi \) (cf. §26 [14]) by using the set constants \( A_i \) which is definable in \( ID_\xi(U_0) \) with the following \( U_0 \);

\[
\begin{align*}
(A.1)_\xi & \quad \forall i < \xi(\forall y \prec_i V \rightarrow A_i \subseteq V) \\
(A.2)_\xi & \quad \forall i < \xi(\forall y \prec_i V \rightarrow A_i \subseteq V) \quad \text{for each abstract } V \text{ in } ID_\xi(U_0)
\end{align*}
\]

where \( U_0 \) is a \( \xi \)-positive operator form defined as \( U_0(X, Y, i, \mu) := F(i, \mu, Y) \forall Y < i \mu(F(i, \mu, Y) \rightarrow X(v)) \) where \( F(i, \mu, Y) := \forall k < i \forall p \subseteq k \mu Y(k, p), Prog(a, \gamma, \beta) := \forall x(a(x) \land \forall y (\gamma(y, x) \land (a(y) \rightarrow \beta(y)) \rightarrow \beta(x)), \text{ and } F_i(\mu) := \forall j < i \forall n < j \mu A_j(\nu) \)

The intended meaning of \( F_i(\mu) \) is that \( \mu \) is an i-fan (cf. Definition 26.16 [14]).

Remember that \( ID^0(U) \) has the mathematical induction of the following form;

\((VJ) \quad V(0), \forall x(V(x) \rightarrow V(y)) \rightarrow V(t)\)

The above \( ID^0(U_0) \) is the specific subsystem of the system \( ID_\xi \) of Inductive Definition in which the induction schemata are used only for the accessibility predicate \( A_i \) of ordinals.

We consider the subsystem \( S_\xi(ID_\xi(U_0)) \) of \( ID_\xi(U_0) \) where each abstract \( V \) in \((A.2)_\xi \), \((TI)_\xi \) and \((VJ) \) is restricted to that of level \( lv(V) \leq k \); where \( lv(V) \) is defined by the definition below.

We introduce the notion of level of \( A(lv(A)) \) for a formula \( A \) to express, roughly speaking, the implicational complexity of \( A \). We assume that the language contains only \( \forall, \lor \) and \( \land \) for the logical connectives in this section.

We first recall the degree \( d \) of a formula in the language of \( ID^0(U) \) defined in Arai [3], which intends to indicate how many times inductive definition is applied.

**Definition 2** (cf. Def 2.4 in Arai [3])

- \( d(t = s) = 0 \) for all term \( t, s \) and predicate variable \( X \).
- \( d(P_{\xi a} t s) = \begin{cases} i + 1 & \text{if } t \text{ is a closed term whose value is } i < \xi, \\ \xi & \text{otherwise} \end{cases} \)
\begin{align*}
d(t_1 \prec s \land P^{d}_t t_2) &= \begin{cases} 
i & \text{if } s \text{ is a closed term whose value is } i \prec \xi \text{ and } t_1 \text{ is a} \\
\xi & \text{closed term representing the same numeral as } t_2. 
\end{cases}
\end{align*}

Definition 3 (level \( lv(A) \) of formula \( A \) in the language of \( ID_{1}^{f}(U) \)) For the formula \( A \) in the language of \( ID_{1}^{f}(U) \), the level \( lv(A) \) of the formula \( A \) is defined inductively as follows:

\begin{align*}
lv(P) &= 0 \text{ for any atom of the language of } PA.
lv(A \land B) &= \max\{lv(A), lv(B)\}
lv(\forall x A) &= \begin{cases} 
\max\{2, lv(A)\} & \text{if } lv(A) \geq 1 \\
0 & \text{if } lv(A) = 0
\end{cases}
lv(A \supset B) &= \begin{cases} 
\max\{lv(A) + 1, lv(B)\} & \text{if } lv(A) \geq 1 \\
0 & \text{if } lv(A) = 0
\end{cases}
lv(P^{d}_t) &= \begin{cases} 
1 & \text{if } d(P^{d}_t) = \xi \\
0 & \text{otherwise}
\end{cases}
lv(P^{d}_t t) &= \begin{cases} 
1 & \text{if } d(P^{d}_t) = \xi \\
0 & \text{otherwise}
\end{cases}
\end{align*}

The subsystems \( S_{k}(ID_{1}^{f}(U)) \) and \( S_{k}(ID_{1}^{f}(U)) \) of \( ID_{1}^{f}(U) \) and \( ID_{1}^{f}(U) \) are defined in terms of level \( lv \) as follows:

Definition 4 (the subsystem \( S_{k}(ID_{1}^{f}(U)) \) of \( ID_{1}^{f}(U) \)) \( S_{k}(ID_{1}^{f}(U)) \) is \( ID_{1}^{f}(U) \) except that for every abstract \( V \) in \( (A.2)_{E}, (T)_{E} \text{ and } (VJ), lv(V) \leq k \) holds.

\[ S_{k}(ID_{1}^{f}(U)) := \bigcup\{S_{k}(ID_{1}^{f}(U)) \mid U \text{ is a positive operator form}\} \]

The following notation is introduced;

Notation 1 Let \( TI[\alpha, \gamma, \mu] \) denote the schema defined as \( TI[\alpha, \gamma, \mu] := \alpha(\mu) \land (Prog[\alpha, \gamma, V] \rightarrow \forall \nu(\gamma(\mu, \nu) \land \alpha(\nu) \rightarrow V(\nu))) \). And \( TI[\alpha, \gamma, \mu]Q \) is the result of \( TI[\alpha, \gamma, \mu] \) by substituting \( Q \) for \( V \).

Notation 2 \( \omega(\xi, 0, \alpha) := \alpha \) and \( \omega(\xi, n + 1, \alpha) := (\xi, \omega(\xi, n, \alpha)) \).

Then by checking Arai's well ordering proof of \( O(\xi + 1, 1) \) [1] carefully, Proposition 1 is easily observed.

Proposition 1 For a formula \( Q \) with \( lv(Q) \leq 2 \) and \( k > 2 \), \( TI[F_0, <_0, \omega(\xi, k, 0)] \) is provable in \( S_{k}(ID_{1}^{f}(U_0)) \). Namely, the ordinal \( \omega(\xi, k, 0) \) is accessible in \( S_{k}(ID_{1}^{f}(U_0)) \) with respect to \( <_0. \)

Proof.

We only consider the case in which \( \xi \) is a limit. (See Remark after Proposition 2 for the successor case.) Let \( \bigcap_{k < \xi} A_k := \{\mu \mid \forall k < i A_k(\mu)\}. \) In Lemma 3 of [1] \( (T)_{E} \) is used with the abstract \( \{i\} Prog[F_i, <_i, \bigcap_{k < i} A_k] := \{i\} \forall \nu(F_i(\nu) \land \nu \prec x (F_i(y) \rightarrow \bigcap_{k < i} A_k(y))) - \bigcap_{k < i} A_k(x)) \), here \( lv(Prog[F_i, <_i, \bigcap_{k < i} A_k(\mu)]) = 3. \) Let \( A := \bigcap_{i < \xi} A_i \) and \( R(\nu) := \forall \mu \prec \xi (i, \nu)(F_\xi(\mu) \rightarrow A(\mu)) \). In Lemma 4 of [1] \( (A.2)_{E} \) is used with the abstract \( \{x\} R(\mu) := \forall \mu \prec \xi (i, x)(F_\xi(\mu) \rightarrow A(\mu)) \) (with \( lv(R(\xi)) = 2 \)) and \( (T)_{E} \) is used with the abstract \( \{i\} R(0) := \forall \mu \prec \xi (i, 0)(F_\xi(\mu) \rightarrow A(\mu)) \) (with \( lv(R(0)) = 2 \)).

Then in Lemma 5 of [1] it is shown that \( TI[F_\xi, <_\xi, (\xi, 0)]Q \) is provable in \( ID_{1}^{f}(U_0) \) for each unary predicate \( Q(x) \) in \( ID_{1}^{f}(U) \); In the case where \( \lim(\xi) \), \( (A.2)_{E} \) are used
with the abstract \( \{ x \}(x < \xi (i, 0) \rightarrow Q(x)) \) for all \( i < \xi \) (with level \( lv(Q) \)). In the case where \( \text{Suc}(\xi) \), \((A.2)_\xi \) is used with the abstract \( \{ x \}(x < \xi (\xi, 0) \rightarrow Q(x)) \) (with level \( lv(Q) \)).

Hence until now it is observed that

\[
(I) \quad S_{Max(3, lv(Q))}(ID_\xi(U_0)) \vdash T(F_\xi, <\xi, (\xi, 0))Q.
\]

From \((I)\) it is derived in the way familiar by Gentzen [8] that

\[
(II) \quad S_{k+3}(ID_\xi(U_0)) \vdash T(F_\xi, <\xi, (\omega(\xi, k + 3, 0))Q \text{ with } lv(Q) \leq 2 \text{ and } k \geq 0.
\]

Let us observe the proof of \((II)\). In Lemma 7 of [1] it is shown that \( \text{Prog}[F_\xi, <\xi, Q] \rightarrow \text{Prog}[F_\xi, <\xi, s[Q]] \), where \( s[Q] \) is a jump operator defined as \( s[Q](\mu) := \forall \rho(\mu) \rightarrow \forall \nu <\xi \rho(\mu) \rightarrow Q(\nu) \) and \( \text{Prov}(\mu) \equiv (F_\xi(\mu) \rightarrow Q(\mu)) \), where \( \lambda \mu. \mu + \nu^\xi \) is a primitive recursive function which is a generalization of \( \lambda \nu. \nu + \omega^\mu \) of Gentzen [8] and defined in [1] as follows:

- If \( \mu = 0 \), then \( \mu + \nu^\xi = \nu + \mu^\xi = \nu \)
- Suppose \( \mu \neq 0 \) and \( \nu \neq 0 \) and
  \( \mu = \mu_1 \# \cdots \# \mu_m \) with \( \mu_1 \geq \xi \cdots \mu_m \neq 0 \)
  \( \nu = \nu_1 \# \cdots \# \nu_n \) with \( \nu_1 \geq \xi \cdots \nu_n \neq 0 \)
  Let \( I \) be the number whose \( \nu_1 \leq I \leq m \) and \( \mu_1 \leq \xi \nu_1 < \xi \mu_i + 1 \),
  \begin{align*}
  \mu + \nu^\xi &= \mu_1 \# \cdots \# \mu_i \# \cdots \# \nu_n \\
  \text{Note that } lv(s^n[Q]) &= n + Max(2, lv(Q)) \text{ with } n \geq 1, \text{ where } s^n[Q] := s[s[s[Q]] \cdots].
  \end{align*}

Let us sketch the proof of \( \text{Prog}[F_\xi, <\xi, Q] \rightarrow \text{Prog}[F_\xi, <\xi, s[Q]] \) due to Gentzen [8], where a mathematical induction of the level \( \leq lv(Q) \) is used;

Assume
\[
\begin{align*}
&\text{Prog}[F_\xi, <\xi, Q] \quad \ldots (1) \\
&F_\xi(z) \land \forall y <\xi z(F_\xi(y) \rightarrow s[Q](y)) \quad \ldots (2)
\end{align*}
\]

We have to show \( s[Q](x) \). So assume further
\[
\begin{align*}
&F_\xi(\rho) \quad \ldots (3) \\
&\forall \nu <\xi \rho(\mu(\xi, \xi, \nu \rho)) \quad \ldots (4) \\
&\nu <\xi \rho(\xi, x, \nu) \quad \ldots (5)
\end{align*}
\]

Under the above assumptions \((1) \sim (5)\), we have to show \( Q(\nu) \).

Consider the case where \( x \neq 0 \). Since \( \nu <\xi \rho(\xi, x, \nu) \), there exists primitive recursive functions \( f \) and \( g \) such that \( \nu <\xi \rho(\xi, f(x, \nu, \nu) \rho) \cdot g(x, \nu, \nu) \rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu \) holds.

From \((3)\), \( s[Q](f(x, \nu, \nu)) \) holds. Then a universal instantiation with \( \rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu \) (note that \( \rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu <\xi \rho(\xi, x, \nu) \)) holds for an arbitrary \( n \) allows the following:
\[
\begin{align*}
&F_\xi(\rho) \cdot \nu(n(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi \\
&\rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu(F_\xi(\eta) \rightarrow Q(\eta)) \ldots (6)
\end{align*}
\]

From \( F_\xi(\rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu) \) (from \((5)\)) and the property of \( \text{Suc} \), the following holds;
\[
\begin{align*}
&\forall \eta <\xi \rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi \\
&\rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu(F_\xi(\eta) \rightarrow Q(\eta)) \ldots (7)
\end{align*}
\]

Then mathematical induction with abstract \((1)\) \( (\forall \eta <\xi \rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu(F_\xi(\eta) \rightarrow Q(\eta))) \), whose level is \( Max(2, lv(Q)) \), implies (with \((4)\)) \( \forall \eta <\xi \rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu(F_\xi(\eta) \rightarrow Q(\eta)) \). Hence from \((5)\), \( Q(\nu) \) holds.

Consider the case where \( x = 0 \). For each formula \( Q \), \( s[Q] \) denotes the formula of the following form; \( s[Q](\mu) := \forall \nu(\rho(F(\mu) \rightarrow \forall \nu <\xi \rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu(F_\xi(\eta) \rightarrow Q(\eta)))) \). Then we can prove without \((A.1)_\xi, (A.2)_\xi, T(F_\xi) \) and the mathematical induction that \( \text{Prog}[F_\xi, <\xi, Q] \rightarrow \text{Prog}[F_\xi, <\xi, s[Q]] \). As is shown

\[
\text{Suppose } \rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu(<\xi \rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi)
\]

\[
\text{Consider } \rho(\xi, f(x, \nu, \nu) \rho) \cdot \nu(F_\xi(\eta) \rightarrow Q(\eta)) \ldots (7)
\]
above, in Lemma 7 of [1] all the mathematical inductions used are restricted to those of level \( \leq \text{Max}(2, lv(Q)) \).

From now we assume \( lv(Q) \leq 2 \). With the help of \( \text{Prog}[F_\xi, <_\xi, Q] \rightarrow \text{Prog}[F_\xi, <_\xi, s(Q)] \) and \( \text{Prog}[F_\xi, <_\xi, s(Q)] \rightarrow \text{Prog}[F_\xi, <_\xi, s^2(Q)] \), in which proof all mathematical inductions are restricted to those of level \( \leq 3 \), (I) implies the following (II)_0:

\[ (II)_0 \quad S_3(ID_\xi(U_0)) \vdash TI[F_\xi, <_\xi, \omega(\xi, 3, 0)]_Q \]

By replying this method, the above (II) is obtained.

Then following Arai [1], the next proposition is derived from (II).

\[ S_{k+3}(ID_\xi(U_0)) \vdash TI[F_0, <_0, \omega(\xi, k + 3, 0)]_Q \text{ with } lv(Q) \leq 2 \text{ and } k \geq 0. \]

Hence the proposition holds.

\[ \Box \]

Using the above, Proposition 2 follows:

**Proposition 2** For \( k > 2 \), the ordinal up to \( \omega(\xi, k+1, 0) \) is accessible in \( S_k(ID_\xi(U_0)) \) with respect to \( <_0 \).

**Remark 1:**
From the case in which \( \xi \) is a successor ordinal, the transfinite induction formula \( \{i\} \text{Prog}[F_i, <_i, \bigcap_{k \leq i} A_i] \) at the beginning of the proof of Proposition 1 above is replaced by \( \{i\} \text{Prog}[F_i, <_i, A_i] \), which has level 2, instead of 3. Hence, the Propositions 1 and 2 hold for \( k > 1 \).

## 3 Unprovability of the transfinite induction up to \( \omega(\xi, k + 1, 0) \) in system \( S_k(AI^-_\xi) \)

Our aim in this chapter is to prove the estimation we have observed in previous chapter is sharp one;

\[ S_k(ID_\xi) \nvdash TI[F_0, <_0, \omega(\xi, k + 1, 0)] \text{ for } k > 2 \]

On the whole segment of \( ID_\xi = \bigcup_n S_n(ID_\xi) \), Arai [3] proves that \( ID_\xi \nvdash TI[F_0, <_0, O(\xi + 1, 1)] \). Note that \( O(\xi + 1, 1) := \bigcup_\omega \omega(\xi, k, 0) \). He shows that the consistency of \( ID_\xi \) is provable using transfinite induction up to \( O(\xi + 1, 1) \) by the proof reduction method which is originally due to Gentzen-Takeuti. In this section we modify his consistency proof in more delicate manner and prove the following by the cut elimination (proof reduction) method;

\[ TI[F_0, <_0, \omega(\xi, k + 1, 0)] \vdash Cons(S_k(ID_\xi)) \text{ for } k > 2 \]

Our crucial point is to introduce a \( \eta \)-height \( h_\eta \) for each \( \eta \leq \xi \) (Definition 11) and consider an ordinal assignment to a proof \( < P, \{h_\eta\}_{\eta \leq \xi}, d > \) with \( \xi \)-sort of height (Definition 13).

For the Gentzen-Takeuti cut elimination procedure to work, Arai [3] formulates his system \( AI^-_\xi \) of \( \xi \)-times iterated inductive definition in the form of iterated comprehension axiom by using second order free variables. System \( AI^-_\xi \) is defined by adding the following principles based on \( PA \).
Definition 5 (System $\text{AI}_\xi^-$, cf. Arai [3])

For any arithmetical form $B$, the following axioms schemata are added.

\[
\begin{align*}
(Q^B : \text{right}) \quad & \quad \Gamma \vdash \Delta, B(X, Q^B_{\xi t}, t, s) \\
\end{align*}
\]

where $Q^B_{\xi t} := \{ (x, y) | (x < t \land Q^B xy) \}$

\[
\begin{align*}
(Q^B : \text{left}) \quad & \quad t < \xi, Q^B ts \rightarrow B(V, Q^B_{\xi t}, t, s)
\end{align*}
\]

We assume that the language contains only $\forall, \land$ and $\land$ for the logical connectives. Then, the definition of $lv$ in the previous section is modified as follows;

Definition 6 ($\eta$-level $lv_\eta(A)$ of a formula $A$ with $\eta < \xi$) For the formula $A$ in the language of $\text{AI}_\xi^-$ and an ordinal $\eta < \xi$, the $\eta$-level $lv_\eta(A)$ of the formula $A$ is defined inductively as follows, where $d$ is defined in Definition 2 of previous section with using $Q^B$ instead of $pU$ and $d(Xt) := 0$ (for $X$ a predicate variable):

\[
\begin{align*}
lv_\eta(P) & := 0 \text{ for any atom of } L_{PA}. \\
lv_\eta(A \land B) & := \max\{lv_\eta(A), lv_\eta(B)\} \\
lv_\eta(\forall x.A) & := \begin{cases} 
\max\{2, lv_\eta(A)\} & \text{if } lv_\eta(A) \geq 1 \\
0 & \text{if } lv_\eta(A) = 0
\end{cases} \\
lv_\eta(\exists x.A) & := \begin{cases} 
lv_\eta(A) + 1 & \text{if } lv_\eta(A) \geq 1 \\
0 & \text{if } lv_\eta(A) = 0
\end{cases} \\
lv_\eta(Q^B) & := \begin{cases} 
1 & \text{if } d(Q^B) = \eta \\
0 & \text{otherwise}
\end{cases} \\
lv_\eta(t < s \land Q^B) & := \begin{cases} 
1 & \text{if } d(t < s \land Q^B) = \eta \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Note that $lv_\eta$ for $\eta = \xi$ is the same as $lv$ of the previous section (with using $Q^B$ instead of $P^u$ in the definition of $lv$ of the previous section with replacing $\lor$ by $\land$.)

We can define the fragments $S_k(\text{AI}_\xi^-)$ in the same manner as $S_k(\text{ID}_\xi)$ as follows.

Definition 7 (the subsystem $S_k(\text{AI}_\xi^-)$ of $\text{AI}_\xi^-$) $S_k(\text{AI}_\xi^-)$ is $\text{AI}_\xi^-$ except that for every abstract $V$ in $Q^B:\text{left}$ and $(VJ), lv_\xi(V) \leq k$ holds.

$\text{ID}_\xi$ is obtained from $\text{ID}_\xi'$ in the previous section by changing the underlying logic from the intuitionistic to the classical. For each formula $F$ of the language of $\text{ID}_\xi$, we define a formula $F^*$ of the language of $\text{AI}_\xi$ by substituting $Q^B$ for all occurrences of $P^u$, where

\[
B(X, Y, c_0, c_1) := \forall y\left( U(X, Y, c_0, y) \rightarrow X \land Y \right) \rightarrow X \vee Y.
\]

It is well known that by this $\ast$, ID$_\xi$ is embeddable into AI$_\xi$ (cf. [3]). Obviously $lv(F) = lv_\xi(F^*)$ holds i.e., $\xi$-level of a formula remains the same through the above interpretation.

Until the end of this section, we assume that all formulas occurring in a proof figure of $\text{AI}_\xi^-$ are of the following normal form:

Lemma 1 (the normal form of a formula in $\text{AI}_\xi^-$) For arbitrary formula $A$ of the language of $\text{AI}_\xi^-$, there exists a formula of the following form, called a normal formula, which is equivalent to $A$ (in $\text{LK}$):

\[
\forall x_1 \ldots \forall x_n \forall y \enspace \forall y[D(Q^Bt_1s_1, \ldots, Q^Bt_ms_m)]
\]

where $D[*_1, \ldots, *_m]$ is a context of the language of $PA$, and no quantifier occurring in $D$ bounds any $*_i$ ($1 \leq i \leq m$) and $lv_\eta(D[Q^Bt_1s_1, \ldots, Q^Bt_ms_m]) \leq 2$ for any $\eta < \xi$. 
Definition 8 (normal proofs) Let $S$ be a sequent of normal formulas. A normal proof of $S$ is a proof in which $\forall$-left rules are used, instead of $\forall$-left rules in a proof:

$$
\Gamma \vdash \Delta, A(t_1, \ldots, t_n) \\
\forall x_1 \cdots x_n \vdash A(x_1, \ldots, x_n), \Gamma \vdash \Delta \\
\forall$-left
$$

Note that the original $\forall$-left rule may also appear in a normal proof.

Lemma 2 Any provable sequent of normal formulas has a normal proof.

From now on we assume any $\Sigma_k(\mathcal{L}_e^-)$-proof to be normal by virtue of the above two lemmata.

Definition 9 For each formula $\varphi(A)$, $\eta(A) \leq \xi$ is defined as $\eta(A) := \text{Max}\{\eta \mid lv_\eta(A) \neq 0\}$.

Definition 10 ($g_\eta(A)$ with $\eta < \xi$)

$$
g_\eta(A) := \begin{cases} 
g(A) & \text{if } \eta(A) \geq \eta \\
0 & \text{if } \eta(A) < \eta 
\end{cases}
$$

where $g(A)$ denotes the number of logical symbols in $A$.

We modify the notion of proof with degree $< P, d >$ of Arai [3] into $< P, \{h_\eta\}_{\eta \leq \xi}, d >$ by introducing $\xi$-sort of height $\{h_\eta\}_{\eta \leq \xi}$, as follows:

Definition 11 (A proof with $\xi$-sort of height $< P, \{h_\eta\}_{\eta \leq \xi}, d >$) A proof $< P, d >$ (with degree $d$) is called a proof with $\xi$-sort of height $< P, \{h_\eta\}_{\eta \leq \xi}, d >$ if for each sequent $S$ of $P$ and each ordinal $\eta \leq \xi$, a natural number $h_\eta(S)$ satisfying the following condition is assigned. We call $h_\eta$ a $\eta$-height.

0. $h_\eta(S) = 0$ for every $\eta \leq \xi$ if $S$ is the end sequent of $P$.

For the last inference $I$ of the form

$$
I \quad \frac{S}{S'}
$$

1. $h_\eta(S) = 0$ for every $\eta \leq \xi$ if $I$ is a substitution.

2. $h_\eta(S) = h_\eta(S')$ for every $\eta \leq \xi$ if $I$ is an inference except substitution, induction and cut.

3. \begin{align*}
&\begin{cases} 
h_\eta(S) \geq \text{Max}\{h_\eta(S'), g_\eta(D)\} & \text{for } \eta < \xi \\
h_\xi(S) = \text{Max}\{h_\xi(S'), lv_\xi(D)\} & \text{if } I \text{ is a cut, where } D \text{ is the cut formula of the inference } I.
\end{cases}
\end{align*}

4. \begin{align*}
&\begin{cases} 
h_\eta(S) \geq \text{Max}\{h_\eta(S'), g_\eta(D)\} + 1 & \text{for } \eta < \xi \\
h_\xi(S) = \text{Max}\{h_\xi(S'), lv_\xi(D)\} + 1 & \text{if } I \text{ is an induction.}
\end{cases}
\end{align*}

Definition 12 For each sequent $S$ of $< P, \{h_\eta\}_{\eta \leq \xi}, d >$, $\eta(S) \leq \xi$ is defined as

$$
\eta(S) := \begin{cases} 
d(I) & \text{if } S \text{ is the upper sequent of the substitution } I \\
\text{Max}\{\eta \mid h_\eta(S) \neq 0\} & \text{otherwise}
\end{cases}
$$

The following is an immediate consequence from Definition 12.
Lemma 3 For any proof with \( \xi \)-sort of height \(< P, \{ h_n \}_{n \leq \xi}, d > \) and for any inference \( I \) (with a lower sequent \( S' \) and an upper sequent \( S \)) in \(< P, \{ h_n \}_{n \leq \xi}, d >\),

\[ \eta(S) \geq \eta(S') \]

holds.

Notation 3 For \( i \leq \xi \) and an ordinal diagram \( \alpha \), an ordinal diagram \( \omega(i, n, \alpha) \) is defined inductively as follows.

1. \( \omega(i, 0, \alpha) := \alpha \)
2. \( \omega(i, n + 1, \alpha) := (i, \omega(i, n, \alpha)) \)

Definition 13 (ordinal assignment) Let \( I \) be an inference of the form

\[
\begin{array}{c}
\text{I} \\
\hline
S_1 \quad S_2 \\
\hline
S
\end{array}
\]

Then \( O(S) \) is defined as follows:

1. When \( I \) is a cut,

\[ O(S) := \omega(\eta(S), k - h_{\eta(S)}(S), \omega(\eta(S_1), h_{\eta(S_1)}(S_1), O(S_1) \# O(S_2))) \]

Here \( k := \operatorname{Max}\{h_{\eta(T)}(T) \mid T \text{ is above } I\} \) and \( c[*] := \omega(\gamma_1, k_1, \omega(\gamma_2, k_2, \ldots, \omega(\gamma_n, k_n, *))\), where \( \{ \gamma_1, \ldots, \gamma_n \} := \{ \gamma \mid \eta(S) < \gamma < \eta(S_1) \text{ and } h_\eta(T) \neq 0 \text{ for some } T \text{ above } I \} \) with \( \gamma_1 < \cdots < \gamma_n \) and \( k_i := \operatorname{Max}\{h_\eta(T) \mid T \text{ is above } I\}. \)

2. When \( I \) is a logical inference,

\[ O(S) := O(S_1) \# O(S_2) \# 0 \]

3. When \( I \) is a structural inference,

\[ O(S) := O(S_1) \# O(S_2) \]

4. When \( I \) is a substitution,

\[ O(S) := (d(I), O(S_1)) \]

Theorem 1 The transfinite induction on \( \omega(\xi, k + 1, 0) \) is unprovable in \( S_k(AI_\xi^-) \) for \( k > 2 \).

Proof.

We refine the proof reduction process of Arai [3] to define the reduction process for \( S_k(AI_\xi^-) \) (\( k > 2 \)), and show that the well-orderness of \( \omega(\xi, k + 1, 0) \) implies the termination of the reduction process, hence the consistency of \( S_k(AI_\xi^-) \). Then the above theorem follows from Gödel's incompleteness theorem.

(preparation)

Without loss of generality, we assume that all logical initial sequents of the form \( p \rightarrow p \) where \( p \) is an atomic and that there exists no free variables which is not used as an eigenvariable.

(elimination of initial sequents in the end-piece) As usual.

(elimination of weakening) elimination of weakening known in the usual way (cf. Takeuti [14]) does not only for a weakening in end-piece but also for a more general weakening with such a weakening formula \( D \) as the bundle \( I \) (cf. p78 of [14]) which begins with \( D \) ends with a cut formula \( D \) and no logical inference affect \( I \).
(elimination of the mathematical induction rule) As usual.
Then from sublemma 12.9 of [14], there exists a suitable cut \( J \) in the end piece of
\(< P, \{ h_\eta \}_{\eta \leq \xi}, d > \). Let \( I_1 \) and \( I_2 \) be boundary logical inferences whose principal formulas are ancestors of left and right cut formulas of \( J \).
We shall demonstrate following three essential cases both for limit ordinal \( \xi \) and for successor ordinal \( \xi' \);
(Case 1) The case where the cut formula \( C := A \land B \) with \( \eta(C) < \xi' \);
Let \( K \) (whose lower sequent is \( T \) and whose upper sequent is \( T_1 \)) denotes the uppermost inference below \( J \) such that either (i) or (ii) holds;
\[
\eta(T) = \eta(A) \land (h_{\eta(A)}(S_1) > h_{\eta(A)}(T)) \quad \text{(i)}
\]
\[
\eta(T) < \eta(A) \quad \text{(ii)}
\]
where \( A \) is the auxiliary formula of \( I_1 \) and \( I_2 \)
\(< P, \{ h_\eta \}_{\eta \leq \xi}, d > \) is as follows:
\[
\begin{array}{c}
S_{I_1}^1 \quad S_{I_1}^2 \quad I_1 \quad S_{I_2}^2 \quad I_2 \\
S_{I_1}^1 \quad S_{I_2}^1 \\
S_{J}^1 \quad S_{J}^2 \quad J
\end{array}
\]
\[
\begin{array}{c}
S_{I_1}^1 : \quad \Gamma_1 \land \Delta_1, A_1 \\
S_{I_2}^1 : \quad \Gamma_2 \land \Delta_2, B_1 \\
S_{I_1}^2 : \quad \Gamma_1, \Delta_1, A_1 \land B_1 \\
S_{I_2}^2 : \quad A_3, \Pi_3 \land \Lambda_3 \\
S_{J}^1 : \quad \Gamma \land \Delta, A \land B \\
S_{J}^2 : \quad A \land B, \Pi \land \Lambda \\
T_1 : \quad \Phi \land \Psi
\end{array}
\]
\(< P', \{ h_\eta \}_{\eta \leq \xi}, d' > \) is as follows, where \( I_1' \) and \( I_2' \) are weakening-right (with a weakening formula \( A_1 \)) and weakening-left (with a weakening formula \( A_3 \)) respectively;
\[
\begin{array}{c}
S_{I_1}^1 \quad I_1' \quad I_2 \quad S_{I_2}^2 \quad I_2' \\
S_{I_1}^1 \quad S_{I_2}^1 \\
S_{J}^1 \quad S_{J}^2 \quad J
\end{array}
\]
\[
\begin{array}{c}
S_{I_1}^1 : \quad \Gamma_1 \land \Delta_1, A_1 \land B_1 \\
S_{I_2}^1 : \quad \Gamma \land \Delta, A \land B \\
S_{I_1}^2 : \quad A_3, A_3 \land B_3, \Pi_3 \land \Lambda_3 \\
S_{I_2}^2 : \quad A \land B, A, \Pi \land \Lambda \\
T_1' : \quad \Phi \land \Psi, A \quad T_2' : \quad A, \Phi \land \Psi \\
T_1'' : \quad \Phi, \Psi \land \Phi
\end{array}
\]
(case 1.1): The case where (i) holds. Then for any sequent \( T' \) between \( S_1 \) and \( T \),
\( \eta(T') \geq \eta(A) \) holds.
(case 1.1.1) \( \eta(T_1) = \eta(T) \)
\( O_{P'}(T^*) <_0 O_P(T) \) is checked as usual way.
(case 1.1.2) \( \eta(T_1) > \eta(T) \)
special case of (case 1.2)
(Case 1.2): The case where (ii) holds. Then \( \eta(T) < \eta(A) \leq \eta(T_1) \) holds. We assign
\[
h'_\eta(U_1) := \begin{cases} 
    g(A) & \text{if } \eta = \eta(A), \quad \text{and } h'_\eta(T_1') := h'_\eta(T_1'^*) := h_\eta(T_1) \text{ for all } \eta \leq \xi.
    \\
    0 & \text{otherwise}
\end{cases}
\]
Hence \( \eta(U_1) = \eta(A) \) holds. On the other hand, there exist contexts \( a \) and \( b \) such
that \( O_p(T) = \omega(\eta(T), k - h_{\eta(T)}(T), \alpha_1 \# \alpha_2) \),
\( O_p(U_1) = \omega(\eta(U_1), m - h_{\eta(U_1)}(U_1), \alpha_1 \# \alpha_2) = \omega(\eta(A), m - g(A), \alpha_1 \# \alpha_2) \) and
\( O_p(T^*) = \omega(\eta(T), k' - h_{\eta(T)}(T), \alpha_1 \# \alpha_2) \).
Since \( \omega(\eta(A), k, \alpha_1 \# \alpha_2) > \omega(\eta(U_1), g(A), O_p(U_1) \# O_p(U_2)) \),
\( O_p(T^*) <_0 O_p(T) \) holds.

(Case 2) The case where cut formula is \( \forall \bar{x} \neg B(\bar{x}) \):

\[< P', \{ h_{\eta} \}_{\eta \leq \xi}, d > \] is as follows: here \( I_2 \) is \( \forall \neg \)-left.

\[\begin{array}{c|c}
S_{11}^2 & I_1 \\
\hline
S_{11}^2 & I_2 \\
\hline
S_1^j & J \\
\hline
T & K \\
\end{array}\]

\[\begin{array}{c|c}
S_{11}^2 & I_1 \\
\hline
S_{11}^2 & I_2 \\
\hline
S_i^1 & \Gamma_1 \rightarrow \Delta_1, \neg B(\bar{x}) \\
\hline
S_i^2 & \Gamma_1 \rightarrow \Delta_1, \forall \bar{x} \neg B(\bar{x}) \\
\hline
S_i^3 & \Pi_1 \rightarrow \Lambda_1, \neg B(\bar{t}) \\
\hline
S_i^4 & \forall \bar{x} \neg B(\bar{x}), \Pi_1 \rightarrow \Lambda_1 \\
\hline
T & \Phi \rightarrow \Psi \\
\end{array}\]

\[< P', \{ h_{\eta} \}_{\eta \leq \xi}, d' > \] is as follows, where \( I_1 \) and \( I_2 \) are weakening-right and weakening-left (respectively) with weakening formulas \( \forall \bar{x} \neg B(\bar{x}) \). Note that by virtue of (preparation) and (elimination of weakening), any formula of the form \( \neg B(\bar{x}) \) which is an ancestor of the auxiliary formula of \( I_1 \) is a descendant of principal formulas of an inference \( \neg \)-right. Hence the following \( S_{11}^2(\bar{x}) \) can be obtained.

\[\begin{array}{c|c}
S_{11}^2(t) & I_1 \\
\hline
S_{11}^2(t) & I_2 \\
\hline
S_i^1 & \Gamma_1 \rightarrow \Delta_1, \neg B(\bar{x}) \\
\hline
S_i^2 & \Gamma_1 \rightarrow \Delta_1, \forall \bar{x} \neg B(\bar{x}) \\
\hline
S_i^3 & \Pi_1 \rightarrow \Lambda_1, \neg B(\bar{t}) \\
\hline
S_i^4 & \forall \bar{x} \neg B(\bar{x}), \Pi_1 \rightarrow \Lambda_1 \\
\hline
T & \Phi \rightarrow \Psi \\
\end{array}\]

Since \( lv_{\eta(B(\bar{z}))}(\forall \bar{x} \neg B(\bar{x})) = lv_{\eta(B(\bar{t}))}(B(\bar{z})) \) holds, \( O(P') <_0 O(P) \) is checked as the usual way.

(Case 3) The case where the cut formula of \( J \) is \( QB ts \):

\[\begin{array}{c|c}
S_{11}^2(\bar{x}) & B(\bar{x}), \Gamma_1 \rightarrow \Delta_1 \\
\hline
S_i^1 & B(\bar{x}), \Gamma_1 \rightarrow \Delta_1, \forall \bar{x} \neg B(\bar{x}) \\
\hline
S_i^2 & B(\bar{x}), \Gamma \rightarrow \Delta, \forall \bar{x} \neg B(\bar{x}) \\
\hline
S_i^3 & B(\bar{x}), \Gamma, \Pi \rightarrow \Delta, \Lambda \\
\hline
S_i^4 & \forall \bar{x} \neg B(\bar{x}), \Pi \rightarrow \Lambda, \Lambda \\
\hline
U_1(\bar{x}) & B(\bar{x}), \Phi \rightarrow \Psi \\
\hline
U_2 & \Phi \rightarrow \Psi, B(\bar{t}) \\
\hline
T & \Phi, \Phi \rightarrow \Psi, \Psi \\
\end{array}\]

Since \( \text{lv}_{\eta(B(\bar{z}))}(\forall \bar{x} \neg B(\bar{x})) > \text{lv}_{\eta(B(\bar{t}))}(B(\bar{z})) \) holds, \( O(P') <_0 O(P) \) is checked as the usual way.

(Case 3) The case where the cut formula of \( J \) is \( QB ts \):

\[< P', \{ h_{\eta} \}_{\eta \leq \xi}, d > \] is as follows, where \( K \) (with the lower sequent \( T \)) denotes the upper most inference below \( J \) such that \( \eta(T) \leq d(B(X, Q < t, t, s)) := i; \)
Let $T_1$ denote such upper sequent of $K$ that is below $J$.

\[
\frac{S^J_1}{S^J} \quad J \\
\frac{S^J_1}{S^J} \quad T_1 \quad (T_2) \\
\frac{S^J_1}{S^J} \quad T \\
\frac{S}{S} \quad S
\]

\[
S : \quad t_2 \prec \xi, Q_{t_2} s_2 \rightarrow B(V, Q_{s_2}, t_2, s_2) \\
S^J_1 : \quad \Gamma_1 \rightarrow \Delta_1, B(X, Q_{t_1}, t_1, s_1) \\
S^J_1 : \quad \Gamma_1 \rightarrow \Delta_1, Q_{t_1} s_1 \\
S^J_1 : \quad \Gamma_2 \rightarrow \Delta_2, Qts \\
S^J_1 : \quad Qts, \Pi \rightarrow \Lambda \\
S^J_1 : \quad \Gamma_2, \Pi \rightarrow \Delta_2, \Lambda_2 \\
T_1 : \quad \Phi_1 \rightarrow \Psi_1 \\
T_2 : \quad \Phi \rightarrow \Psi
\]

\[O_P(T) = \omega(\eta(T)), k - h_{\eta(T)}(T), e[\omega(\eta(T_1)), h_{\eta(T_1)}(T_1), O_P(T_1) \# O_P(T_2))]\]

$< P', (h'_{\eta})_{\eta \leq \xi}, d' >$ is as follows, where $I_1$ is weakening-right with a weakening formula $Q_{t_1} s_1$:

\[
\frac{S^J_1}{S^J} \quad I_1 \quad S \\
\frac{S^J_1}{S^J} \quad T_1 \quad (T_2) \\
\frac{S^J_1}{S^J} \quad T \\
\frac{S}{S} \quad S
\]

We assign $(h'_{\eta})_{\eta \leq \xi}$ as follows:

- $h'_{\eta}(T^*) := h_{\eta}(S)$ for all $\eta \leq \xi$
- $h'_{\eta}(T^*) := h_{\eta}(T_1)$ for all $\eta \leq \xi$.
- $h'_{\eta}(T^*) := \begin{cases} h_{\eta}(T) & \text{if } \eta \leq \eta(T) \\ 0 & \text{otherwise} \end{cases}$

\[O_P(T^*) = (i, O_P(T^*))\]
Note that $\eta(T^*) = i$. Obviously $k = l$ from the figure of $P'$. And from the above assignment $h', c[*] = \omega(\gamma_1, k_1, \ldots, \omega(\gamma_s, k_s, d[*]))$ with $\gamma_s < i = \gamma_s + 1$. Hence $O(P') <_0 O(P)$. Hence $O(P') <_0 O(P)$ holds.

The following Corollary is immediate from the above theorem and the fact that $S_k(ID^*_{\xi})$ is a subsystem of $S_k(\Lambda_\xi^*)$ under the interpretation $\ast$ (cf. the paragraph after Definition 7).

**Corollary 1** The transfinite induction on $\omega(\xi, k + 1, 0)$ is unprovable in $S_k(ID^*_{\xi})$ for $k > 2$.

**Proof.** As remarked after Definition 7, $\xi$-level does not change under the interpretation of an $\Lambda_k$-formula to an $ID^*_{\xi}$-formula. Hence the Corollary is obvious.

**Theorem 2** (Main Theorem)

$$|S_k(ID^*_{\xi}([0]))| = |S_k(ID^*_{\xi})| = |S_k(\Lambda^*_\xi)| = |\omega(\xi, k + 1, 0)|_{<_0}$$

with $k > 2$.

**Remark 2:** Our system $S_k(ID^*_{\xi})$ can be reformulated by means of the alternation complexity of quantifiers when we include $\exists$ in our language. Here, a normal formula is of the form $Q_1 x_1 Q_1 y_1 \ldots Q_n x_n Q_n y_n \forall y D[P^t_{s_1} \ldots P^t_{s_m}]$, where $D[*_1, \ldots, *_m]$ is a context of the language of $PA$ with no quantifier occurring in $D$ bounds any $*_i$ $(1 \leq i \leq m)$, and $\{Q_j, Q_j\} = \{\forall, \exists\}$ $(j = 1, \ldots, m)$. $lu$ is essentially the same as $lu_\xi$ except that we measure the alternation complexity of quantifiers instead of nestedness complexity of negations; namely,

$$lu(D[P^t_{s_1} \ldots P^t_{s_m}]) := \begin{cases} 1 & \text{if all } P^t_{s_i} (i = 1, \ldots, m) \text{ is positive in } D \\ 2 & \text{otherwise} \end{cases}$$

Then the $lu$ of above normal formula is $n + i$ if $Q_n = \forall$ and $n + 1 + i$ if $Q_n = \exists$, where $i := lu(D[P^t_{s_1} \ldots P^t_{s_m}])$. $S^*_\xi(ID^*_{\xi})$ is defined in the same way as the former definition of $S^*_k(ID^*_{\xi})$ with using the above new notation of $lu$. It is easily seen that $S^*_\xi(ID^*_{\xi})$ is equivalent to $S_k(ID^*_{\xi})$. In particular $|S^*_\xi(ID^*_{\xi})| = |\omega(\xi, k + 1, 0)|_{<_0}$ with $k > 2$.

**References**


