A Hierarchy of the Fragments of the System of Inductive Definition (Preliminary Report)

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1 introduction

Gentzen [7] proved the consistency of PA (Peano Arithmetic) by using the transfinite induction up to the first epsilon number \( \epsilon_0 \). Here \( \epsilon_0 = \lim_k \omega_k \), where \( \omega_0 = 0 \) and \( \omega_k+1 = \omega^\omega \). Later in [8] he proved that the accessibility (i.e., transfinite induction) proof up to any ordinal less than \( \epsilon_0 \), e.g., \( \omega_k \) for any natural number \( k \), is provable in PA.

In his [8] the nestedness complexity of implications used in the accessibility proof increases by one while the accessibility of one higher \( \omega \)-tower \( \omega_{k+1} \) is proved from the accessibility of \( \omega_k \). Hence by considering Gentzen’s work [7, 8] a natural question arises; does the hierarchy of \( \omega \)-towers, \( \{\omega_k\}_{k=1,2,...} \), correspond exactly to a certain hierarchy of fragments of PA?

Mints [10] answered this question by estimating the least upper bounds of accessibility ordinals for the fragments of PA, where the fragments are defined by means of the number of alternations of quantifiers, using one quantifier system developed in his former paper [9]. (Shirai [13] also gave a similar result by means of the number of quantifiers.)

The purpose of our paper is to investigate in a similar correspondence (between the hierarchy of critical ordinals and the hierarchy of fragment systems) for the system of \( \xi \)-iterated Inductive Definition \( ID_\xi \) [6]. We first analyze in Section 2 Arai’s optimal accessibility proof for \( ID_\xi \) ([3]) to obtain a hierarchy of accessible ordinals for the fragments of intuitionistic \( ID_\xi \), where the fragments are defined in terms of the nestedness complexity of implications. Then we show in Section 3 the least upper bounds of accessible ordinals (i.e., the critical ordinals) for those fragments, by analyzing Takeuti-Arai’s consistency proofs of \( ID_\xi \) ([3]). In fact, for the upper bounds proof we use the fragments of classical \( ID_\xi \) in terms of the nestedness complexity of classical negations. Since the fragments of \( ID_\xi \) obtained by means of the number of alternations of quantifiers (in a prenex normal form) are also characterized by the nestedness complexity of negations with the help of universal quantifiers (by representing an existential quantifier \( \exists \) by means of \( \neg \forall \neg \)), our result for \( ID_\xi \) corresponds to Mints’ ([10]) for PA.

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2 Provability of transfinite inductions on \( \omega(\xi, k, 0) \) in subsystems of \( S_k(ID^k_\xi(U_0)) \)

Let \( (I, \prec) \) be the well ordered system whose order type is ordinal \( \xi + 1 \). Arai [1] proved the well ordering of Takeuti's system of ordinal diagram \( O(\xi + 1, 1) \) in the system \( ID^k_\xi \) (the intuitionistic system of \( \xi \)-times iterated inductive definition).

In this chapter we introduce a hierarchy of fragments \( S_k(ID^k_\xi(U_0)) \) of \( ID^k_\xi \) based on the nestedness complexity of implications, and observe Arai's well ordering proof of [1] on these fragments.

Now we recall the definitions of \( ID^k_\xi(U) \) and \( ID^k_\xi \) of Feferman [6].

Definition 1 (System \( ID^k_\xi(U) \) and \( ID^k_\xi \), cf. Feferman [6])

For any positive operator form \( U \), \( ID^k_\xi(U) \) is obtained from \( PA \) by adding the following axiom schemata.

\[
(P_\xi 1) \quad \forall x < \xi (A(P_\xi, P_{\xi+1}^U(x), x) \subseteq P_{\xi+1}^U)
\]

\[
(P_\xi 2) \quad \forall x < \xi (\forall (V, P_{\xi+1}^U(x), x) \subseteq V \supset P_{\xi+1}^U \subseteq V)
\]

\[
(TI)_\xi \quad \text{Prog}[I, \prec, V] \rightarrow (I \subseteq V)
\]

where \( P_{\xi+1}^U := \{ x, y \} | x < a \wedge P_{\xi+1}^U \}

\( ID^k := \bigcup \{ ID^k_\xi(U) \mid U \text{ is a positive operator form} \} \)

The starting point of Arai's well ordering proof is to define the notion of accessibility with respect to \( \prec_i \) for \( i < \xi \) (cf. §26 [14]) by using the set constants \( A_i \) which is definable in \( ID^k_\xi(U) \) with the following \( U_0 \);

\[
(A.1)_\xi \quad \forall i < \xi (\forall \text{Prog}_i) < i, V \rightarrow A_i \subseteq V
\]

\[
(A.2)_\xi \quad \forall i < \xi (\forall \text{Prog}_i) < i, V \rightarrow A_i \subseteq V
\]

where \( U_0 \) is a \( X \)-positive operator form defined as \( U_0(X, Y, i, \mu) := \mathcal{F}(i, \mu, Y) \wedge \forall \nu < i \mu(\mathcal{F}(i, \nu, Y) \rightarrow X(\nu)) \) where \( \mathcal{F}(i, \mu, Y) := \forall k < i \nu \in C_i \mu Y(k, p) \), \( \text{Prog}[a, \gamma, \beta] := \forall x(\alpha(x) \wedge \forall y(\gamma(y, x) \wedge \alpha(y) \rightarrow \beta(y)) \rightarrow \beta(x)) \), and \( F_i(\mu) := \forall j < i \nu \in C_j \mu A_j(\nu) \) (the intended meaning of \( F_i(\mu) \) is that \( \mu \) is an \( i \)-fan (cf. Definition 26.16 [14])).

Remember that \( ID^k_\xi(U) \) has the mathematical induction of the following form;

\[
(VJ) \quad \mathcal{V}(V(0), \mathcal{V}(V(z) \rightarrow V(z'))) \rightarrow V(t)
\]

The above \( ID^k_\xi(U_0) \) is the specific subsystem of the system \( ID^k_\xi \) of Inductive Definition in which the induction schemata are used only for the accessibility predicate \( A_i \) of ordinals.

We consider the subsystem \( S_k(ID^k_\xi(U_0)) \) of \( ID^k_\xi(U_0) \) where each abstract \( V \) in \( (A.2)_\xi, (TI)_\xi \) and \( (VJ) \) is restricted to that of level \( lv(V) \leq k \); where \( lv(V) \) is defined by the definition below.

We introduce the notion of level of \( A (lv(A)) \) for a formula \( A \) to express, roughly speaking, the implicational complexity of \( A \). We assume that the language contains only \( \forall, \supset \) and \( \wedge \) for the logical connectives in this section.

We first recall the degree \( d \) of a formula in the language of \( ID^k_\xi(U) \) defined in Arai [3], which intends to indicate how many times inductive definition is applied.

Definition 2 (cf. Def 2.4 in Arai [3])

- \( d(t = s) = 0 \) for all term \( t, s \) and predicate variable \( X \).

- \[
d(P_{\xi+1}^U(t)) = \begin{cases} 
  i + 1 & \text{if } t \text{ is a closed term whose value is } i < \xi, \\
  \xi & \text{otherwise}
\end{cases}
\]
Definition 3 (level \(lv(A)\) of formula \(A\) in the language of \(ID_\xi(U)\)) For the formula \(A\) in the language of \(ID_\xi(U)\), the level \(lv(A)\) of the formula \(A\) is defined inductively as follows:

\[
\begin{align*}
lv(P) & := 0 \text{ for any atom of the language of } PA. \\
lv(A \land B) & := \max\{lv(A), lv(B)\} \\
lv(\forall x A) & := \begin{cases} 
\max\{2, lv(A)\} & \text{ if } lv(A) \geq 1 \\
0 & \text{ if } lv(A) = 0
\end{cases} \\
lv(A \supset B) & := \begin{cases} 
\max\{lv(A) + 1, lv(B)\} & \text{ if } lv(A) \geq 1 \\
0 & \text{ if } lv(A) = 0
\end{cases} \\
lv(P^\xi) & := \begin{cases} 
1 & \text{ if } d(P^\xi) = \xi \\
0 & \text{ otherwise}
\end{cases} \\
lv(t \prec s \land P^\xi) & := \begin{cases} 
1 & \text{ if } d(P^\xi) = \xi \\
0 & \text{ otherwise}
\end{cases}
\end{align*}
\]

The subsystems \(S_k(ID_\xi(U))\) and \(S_k(ID_\xi^f)\) of \(ID_\xi(U)\) and \(ID_\xi^f\) are defined in terms of level \(lv\) as follows;

Definition 4 (the subsystem \(S_k(ID_\xi(U))\) of \(ID_\xi(U)\)) \(S_k(ID_\xi(U))\) is \(ID_\xi(U)\) except that for every abstract \(V\) in \((A.2)\xi\), \((TI)\xi\) and \((VJ)\), \(lv(V) \leq k\) holds.

\(S_k(ID_\xi^f) := \bigcup \{S_k(ID_\xi(U)) \mid U \text{ is a positive operator form}\}\)

The following notation is introduced;

Notation 1 Let \(TI[\alpha, \gamma, \mu]\) denote the schema defined as \(TI[\alpha, \gamma, \mu] := \alpha(\mu) \land (\text{Prog}[\alpha, \gamma, V] \rightarrow \forall \nu(\gamma(\mu, \nu) \land \alpha(\nu) \rightarrow V(\nu)))\). And \(TI[\alpha, \gamma, \mu]Q\) is the result of \(TI[\alpha, \gamma, \mu]\) by substituting \(Q\) for \(V\).

Then by checking Arai’s well ordering proof of \(O(\xi + 1, 1)\) [1] carefully, Proposition 1 is easily observed.

Proposition 1 For a formula \(Q\) with \(lv(Q) \leq 2\) and \(k > 2\), \(TI[F_0, <_0, \omega(\xi, k, 0)]\) is provable in \(S_k(ID_\xi(U_0))\). Namely, the ordinal \(\omega(\xi, k, 0)\) is accessible in \(S_k(ID_\xi(U_0))\) with respect to \(<_0\).

Proof. We follow Arai’s [1].

We only consider the case in which \(\xi\) is a limit. (See Remark after Proposition 2 for the successor \(\xi\) case.) Let \(\bigcap_{k < \xi} A_k := \{i\} \forall k < i A_k(\mu)\). In Lemma 3 of [1] \((TI)\xi\) is used with the abstract \(\{i\} \text{Prog}[F_i, <_i, \bigcap_{k < i} A_k(x)] := \{i\} \forall x(F_i(x) \land \forall y <_i x(F_i(y) \rightarrow \bigcap_{k < i} A_k(y)) \rightarrow \bigcap_{k < i} A_k(x(x)))\), here \(lv(\text{Prog}[F_i, <_i, \bigcap_{k < i} A_k(\mu)]) = 3\). Let \(A := \bigcap_{j < \xi} A_j\) and \(R_0(\nu) := \forall \mu <_\xi (i, \nu)(F_0(\mu) \rightarrow A(\mu))\). In Lemma 4 of [1] \((A.2)\xi\) is used with the abstract \(\{x\} R_0(x) := \forall \mu <_\xi (i, x)(F_0(\mu) \rightarrow A(\mu))\) (with \(lv(R_0(x)) = 2\)) and \((TI)\xi\) is used with the abstract \(\{i\} R_0(0) := \forall \mu <_\xi (i, 0)(F_0(\mu) \rightarrow A(\mu))\) (with \(lv(R_0(0)) = 2\)).

Then in Lemma 5 of [1] it is shown that \(TI[F_\xi, <_\xi, (\xi, 0)]Q\) is provable in \(ID_\xi(U_0)\) for each unary predicate \(Q(x)\) in \(ID_\xi(U)\); In the case where \(\lim(\xi)\), \((A.2)\xi\) are used
with the abstract \(\{x\}(x \prec \xi (i, 0) \rightarrow Q(x))\) for all \(i < \xi\) (with level \(lv(Q)\)). In the case where \(\text{Suc}(\xi)\), (A.2)\(_\xi\) is used with the abstract \(\{x\}(x \prec \xi (\xi, 0) \rightarrow Q(x))\) (with level \(lv(Q)\)).

Hence until now it is observed that

\[
(I) \quad S_{\text{Max}(3, lv(Q))}(ID^\xi_\ell(U_0)) \vdash TI[F_\xi, <\xi, (\xi, 0)]_q.
\]

From (I) it is derived in the way familiar by Gentzen [8] that

\[
(II) \quad S_{k+3}(ID^\xi_\ell(U_0)) \vdash TI[F_\xi, <\xi, \omega(\xi, k + 3, 0)]_q \text{ with } lv(Q) \leq 2 \text{ and } k \geq 0.
\]

Let us observe the proof of (II). In Lemma 7 of [1] it is shown that \(\text{Prog}[F_\xi, <\xi, Q] \rightarrow \text{Prog}[F_\xi, <\xi, s(Q)]\), where \(s(Q)\) is a jump operator defined as \(s(Q)[\mu] := \forall \rho(F_\xi(\rho) \rightarrow \forall \nu <\xi \rho(F_\xi(\nu) \rightarrow Q(\nu)) \rightarrow \forall \nu <\xi \rho + (\xi, \mu)\xi(F_\xi(\nu) \rightarrow Q(\nu)))\), where \(\lambda \nu \mu. \mu + \nu^\xi\) is a primitive recursive function which is a generalization of \(\lambda \nu \mu. \nu + \omega^\mu\) of Gentzen [8] and defined in [1] as follows;

- If \(\mu = 0\), then \(\mu + \nu^\xi = \nu\)
- Suppose \(\mu \neq 0\) and \(\nu \neq 0\) and
  \[
  \mu = \mu_1 \# \cdots \# \mu_m \text{ with } \mu_1 \geq \xi \cdots \geq \xi \mu_m \neq 0
  \]
  \[
  \nu = \nu_1 \# \cdots \# \nu_n \text{ with } \nu_1 \geq \xi \cdots \geq \nu_n \neq 0
  \]
  Let \(l\) be the number such that \(0 \leq l \leq m\) and \(\mu_1 \leq \xi \nu_1 < \xi \mu_{l+1}\), then \(\mu + \nu^\xi = \mu_1 \# \# \# \mu_l \# \cdots \# \# \nu_n\)

Note that \(lv(s^n[Q]) = n + \text{Max}(2, lv(Q))\) with \(n \geq 1\), where \(s^n[Q] := [s\cdots s\{Q\} \cdots].\)

Let us sketch the proof of \(\text{Prog}[F_\xi, <\xi, Q] \rightarrow \text{Prog}[F_\xi, <\xi, s(Q)]\) due to Gentzen [8], where a mathematical induction of the level \(\leq lv(Q)\) is used;

Assume

\[
\begin{align*}
\text{Prog}[F_\xi, <\xi, Q] \quad & \quad \cdots (1) \\
F_\xi(x) \wedge \forall y <\xi x(F_\xi(y) \rightarrow s(Q)(y)) \quad & \quad \cdots (2)
\end{align*}
\]

We have to show \(s(Q)(x)\). So assume further

\[
\begin{align*}
F_\xi(\rho) \quad & \quad \cdots (3) \\
\forall \nu <\xi \rho(F_\xi(\nu) \rightarrow Q(\nu)) \quad & \quad \cdots (4) \\
\nu <\xi \rho + (\xi, x)^\xi \wedge F_\xi(\nu) \quad & \quad \cdots (5)
\end{align*}
\]

Under the above assumptions (1) \(\sim (5)\), we have to show \(Q(\nu)\).

Consider the case where \(x \neq 0\). Since \(\nu <\xi \rho + (\xi, x)^\xi\), there exists primitive recursive functions \(f\) and \(g\) such that \(\nu <\xi \rho + (\xi, f(x, \nu, \rho))^\xi \wedge g(x, \nu, \rho)\) with \(f(x, \nu, \rho) <\xi x\) and \(F_\xi(f(x, \nu, \rho))\). From (2), \(s(Q)(f(x, \nu, \rho))\) holds. Then a universal instantiation with \(\rho + (\xi, f(x, \nu, \rho))^\xi \circ n\) (note that \(\rho + (\xi, f(x, \nu, \rho))^\xi \cdot n <\xi \rho + (\xi, x)^\xi\)) for an arbitrary \(n\) allows the following:

\[
\begin{align*}
F_\xi(\rho + (\xi, f(x, \nu, \rho))^\xi \circ n) \rightarrow \forall \eta <\xi \rho + (\xi, f(x, \nu, \rho))^\xi \cdot n(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi (\rho + (\xi, f(x, \nu, \rho))^\xi \cdot n)(\xi, f(x, \nu, \rho))^\xi(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi \\
F_\xi(\rho + (\xi, f(x, \nu, \rho))^\xi \circ n) \rightarrow (F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi (\rho + (\xi, f(x, \nu, \rho))^\xi \cdot n)(\xi, f(x, \nu, \rho))^\xi(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi \\
F_\xi(\rho + (\xi, f(x, \nu, \rho))^\xi \circ n)(\xi, f(x, \nu, \rho))^\xi(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi (\rho + (\xi, f(x, \nu, \rho))^\xi \cdot n)(\xi, f(x, \nu, \rho))^\xi(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi \\
F_\xi(\rho + (\xi, f(x, \nu, \rho))^\xi \circ n)(\xi, f(x, \nu, \rho))^\xi(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi (\rho + (\xi, f(x, \nu, \rho))^\xi \cdot n)(\xi, f(x, \nu, \rho))^\xi(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi \\
F_\xi(\rho + (\xi, f(x, \nu, \rho))^\xi \circ n)(\xi, f(x, \nu, \rho))^\xi(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi (\rho + (\xi, f(x, \nu, \rho))^\xi \cdot n)(\xi, f(x, \nu, \rho))^\xi(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi \\
F_\xi(\rho + (\xi, f(x, \nu, \rho))^\xi \circ n)(\xi, f(x, \nu, \rho))^\xi(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi (\rho + (\xi, f(x, \nu, \rho))^\xi \cdot n)(\xi, f(x, \nu, \rho))^\xi(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta <\xi
\end{align*}
\]

Then mathematical induction with abstract \(\{n\}(\forall \eta <\xi \rho + (\xi, f(x, \nu, \rho))^\xi \cdot n(F_\xi(\eta) \rightarrow Q(\eta)))\), whose level is \(\text{Max}(2, lv(Q))\), implies (with (4)) \(\forall \eta <\xi \rho + (\xi, f(x, \nu, \rho))^\xi \cdot g(x, \nu, \rho)(F_\xi(\eta) \rightarrow Q(\eta))\). Hence from (5), \(Q(\nu)\) holds.

Consider the case where \(x = 0\). For each formula \(Q\), \(s[Q]\) denotes the formula of the following form; \(s[Q](\mu) := \forall \nu \rho(F_\xi(\rho) \rightarrow \forall \nu <\xi \rho(F_\xi(\nu) \rightarrow Q(\nu)) \rightarrow \forall \nu <\xi (\rho + \mu)^\xi(F_\xi(\nu) \rightarrow Q(\nu)))\). Then we can prove without \((A.1)\xi, (A.2)\xi, TI_\xi\) and the mathematical induction that \(\text{Prog}[F_\xi, <\xi, Q] \rightarrow \text{Prog}[F_\xi, <\xi, s[Q]]\). As is shown
above, in Lemma 7 of [1] all the mathematical inductions used are restricted to those of level \( \leq \text{Max}(2, lv(Q)) \).

From now we assume \( lv(Q) \leq 2 \). With the help of \( \text{Prog}[F_\xi, <\xi, Q] \rightarrow \text{Prog}[F_\xi, <\xi, s(Q)] \) and \( \text{Prog}[F_\xi, <\xi, s(Q)] \rightarrow \text{Prog}[F_\xi, <\xi, s^2(Q)] \), in which proof all mathematical inductions are restricted to those of level \( \leq 3 \), (I) implies the following (II):

\[
\text{(II)}_0 : S_3(ID_\xi(U_0)) \vdash TI[F_\xi, <\xi, \omega(\xi, 3, 0)]_Q
\]

By replying this method, the above (II) is obtained.

Then following Arai [1], the next proposition is derived from (II).

\[
S_{k+3}(ID_\xi(U_0)) \vdash TI[F_0, <\omega, \omega(\xi, k + 3, 0)]_Q \text{ with } lv(Q) \leq 2 \text{ and } k \geq 0.
\]

Hence the proposition holds.

Using the above, Proposition 2 follows;

**Proposition 2** For \( k > 2 \), the ordinal up to \( \omega(\xi, k+1, 0) \) is accessible in \( S_k(ID_\xi(U_0)) \) with respect to \( <\omega \).

Remark 1:
From the case in which \( \xi \) is a successor ordinal, the transfinite induction formula \( \{i\} \text{Prog}[F_i, <\xi, \bigcap_{\xi < i} A_i] \) at the beginning of the proof of Proposition 1 above is replaced by \( \{i\} \text{Prog}[F_i, <\xi, A_i] \), which has level 2, instead of 3. Hence, the Propositions 1 and 2 hold for \( k > 1 \).

### 3 Unprovability of the transfinite induction up to \( \omega(\xi, k+1, 0) \) in system \( S_k(AI^-) \)

Our aim in this chapter is to prove the estimation we have observed in previous chapter is sharp one;

\[
S_k(ID_\xi) \not\vdash TI[F_0, <\omega, \omega(\xi, k + 1, 0)] \text{ for } k \geq 2
\]

On the whole segment of \( ID_\xi = \bigcup_n S_n(ID_\xi) \), Arai [3] proves that \( ID_\xi \not\vdash TI[F_0, <\omega, O(\xi + 1, 1)] \). Note that \( O(\xi + 1, 1) := \bigcup_\xi \omega(\xi, k + 1, 0) \). He shows that the consistency of \( ID_\xi \) is provable using transfinite induction up to \( O(\xi + 1, 1) \) by the proof reduction method which is originally due to Gentzen-Takeuti. In this section we modify his consistency proof in more delicate manner and prove the following by the cut elimination (proof reduction) method;

\[
TI[F_0, <\omega, \omega(\xi, k + 1, 0)] \vdash \text{Cons}(S_k(ID_\xi)) \text{ for } k > 2
\]

Our crucial point is to introduce a \( \eta \)-height \( h_\eta \) for each \( \eta \leq \xi \) (Definition 11) and consider an ordinal assignment to a proof \( \langle P, \{h_\eta\}_{\xi \leq \eta} \rangle \) with \( \xi \)-sort of height (Definition 13).

For the Gentzen-Takeuti cut elimination procedure to work, Arai [3] formalises his system \( AI^- \) of \( \xi \)-times iterated inductive definition in the form of iterated comprehension axiom by using second order free variables. System \( AI^- \) is defined by adding the following principles based on \( PA \).
Definition 5 (System $AL_\xi$).\textsuperscript{[3]} For any arithmetical form $B$, the following axioms schemata are added.

\[ \Gamma \vdash \Delta, B(X, Q^B_{\xi_1}, t, s) \]

where $Q^B_{\xi_1} := \{x, y) : (x \prec \xi \land Q^B x y)\}$

For any logical connective $\prec$.

We assume that the language contains only $V$, $\land$, and $\lor$ for the logical connectives.

Definition 6 ($\eta$-level $l_{\eta}(A)$ of a formula $A$ with $\eta \leq \xi$). For the formula $A$ in the language of $AL_\xi$ and an ordinal $\eta \leq \xi$, the $\eta$-level $l_{\eta}(A)$ of the formula $A$ is defined inductively as follows, where $d$ is defined in Definition 2 of the previous section with $Q^\beta$ instead of $p^\beta$ and $d(Xt) := 0$ (for $X$ a predicate variable):

\[
\begin{align*}
l_{\eta}(P) & := 0 \text{ for any atom of } L_{PA}, \\
l_{\eta}(A \land B) & := \max\{l_{\eta}(A), l_{\eta}(B)\} \\
l_{\eta}(\forall x A) & := \left\{ \begin{array}{ll} 
\max\{2, l_{\eta}(A)\} & \text{if } l_{\eta}(A) \geq 1 \\
0 & \text{if } l_{\eta}(A) = 0 
\end{array} \right. \\
l_{\eta}(\neg A) & := \left\{ \begin{array}{ll} 
l_{\eta}(A) + 1 & \text{if } l_{\eta}(A) \geq 1 \\
0 & \text{if } l_{\eta}(A) = 0 
\end{array} \right. \\
l_{\eta}(Q^\beta) & := \left\{ \begin{array}{ll} 
1 & \text{if } d(Q^\beta) = \eta \\
0 & \text{otherwise} 
\end{array} \right. \\
l_{\eta}(t \prec s \land Q^\beta) & := \left\{ \begin{array}{ll} 
1 & \text{if } d(t \prec s \land Q^\beta) = \eta \\
0 & \text{otherwise} 
\end{array} \right. 
\end{align*}
\]

Note that $l_{\eta}$ for $\eta = \xi$ is the same as $l_{\eta}$ of the previous section (with using $Q^\beta$ instead of $P^\beta$ in the definition of $l_{\eta}$ of the previous section with replacing $\supset$ by $\preceq$).

We can define the fragments $S_k(AL_\xi)$ in the same manner as $S_k(ID_{\xi})$ as follows.

Definition 7 (the subsystem $S_k(AL_\xi)$ of $AL_\xi$). $S_k(AL_\xi)$ is $AL_\xi$ except that for every abstract $\forall$ in $Q^\beta$; left and $(\forall V)$, $l_{\xi}(V) \leq k$ holds.

$ID_\xi$ is obtained from $ID_{\xi}$ in the previous section by changing the underlying logic from the intuitionistic to the classical. For each formula $F$ of the language of $ID_\xi$, we define a formula $F^*$ of the language of $AL_\xi$ by substituting $Q^\beta$ for all occurrences of $P^\beta$, where

\[ B(X, Y, \xi, c_0, c_1) := \forall y(U(X, Y, y, c_0, y) \prec Xy) \prec Xc_1. \]

It is well known that by this $*$, $ID_\xi$ is embeddable into $AL_\xi$ (cf. [3]). Obviously $l_{\eta}(F) = l_{\xi}(F^*)$ holds i.e., $\xi$-level of a formula remains the same through the above interpretation.

Until the end of this section, we assume that all formulas occurring in a proof figure of $AL_\xi$ are of the following normal form:

Lemma 1 (the normal form of a formula in $AL_\xi$). For arbitrary formula $A$ of the language of $AL_\xi$, there exists a formula of the following form, called a normal formula, which is equivalent to $A$ (in $LK$):

\[ \forall \bar{z}_1 \ldots \forall \bar{z}_n \forall \bar{y} \exists \bar{y} D[Q^\theta_{\xi_1} s_1, \ldots, Q^\theta_{\xi_m} s_m] \]

where $D[\bar{s}_1, \ldots, \bar{s}_m]$ is a context of the language of $PA$, and no quantifier occurring in $D$ bounds any $\xi_i$ ($1 \leq i \leq m$) and $l_{\eta}(D[Q^\theta_{\xi_1} s_1, \ldots, Q^\theta_{\xi_m} s_m]) \leq 2$ for any $\eta \leq \xi$. 

Definition 8 (normal proofs) Let $S$ be a sequent of normal formulas. A normal proof of $S$ is a proof in which $\forall$-left rules are used, instead of $\forall$-left rules in a proof:

$$
\Gamma \rightarrow \Delta, A(t_1, \ldots, t_n)~
\forall x_1 \cdot \cdot \cdot x_n \rightarrow A(x_1, \ldots, x_n), \Gamma \rightarrow \Delta
$$

Note that the original $\forall$-left rule may also appear in a normal proof.

Lemma 2 Any provable sequent of normal formulas has a normal proof.

From now on we assume any $S_k(\Lambda l^{-})$-proof to be normal by virtue of the above two lemmata.

Definition 9 For each formula $A$, $\eta(A) \leq \xi$ is defined as $\eta(A) := \max \{ \eta \mid l\eta(A) \neq \emptyset \}$.

Definition 10 ($g_{\eta}$ with $\eta < \xi$)

$$
g_{\eta}(A) := \begin{cases} 
g(A) & \text{if } \eta(A) \geq \eta \\
0 & \text{if } \eta(A) < \eta 
\end{cases}
$$

where $g(A)$ denotes the number of logical symbols in $A$.

We modify the notion of proof with degree $< P, d >$ of Arai [3] into $< P, \{ h_{\eta} \}_{\eta \leq \xi}, d >$ by introducing $\xi$-sort of height $\{ h_{\eta} \}_{\eta \leq \xi}$, as follows:

Definition 11 (A proof with $\xi$-sort of height $< P, \{ h_{\eta} \}_{\eta \leq \xi}, d >$) A proof $< P, d >$ with degree $d$ is called a proof with $\xi$-sort of height $< P, \{ h_{\eta} \}_{\eta \leq \xi}, d >$ if for each sequent $S$ of $P$ and each ordinal $\eta \leq \xi$, a natural number $h_{\eta}(S)$ satisfying the following condition is assigned. We call $h_{\eta}$ a $\eta$-height.

0. $h_{\eta}(S) = 0$ for every $\eta \leq \xi$ if $S$ is the end sequent of $P$.

For the last inference $I$ of the form

$$
I \quad \frac{S}{S'}
$$

1. $h_{\eta}(S) = 0$ for every $\eta \leq \xi$ if $I$ is a substitution.

2. $h_{\eta}(S) = h_{\eta}(S')$ for every $\eta \leq \xi$ if $I$ is an inference except substitution, induction and cut.

3. $\begin{cases} 
1 & h_{\eta}(S) \geq \max \{ h_{\eta}(S'), g_{\eta}(D) \} \text{ for } \eta < \xi \\
2 & h_{\xi}(S) = \max \{ h_{\xi}(S'), l\xi(D) \} 
\end{cases}$

if $I$ is a cut, where $D$ is the cut formula of the inference $I$.

4. $\begin{cases} 
1 & h_{\eta}(S) \geq \max \{ h_{\eta}(S'), g_{\eta}(D) \} + 1 \text{ for } \eta < \xi \\
2 & h_{\xi}(S) = \max \{ h_{\xi}(S'), l\xi(D) \} + 1 
\end{cases}$

if $I$ is an induction.

Definition 12 For each sequent $S$ of $< P, \{ h_{\eta} \}_{\eta \leq \xi}, d >$, $\eta(S) \leq \xi$ is defined as

$$
\eta(S) := \begin{cases} 
d(I) & \text{if } S \text{ is the upper sequent of the substitution } I \\
\max \{ \eta \mid h_{\eta}(S) \neq 0 \} & \text{otherwise}
\end{cases}
$$

The following is an immediate consequence from Definition 12.
Lemma 3 For any proof with $\xi$-sort of height $< P, \{h_n\}_{n \leq \xi}, d >$ and for any inference $I$ (with a lower sequent $S'$ and an upper sequent $S$) in $< P, \{h_n\}_{n \leq \xi}, d >$,

$$\eta(S) \geq \eta(S')$$

holds.

Notation 3 For $i \leq \xi$ and an ordinal diagram $\alpha$, an ordinal diagram $\omega(i, n, \alpha)$ is defined inductively as follows.

- $\omega(i, 0, \alpha) := \alpha$
- $\omega(i, n + 1, \alpha) := (i, \omega(i, n, \alpha))$

Definition 13 (ordinal assignment) Let $I$ be an inference of the form

$$I \quad \frac{S_1}{S} \quad S_2$$

Then $O(S)$ is defined as follows:

1. When $I$ is a cut,

$$O(S) := \omega(\eta(S), k - h_{\eta(S)}(S), c[\omega(\eta(S_1), h_{\eta(S)}(S_1), O(S_1)\#O(S_2))])$$

Here $k := \text{Max}\{h_{\eta}(T) \mid T \text{ is above } I\}$ and $c[\ast] := \omega(\gamma_1, k_1, \omega(\gamma_2, k_2, \ldots, \omega(\gamma_n, k_n, \ast))),$

where \{$\gamma_1, \ldots, \gamma_n$\} := \{$\gamma \mid \eta(S) < \gamma < \eta(S_1)$ and $h_{\eta}(T) \neq 0$ for some $T$ above $I$\}

with $\gamma_1 < \cdots < \gamma_n$ and $k_i := \text{Max}\{h_{\eta}(T) \mid T \text{ is above } I\}$.

2. When $I$ is a logical inference,

$$O(S) := O(S_1)\#O(S_2)\#0$$

3. When $I$ is a structural inference,

$$O(S) := O(S_1)\#O(S_2)$$

4. When $I$ is a substitution,

$$O(S) := (d(I), O(S_1))$$

Theorem 1 The transfinite induction on $\omega(\xi, k + 1, 0)$ is unprovable in $S_k(\lambda I^-)$ for $k > 2$.

Proof.

We refine the proof reduction process of Arai [3] to define the reduction process for $S_k(\lambda I^-)$ ($k > 2$), and show that the well-orderedness of $\omega(\xi, k + 1, 0)$ implies the termination of the reduction process, hence the consistency of $S_k(\lambda I^-)$. Then the above theorem follows from Gödel's incompleteness theorem.

(preparation)

Without loss of generality, we assume that all logical initial sequents of the form $p \rightarrow p$ where $p$ is an atomic and that there exists no free variables which is not used as an eigenvariable.

(elimination of initial sequents in the end-piece) As usual.

(elimination of weakening) elimination of weakening known in the usual way (cf. Takeuti [14]) does work not only for a weakening in end-piece but also for a more general weakening with such a weakening formula $D$ as the bundle $I$ (cf. p78 of [14]) which begins with $D$ ends with a cut formula $D$ and no logical inference affect $I$.  

\[1\text{In the case where } \eta(S) = \eta(S_1), \text{ } c[\ast] \text{ is } \ast \text{ and } O(S) := \omega(\eta(S), k + h_{\eta(S)}(S_1) - h_{\eta(S)}(S), O(S_1)\#O(S_2)).\]
(elimination of the mathematical induction rule) As usual. Then from sublemma 12.9 of [14], there exists a suitable cut $J$ in the end piece of $< P, \{h_\eta\}_{\eta \leq \xi}, d >$. Let $I_1$ and $I_2$ be boundary logical inferences whose principal formulas are ancestors of left and right cut formulas of $J$. We shall demonstrate following three essential cases both for limit ordinal $\xi$ and for successor ordinal $\xi$.

(Case 1) The case where the cut formula $C := A \land B$ with $\eta(C) < \xi$; Let $K$ (whose lower sequent is $T$ and whose upper sequent is $T_1$) denotes the uppermost inference below $J$ such that either (i) or (ii) holds;

$$\eta(T) = \eta(A) \land (h_{\eta(A)}(S_1) > h_{\eta(A)}(T)) \cdots (i)$$
$$\eta(T) < \eta(A) \cdots (ii)$$

where $A$ is the auxiliary formula of $I_1$ and $I_2$

$$< P, \{h_\eta\}_{\eta \leq \xi}, d >$$

is as follows:

\[
\begin{array}{cccc}
S_1^1 & S_1^2 & I_1 & S_2^1 \\
S_2^1 & S_2^2 & I_2 & J \\
T_1 & T & K & J \\
\end{array}
\]

(Case 1.1): The case where (i) holds. Then for any sequent $T'$ between $S_1$ and $T$, $\eta(T') \geq \eta(A)$ holds.

(Case 1.1.1) $\eta(T_1) = \eta(T)$

$O_{P'}(T^*) \prec_0 O_P(T)$ is checked as usual way.

(Case 1.1.2) $\eta(T_1) > \eta(T)$

special case of (case 1.2)

(Case 1.2): The case where (ii) holds. Then $\eta(T) < \eta(A) \leq \eta(T_1)$ holds. We assign

$$h'_\eta(U_1) := \begin{cases} h_\eta(T) & \text{if } \eta < \eta(T) \\ g(A) & \text{if } \eta = \eta(A) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h'_\eta(T_1^*) := h'_\eta(T_1^*) := h_\eta(T_1) \quad \text{for all } \eta \leq \xi.$$

Hence $\eta(U_1) = \eta(A)$ holds. On the other hand, there exist contexts $a$ and $b$ such
that $O_P(T) = \omega(\eta(T), k - h_n(T), a[\omega(\eta(A), k_1, b[\alpha_1 \# \alpha_2])]),$

$O_P(U_1) = \omega(\eta(U_1), m - h_n(U_1), b[\alpha_1 \# \alpha_2]) = \omega(\eta(A), m - g(A), b[\alpha_1 \# \alpha_2])$ and

$O_P(T^*) = \omega(\eta(T), k' - \eta(T), a[\omega(\eta(U_1), g(A), O_P(U_1) \# O_P(U_2))])$

Since $\omega(\eta(A), k_1, b[\alpha_1 \# \alpha_2]) > \omega(\eta(U_1), g(A), O_P(U_1) \# O_P(U_2)), O_P(T^*) <_o O_P(T)$ holds.

(Case 2) The case where cut formula is $\forall z \rightarrow B(z)$:

$\langle P, \{h_n\}_{n \le d}, d \rangle$ is as follows; here $I_2$ is $\forall\leftarrow$.

\[
\begin{array}{c}
\begin{array}{c}
S_1^1 \quad S_1^2 \quad I_1 \quad S_2^1 \quad S_2^2 \quad I_2 \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
S_1^3 \quad S_1^4 \quad J \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
S_1^1 : \quad \Gamma_1 \rightarrow \Delta_1, \neg B(\bar{z}) \\
S_1^1 : \quad \Gamma_1 \rightarrow \Delta_1, \forall \bar{z} \rightarrow \neg B(\bar{z}) \\
S_1^2 : \quad \Pi_1 \rightarrow \Lambda_1, \neg B(\bar{z}) \\
S_1^2 : \quad \Pi_1 \rightarrow \Lambda_1, \forall \bar{z} \rightarrow \neg B(\bar{z}) \\
S_2^1 : \quad \forall \bar{z} \rightarrow B(\bar{z}), \Pi_1 \rightarrow \Lambda_1 \\
S_2^1 : \quad \forall \bar{z} \rightarrow B(\bar{z}), \Pi_1 \rightarrow \Lambda_1 \\
T : \quad \Phi \rightarrow \Psi \\
\end{array}
\end{array}
\]

$\langle P', \{h_n\}_{n \le d}, d' \rangle$ is as follows, where $I_1$ and $I_2$ are weakening-right and weakening-left (respectively) with weakening formulas $\forall \bar{z} \rightarrow B(\bar{z})$. Note that by virtue of preparation and (elimination of weakening), any formula of the form $\neg B(\bar{z})$ which is an ancestor of the auxiliary formula of $I_1$ is a descendant of principal formulas of an inference $\neg\rightarrow$. Hence the following $S_1^1(\bar{z})$ can be obtained.

\[
\begin{array}{c}
\begin{array}{c}
S_1^1(t) \quad S_1^2(t) \quad I_1 \quad S_2^1(t) \quad S_2^2(t) \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
S_1^3(t) \quad S_1^4(t) \quad J \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
S_1^1(t) : \quad B(\bar{z}), \Gamma_1 \rightarrow \Delta_1 \\
S_1^1(t) : \quad B(\bar{z}), \Gamma_1 \rightarrow \Delta_1, \forall \bar{z} \neg B(\bar{z}) \\
S_1^2(t) : \quad B(\bar{z}), \Gamma \rightarrow \Delta, \forall \bar{z} \rightarrow B(\bar{z}) \\
S_1^2(t) : \quad \forall \bar{z} \rightarrow B(\bar{z}), \Pi \rightarrow \Lambda, \Lambda \\
S_1^2(t) : \quad \forall \bar{z} \rightarrow B(\bar{z}), \Pi \rightarrow \Lambda, B(\bar{z}) \\
S_1^2(t) : \quad \Pi_1 \rightarrow \Lambda_1, B(\bar{z}) \\
S_2^2(t) : \quad \forall \bar{z} \rightarrow B(\bar{z}), \Pi_1 \rightarrow \Lambda_1, B(\bar{z}) \\
U_1(\bar{z}) : \quad B(\bar{z}), \Phi \rightarrow \Psi \\
U_2 : \quad \Phi \rightarrow \Psi, B(\bar{z}) \\
T^* : \quad \Phi, \Phi \rightarrow \Psi, \Psi \\
\end{array}
\end{array}
\]

Since $I_{\eta(B(\bar{z}))}(\forall \bar{z} \rightarrow B(\bar{z})) > I_{\eta(B(\bar{z}))}(B(\bar{z}))$ holds, $O(P') <_o O(P)$ is checked as the usual way.

(Case 3) The case where the cut formula of $J$ is $Q_{\forall z}$.s:

$\langle P, \{h_n\}_{n \le d}, d \rangle$ is as follows, where $K$ (with the lower sequent $T$) denotes the upper most inference below $J$ such that $\eta(T) \le d(B(X, Q_{\forall z}, t, s)) \iff i$.
Let $T_1$ denote such upper sequent of $K$ that is below $J$.

Let $T_1$ denote such upper sequent of $K$ that is below $J$.

Then $T_1$ is defined as follows. where $I_1$ is weakening-right with a weakening formula $Q^r t_1 s_1$:

We assign $\{h'_\eta\}_{\eta \leq \xi}$ as follows:

- $h'_\eta(T_1) := h_\eta(S)$ for all $\eta \leq \xi$
- $h'_\eta(T_1) := h_\eta(T_1)$ for all $\eta \leq \xi$.
- $h'_\eta(T^*) := \left\{ \begin{array}{ll} h_\eta(T) & \text{if } \eta \leq \eta(T) \\ 0 & \text{otherwise} \end{array} \right.$

$O_P(T) = \omega(\eta(T), k = h_\eta(T_1)(T), c[\omega(\eta(T_1), h_\eta(T_1)(T_1), O_P(T_1) \# O_P(T_2))]).$

$< P', \{h'_\eta\}_{\eta \leq \xi}, d' >$ is as follows, where $I_1$ is weakening-right with a weakening formula $Q^r t_1 s_1$:  

$O_P(T^*) = (i, O_P(T^*))$
\[ O_P(T^*) = \omega(\eta(T^*), l - h^n_{\eta(T^*)}(T^*), d[\omega(\eta(T^*), h^n_{\eta(T^*)}(T^*), O_P(T^*), O_P(T^*)])] \]

Note that \( \eta(T^*) = i \). Obviously \( k = l \) from the figure of \( P' \). And from the above assignment \( h' \), \( c[*] = \omega(\gamma_1, k_1, \ldots, \omega(\gamma_s, k_s, d[*])) \) with \( \gamma_s < i = \gamma_{s+1} \). Hence \( O(P') <_0 O(P) \) holds.

The following Corollary is immediate from the above theorem and the fact that \( S_k(ID^b_\xi) \) is a subsystem of \( S_k(AL^b_\xi) \) under the interpretation * (cf. the paragraph after Definition 7).

**Corollary 1** The transfinite induction on \( \omega(\xi, k + 1) \) is unprovable in \( S_k(ID^b_\xi) \) for \( k > 2 \).

**Proof.** As remarked after Definition 7, \( \xi \)-level does not change under the interpretation of an \( AL^b_\xi \)-formula to an \( ID^b_\xi \)-formula. Hence the Corollary is obvious.

**Theorem 2 (Main Theorem)**

\[ |S_k(ID^b_\xi(\mathbf{I}_0))| = |S_k(ID^b_\xi)| = |S_k(AL^b_\xi)| = |\omega(\xi, k + 1, 0)| <_0 \text{ with } k > 2. \]

**Remark 2:** Our system \( S_k(ID^b_\xi) \) can be reformulated by means of the alternation complexity of quantifiers when we include \( \exists \) in our language. Here, a normal formula is of the form \( Q_1 \alpha_1 Q_1 \beta_1 \ldots Q_n \alpha_n Q_n \beta_n \forall \exists D[P^\alpha_1 t_1 s_1, \ldots, P^\alpha_n t_n s_n] \), where \( D[\alpha_1, \ldots, \alpha_n] \) is a context of the language of \( PA \) with no quantifier occuring in \( D \) bounds any \( \alpha_i \) (\( 1 \leq i \leq n \)), and \( \{Q_j, \beta_j\} = \{\forall, \exists\} \) (\( j = 1, \ldots, m \)). \( lv \) is essentially the same as \( lv_c \) except that we measure the alternation complexity of quantifiers instead of nestedness complexity of negations; namely,

\[ lv(D[P^\alpha_1 t_1 s_1, \ldots, P^\alpha_n t_n s_n]) := \begin{cases} 1 & \text{if all } P^\alpha_i t_i s_i (i = 1, \ldots, m) \text{ is positive in } D \\ 2 & \text{otherwise} \end{cases} \]

Then the \( lv \) of above normal formula is \( n + i \) if \( Q_n = \forall \) and \( n + 1 + i \) if \( Q_n = \exists \), where \( i := lv(D[P^\alpha_1 t_1 s_1, \ldots, P^\alpha_n t_n s_n]) \). \( S'_k(ID^b_\xi) \) is defined in the same way as the former definition of \( S_k(ID^b_\xi) \) with using the above new notation of \( lv \). It is easily seen that \( S'_k(ID^b_\xi) \) is equivalent to \( S_k(ID^b_\xi) \). In particular \( |S'_k(ID^b_\xi)| = |\omega(\xi, k + 1, 0)|_0 \) with \( k > 2 \).

**References**


