A Hierarchy of the Fragments of the System of Inductive Definition (Preliminary Report)

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1 introduction

Gentzen [7] proved the coinsistency of PA (Peano Arithemetic) by using the transfinite induction up to the first *epsilon* number ϵ_0. Here ϵ_0 is $\lim_k \omega_k$, where $\omega_0 = 0$ and $\omega_{k+1} = \omega^{\omega_k}$. Later in [8] he proved that the accessibility (i.e., transfinite induction) proof up to any ordinal less than ϵ_0, eg., ω_k for any natural number k, is provable in PA.

In his [8] the nestedness complexity of implications used in the accessibility proof increases by one while the accessibility of one higher ω-tower ω_{k+1} is proved from the accessibility of ω_k. Hence by considering Gentzen's work [7, 8] a natural question arises; does the hierarchy of ω-towers, $\{\omega_k\}_{k=1,2,\ldots}$, correspond exactly to a certain hierarchy of fragments of PA?

Mints [10] answered this question by estimating the least upper bounds of accessibility ordinals for the fragments of PA, where the fragments are defined by means of the number of alternations of quantifiers, using one quantifier system developed in his former paper [9]. (Shirai [13] also gave a similar result by means of the number of quantifiers.)

The purpose of our paper is to investigate in a similar correspondence (between the hierarchy of critical ordinals and the hierarchy of fragment systems) for the system of ξ-iterated Inductive Definition ID_ξ [6]. We first analyze in Section 2 Arai's optimal accessibility proof for ID_ξ ([3]) to obtain a hierarchy of accessible ordinals for the fragments of intuitionistic ID_ξ, where the fragments are defined in terms of the nestedness complexity of implications. Then we show in Section 3 the least upper bounds of accessible ordinals (i.e., the critical ordinals) for those fragments, by analyzing Takeuti-Arai's consistency proofs of ID_ξ ([3]). In fact, for the upper bounds proof we use the fragments of classical ID_ξ in terms of the nestedness complexity of classical negations. Since the fragments of ID_ξ obtained by means of the number of alternations of quantifiers (in a prenex normal form) are also characterized by the nestedness complexity of negations with the help of universal quantifiers (by representing an existential quantifier $\exists$ by means of $\neg\forall\neg$), our result for ID_ξ corresponds to Mints' ([10]) for PA.

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2 Provability of transfinite inductions on $\omega(\xi, k, 0)$ in subsystems of $S_k(ID^i_\xi(\mathcal{U}_0))$

Let $(I, \prec)$ be the well ordered system whose order type is ordinal $\xi + 1$. Arai [1] proved the well ordering of Takeuti's system of ordinal diagram $O(\xi + 1, 1)$ in the system ID^i_ξ (the intuitionistic system of ξ-times iterated inductive definition).

In this chapter we introduce a hierarchy of fragments $S_k(ID^i_\xi)$ of ID^i_ξ based on the nestedness complexity of implications, and observe Arai's well ordering prqof of [1] on these fragments.

Now we recall the definitions of $ID^i_\xi(\mathcal{U})$ and ID^i_ξ of Feferman [6].

Definition 1 (System $ID^i_\xi(\mathcal{U})$ and ID^i_ξ, cf. Feferman [6])
For any positive operator form $\mathcal{U}$, $ID^i_\xi(\mathcal{U})$ is obtained from PA by adding the following axiom schemata.

$(P_\xi.1)$ $\quad \to \forall x \prec \xi(\mathcal{A}(P^{\mathcal{U}}_x, P^{\mathcal{U}}_{\prec x}, x) \subseteq P^{\mathcal{U}}_x)$

$(P_\xi.2)$ $\quad \to \forall x \prec \xi(\mathcal{U}(V, P^{\mathcal{U}}_{\prec x}, x) \subseteq V \supset P^{\mathcal{U}}_x \subseteq V)$

$(TI)_\xi$ $\quad Prog[I, \prec, V] \to (I \subseteq V)$

where $P^{\mathcal{U}}_{\prec a} := \{x, y\}(x \prec a \wedge P^{\mathcal{U}}xy)$
$ID^i := \bigcup\{ID^i(\mathcal{U}) \mid \mathcal{U}$ *is a positive operator form*$\}$

The starting point of Arai's well ordering proof is to define the notion of accessibility with respect to $<_i$ for $i \prec \xi$ (cf. §26 [14]) by using the set constants A_i which is definable in $ID^i_\xi(\mathcal{U}_0)$ with the following $\mathcal{U}_0$;
$(A.1)_\xi$ $\quad \forall i \prec \xi Prog[F_i, <_i, A_i]$
$(A.2)_\xi$ $\quad \forall i \prec \xi(Prog[F_i, <_i, V) \to A_i \subseteq V)$ for each abstract V in $ID^i_\xi(\mathcal{U}_0)$
where $\mathcal{U}_0$ is a X-positive operator form defined as $\mathcal{U}_0(X, Y, i, \mu) := \mathcal{F}(i, \mu, Y) \wedge \forall \nu \prec_i \mu(\mathcal{F}(i, \nu, Y) \to X(\nu))$ where $\mathcal{F}(i, \mu, Y) := \forall k \prec i \forall \rho \subset_k \mu Y(k, \rho)$, $Prog[\alpha, \gamma, \beta] := \forall x(\alpha(x) \wedge \forall y(\gamma(y, x) \wedge \alpha(y) \to \beta(y)) \to \beta(x))$, and $F_i(\mu) := \forall j \prec i \forall \nu \subset_j \mu A_j(\nu)$ (the intended meaning of $F_i(\mu)$ is that μ is an i-fan (cf. Definition 26.16 [14])).

Remember that $ID^i_\xi(\mathcal{U})$ has the mathematical induction of the following form;
(VJ) $\quad V(0), \forall x(V(x) \to V(x')) \to V(t)$

The above $ID^i_\xi(\mathcal{U}_0)$ is the specific subsystem of the system ID^i_ξ of Inductive Definition in which the induction schenmata are used only for the accesibility predicate A_i of ordinals.

We consider the subsystem $S_k(ID^i_\xi(\mathcal{U}_0))$ of $ID^i_\xi(\mathcal{U}_0)$ where each abstract V in $(A.2)_\xi$, $(TI)_\xi$ and (VJ) is restricted to that of level $lv(V) \leq k$; where $lv(V)$ is defined by the definition below.

We introduce the notion of level of A ($lv(A)$) for a formula A to express, roughly speaking, the implicational complexity of A. We assume that the language contains only $\forall$, $\supset$ and $\wedge$ for the logical connectives in this section.

We first recall the degree d of a formula in the language of $ID^i_\xi(\mathcal{U})$ defined in Arai [3], which intends to indicate how many times inductive definition is applied.

Definition 2 (cf. Def 2.4 in Arai [3])

- $d(t = s) = 0$ *for all term t,s and predicate variable X.*
-

$$d(P^{\mathcal{U}}ts) = \begin{cases} i \oplus 1 & \textit{if } t \textit{ is a closed term whose value is } i \prec \xi. \\ \xi & \textit{otherwise} \end{cases}$$

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$$d(t_1 \prec s \wedge P^{\mathcal{U}} t_2 r) = \begin{cases} i & \text{if } s \text{ is a closed term whose value is } i \prec \xi \text{ and } t_1 \text{ is a closed term representing the same numeral as } t_2. \\ \xi & \text{otherwise} \end{cases}$$

Definition 3 (level $lv(A)$ of formula A in the language of $ID^i_\xi(\mathcal{U})$) *For the formula A in the language of $ID^i_\xi(\mathcal{U})$, the level $lv(A)$ of the formula A is defined inductively as follows;*

$lv(P) := 0$ *for any atom of the language of PA.*
$lv(A \wedge B) := max\{lv(A), lv(B)\}$

$$lv(\forall x A) := \begin{cases} max\{2, lv(A)\} & \text{if } lv(A) \geq 1 \\ 0 & \text{if } lv(A) = 0 \end{cases}$$

$$lv(A \supset B) := \begin{cases} Max\{lv(A)+1, lv(B)\} & \text{if } lv(A) \geq 1 \\ 0 & \text{if } lv(A) = 0 \end{cases}$$

$$lv(P^{\mathcal{U}} t) := \begin{cases} 1 & \text{if } d(P^{\mathcal{U}} t) = \xi \\ 0 & \text{otherwise} \end{cases}$$

$$lv(t \prec s \wedge P^{\mathcal{U}}_t) := \begin{cases} 1 & \text{if } d(P^{\mathcal{U}}_t) = \xi \\ 0 & \text{otherwise} \end{cases}$$

The subsystems $S_k(ID^i_\xi(\mathcal{U}))$ and $S_k(ID^i_\xi)$ of $ID^i_\xi(\mathcal{U})$ and ID^i_ξ are defined in terms of level lv as follows;

Definition 4 (the subsystem $S_k(ID^i_\xi(\mathcal{U}))$ of $ID^i_\xi(\mathcal{U})$) *$S_k(ID^i_\xi(\mathcal{U}))$ is $ID^i_\xi(\mathcal{U})$ except that for every abstract V in $(A.2)_\xi$, $(TI)_\xi$ and (VJ), $lv(V) \leq k$ holds.*
$S_k(ID^i_\xi) := \bigcup\{S_k(ID^i_\xi(\mathcal{U}) \mid$ *U is a positive operator form*$\}$

The following notation is introduced;

Notation 1 *Let $TI[\alpha, \gamma, \mu]$ denote the schema defined as $TI[\alpha, \gamma, \mu] := \alpha(\mu) \wedge (Prog[\alpha, \gamma, V] \rightarrow \forall \nu(\gamma(\mu, \nu) \wedge \alpha(\nu) \rightarrow V(\nu)))$. And $TI[\alpha, \gamma, \mu]_Q$ is the result of $TI[\alpha, \gamma, \mu]$ by substituting Q for V*

Notation 2 $\omega(\xi, 0, \alpha) := \alpha$ *and* $\omega(\xi, n+1, \alpha) := (\xi, \omega(\xi, n, \alpha))$.

Then by checking Arai's well ordering proof of $O(\xi+1, 1)$ [1] carefully, Proposition 1 is easily observed.

Proposition 1 *For a formula Q with $lv(Q) \leq 2$ and $k > 2$, $TI[F_0, <_0, \omega(\xi, k, 0)]$ is provable in $S_k(ID_\xi(\mathcal{U}_0))$. Namely, the ordinal $\omega(\xi, k, 0)$ is accessible in $S_k(ID^i_\xi(\mathcal{U}_0))$ with respect to $<_0$.*

Proof.
We follow Arai's [1].
We only consider the case in which ξ is a limit. (See Remark after Proposition 2 for the successor ξ case.) Let $\bigcap_{k \prec i} A_k := \{\mu\}\forall k \prec i A_k(\mu)$. In Lemma 3 of [1] $(TI)_\xi$ is used with the abstract $\{i\}Prog[F_i, <_i, \bigcap_{k \prec i} A_k] := \{i\}\forall x(F_i(x) \wedge \forall y <_i x(F_i(y) \rightarrow \bigcap_{k \prec i} A_k(y)) \rightarrow \bigcap_{k \prec i} A_k(x)))$, here $lv(Prog[F_i, <_i, \bigcap_{k \prec i} A_k(\mu)]) = 3$. Let $\overline{A} := \bigcap_{j \prec \xi} A_j$ and $R_i(\nu) := \forall \mu <_\xi (i, \nu)(F_\xi(\mu) \rightarrow \overline{A}(\mu))$. In Lemma 4 of [1] $(A.2)_\xi$ is used with the abstract $\{x\}R_i(x) := \forall \mu <_\xi (i, x)(F_\xi(\mu) \rightarrow \overline{A}(\mu))$ (with $lv(R_i(x)) = 2$) and $(TI)_\xi$ is used with the abstract $\{i\}R_i(0) := \forall \mu <_\xi (i, 0)(F_\xi(\mu) \rightarrow \overline{A}(\mu))$ (with $lv(R_i(0)) = 2$).
Then in Lemma 5 of [1] it is shown that $TI[F_\xi, <_\xi, (\xi, 0)]_Q$ is provable in $ID^i_\xi(\mathcal{U}_0)$ for each unary predicate $Q(x)$ in $ID^i_\xi(\mathcal{U})$; In the case where $lim(\xi)$, $(A.2)_\xi$ are used

with the abstract $\{x\}(x \prec_\xi (i,0) \to Q(x))$ for all $i \prec \xi$ (with level $lv(Q)$). In the case where $Suc(\xi)$, $(A.2)_\xi$ is used with the abstract $\{x\}(x <_\xi (\xi,0) \to Q(x))$ (with level $lv(Q)$).
Hence until now it is observed that

$$(I) \qquad S_{Max(3,lv(Q))}(ID^i_\xi(\mathcal{U}_0)) \vdash TI[F_\xi, <_\xi, (\xi,0)]_Q.$$

From (I) it is derived in the way familiar by Gentzen [8] that

$$(II) \qquad S_{k+3}(ID^i_\xi(\mathcal{U}_0)) \vdash TI[F_\xi, <_\xi, \omega(\xi, k+3, 0)]_Q \text{ with } lv(Q) \leq 2 \text{ and } k \geq 0.$$

Let us observe the proof of (II). In Lemma 7 of [1] it is shown that $Prog[F_\xi, <_\xi, Q] \to Prog[F_\xi, <_\xi, s[Q]]$, where $s[Q]$ is a jump operator defined as $s[Q](\mu) := \forall\rho(F_\xi(\rho) \to \forall\nu <_\xi \rho(F_\xi(\nu) \to Q(\nu)) \to \forall\nu <_\xi \rho + (\xi,\mu)^\xi(F_\xi(\nu) \to Q(\nu)))$, where $\lambda\nu\mu.\mu + \nu^\xi$ is a primitive recursive function which is a generarization of $\lambda\nu\mu.\nu + \omega^\mu$ of Gentzen [8] and defined in [1] as follows;

- If $\mu = 0$, then $\mu + \nu^\xi = \nu + \mu^\xi = \nu$
- Suppose $\mu \neq 0$ and $\nu \neq 0$ and
  $\mu \equiv \mu_1 \# \cdots \# \mu_m$ with $\mu_1 \geq_\xi \cdots \geq_\xi \mu_m \not\equiv 0$
  $\nu = \nu_1 \# \cdots \# \nu_n$ with $\nu_1 \geq_\xi \ldots \geq \nu_n \not\equiv 0$
  Let l be the number such that $0 \leq l \leq m$ and $\mu_l \leq_\xi \nu_1 <_\xi \mu_{l+1}$,
  then $\mu + \nu^\xi := \mu_1 \# \cdots \# \mu_l \# \nu_1 \# \cdots \nu_n$

Note that $lv(s^n[Q]) = n + Max(2, lv(Q))$ with $n \geq 1$, where $s^n[Q] := \overbrace{s[\cdots s[}^{n-times} Q] \cdots]$.
Let us sketch the proof of $Prog[F_\xi, <_\xi, Q] \to Prog[F_\xi, <_\xi, s[Q]]$ due to Gentzen [8], where a mathematical induction of the level $\leq lv(Q)$ is used;
Assume
$Prog[F_\xi, <_\xi, Q] \quad \cdots(1)$
$F_\xi(x) \wedge \forall y <_\xi x(F_\xi(y) \to s[Q](y)) \quad \cdots(2)$
We have to show $s[Q](x)$. So assume further
$F_\xi(\rho) \quad \cdots(3)$
$\forall\nu <_\xi \rho(F_\xi(\nu) \to Q(\nu)) \quad \cdots(4)$
$\nu <_\xi \rho \oplus (\xi, x)^\xi \wedge F_\xi(\nu) \quad \cdots(5)$
Under the above assumptions $(1) \sim (5)$, we have to show $Q(\nu)$.
Consider the case where $x \neq 0$. Since $\nu <_\xi \rho \oplus (\xi, x)^\xi$, there exists primitive recursive functions f and g such that $\nu <_\xi \rho \oplus (\xi, f(x,\nu,\rho)) \cdot g(x,\nu,\rho)$ with $f(x,\nu,\rho) <_\xi x$ and $F_\xi(f(x,\nu,\rho))$. From (2), $s[Q](f(x,\nu,\rho))$ holds. Then a universal instantiation with $\rho \oplus (\xi, f(x,\nu,\rho)^\xi) \cdot n$ (note that $\rho \oplus (\xi, f(x,\nu,\rho)^\xi) \cdot n <_\xi \rho \oplus (\xi,x)^\xi$) for an arbitrary n allows the following:
$F_\xi(\rho \oplus (\xi, f(x,\nu,\rho)^\xi) \cdot n) \to \forall\eta <_\xi \rho \oplus (\xi, f(x,\nu,\rho)^\xi) \cdot n(F_\xi(\eta) \to Q(\eta)) \to \forall\eta <_\xi (\rho \oplus (\xi, f(x,\nu,\rho))^\xi \cdot n) \oplus (\xi, f(x,\nu,\rho))^\xi(F_\xi(\eta) \to Q(\eta)) \cdots (6)$
From $F_\xi(\rho \oplus (\xi, f(x,\nu,\rho)^\xi) \cdot n)$ (from (5)) and the property of Suc, the following holds;
$\forall\eta <_\xi \rho \oplus (\xi, f(x,\nu,\rho))^\xi \cdot n(F_\xi(\eta) \to Q(\eta)) \to \forall\eta <_\xi \rho \oplus (\xi, f(x,\nu,\rho)^\xi) \cdot Suc(n)(F_\xi(\eta) \to Q(\eta)) \cdots (7)$
Then mathematical induction with abstract $\{n\}(\forall\eta <_\xi \rho \oplus (\xi, f(x,\nu,\rho))^\xi \cdot n(F_\xi(\eta) \to Q(\eta)))$, whose level is $Max(2, lv(Q))$, implies (with (4)) $\forall\eta <_\xi \rho \oplus (\xi, f(x,\nu,\rho))^\xi \cdot g(x,\nu,\rho)(F_\xi(\eta) \to Q(\eta))$. Hence from (5), $Q(\nu)$ holds.
Consider the case where $x = 0$. For each formula Q, $s[Q]$ denotes the formula of the following form; $s[Q](\mu) := \forall\rho(F_\xi(\rho) \to \forall\nu <_\xi \rho(F_\xi(\nu) \to Q(\nu)) \to \forall\nu <_\xi \rho + \mu^\xi(F_\xi(\nu) \to Q(\nu)))$. Then we can prove without $(A.1)_\xi$, $(A.2)_\xi$, TI_ξ and the mathematocal induction that $Prog[F_\xi, <_\xi, Q] \to Prog[F_\xi, <_\xi, s[Q]]$. As is shown

above, in Lemma 7 of [1] all the mathematical inductions used are restricted to those of level $\leq Max(2, lv(Q))$.

From now we assume $lv(Q) \leq 2$. With the help of $Prog[F_\xi, <_\xi, Q] \rightarrow Prog[F_\xi, <_\xi, s[Q]]$ and $Prog[F_\xi, <_\xi, s[Q]] \rightarrow Prog[F_\xi, <_\xi, s^2[Q]]$, in which proof all mathematical inductions are restricted to those of level ≤ 3, (I) implies the following $(II)_0$;

$$(II)_0 \quad S_3(ID^i_\xi(\mathcal{U}_0)) \vdash TI[F_\xi, <_\xi, \omega(\xi, 3, 0)]_Q$$

By replying this methode, the above (II) is obtained.

Then following Arai [1], the next proposition is derived from (II).

$$S_{k+3}(ID^i_\xi(\mathcal{U}_0)) \vdash TI[F_0, <_0, \omega(\xi, k+3, 0)]_Q \text{ with } lv(Q) \leq 2 \text{ and } k \geq 0.$$

Hence the proposition holds.

□

Using the above, Proposition 2 follows;

Proposition 2 *For $k > 2$, the ordinal up to $\omega(\xi, k+1, 0)$ is accessible in $S_k(ID^i_\xi(\mathcal{U}_0))$ with respect to $<_0$.*

Remark 1;

From the case in which ξ is a succesor ordinal, the transfinite induction formula $\{i\}Prog[F_i, <_i, \bigcap_{k \prec i} A_k]$ at the beginning of the proof of Proposition 1 above is replaced by $\{i\}Prog[F_i, <_i, A_i]$, which has level 2, instead of 3. Hence, the Propositions 1 and 2 hold for $k > 1$.

3 Unprovability of the transfinite induction up to $\omega(\xi, k+1, 0)$ in system $S_k(AI^-_\xi)$

Our aim in this chapter is to prove the estimation we have observed in previous chapter is sharp one;

$$S_k(ID_\xi) \nvdash TI[F_0, <_0, \omega(\xi, k+1, 0)] \text{ for } k > 2$$

On the whole segment of $ID_\xi = \bigcup_n S_n(ID_\xi)$, Arai [3] proves that $ID_\xi \nvdash TI[F_0, <_0, O(\xi+1, 1)]$. Note that $O(\xi+1, 1) := \bigcup_k \omega(\xi, k, 0)$. He shows that the consistency of ID_ξ is provable using tranfinite induction up to $O(\xi+1, 1)$ by the proof reduction method which is originally due to Gentzen-Takeuti. In this section we modify his consistency proof in more delicate manner and prove the following by the cut elimination (proof reduction) method;

$$TI[F_0, <_0, \omega(\xi, k+1, 0)] \vdash Cons(S_k(ID_\xi)) \text{ for } k > 2$$

Our crucial point is to introduce a η-height h_η for each $\eta \preceq \xi$ (Definition 11) and consider a ordinal assignment to a proof $< P, \{h_\eta\}_{\eta \preceq \xi}, d >$ with ξ-sort of height (Definition 13).

For the Gentzen-Takeuti cut elimination procedure to work, Arai [3] formalises his system AI^-_ξ of ξ-times iterated inductive definition in the form of iterated comprehension axiom by using second order free variables. System AI^-_ξ is defined by adding the following principles based on PA.

Definition 5 (System AI_ξ^-, cf. Arai [3])
For any arithmetical form $\mathcal{B}$, the following axioms schemata are added.

$$(Q^\mathcal{B} : right) \quad \frac{\Gamma \to \Delta, \mathcal{B}(X, Q^\mathcal{B}_{\prec t}, t, s)}{\Gamma \to \Delta, Q^\mathcal{B} ts} \quad \text{where } Q^\mathcal{B}_{\prec t} := \{x, y\}(x \prec t \land Q^\mathcal{B} xy)$$

$$(Q^\mathcal{B} : left) \quad t \prec \xi, Q^\mathcal{B} ts \to \mathcal{B}(V, Q^\mathcal{B}_{\prec t}, t, s)$$

We assume that the language contains only $\forall$, $\neg$ and $\land$ for the logical connectives. Then, the definition of lv in the previous section is modified as follows;

Definition 6 (η-level $lv_\eta(A)$ of a formula A with $\eta \preceq \xi$) *For the formula A in the language of AI_ξ^- and an ordinal $\eta \preceq \xi$, the η-level $lv_\eta(A)$ of the formula A is defined inductively as follows, where d is defined in Definition 2 of previous section with using $Q^\mathcal{B}$ instead of $P^\mathcal{U}$ and $d(Xt) := 0$ (for X a predicate variable):*

$lv_\eta(P) := 0$ *for any atom of* L_{PA}.

$lv_\eta(A \land B) := max\{lv_\eta(A), lv_\eta(B)\}$

$$lv_\eta(\forall x A) := \begin{cases} max\{2, lv_\eta(A)\} & \text{if } lv_\eta(A) \geq 1 \\ 0 & \text{if } lv_\eta(A) = 0 \end{cases}$$

$$lv_\eta(\neg A) := \begin{cases} lv_\eta(A) + 1 & \text{if } lv_\eta(A) \geq 1 \\ 0 & \text{if } lv_\eta(A) = 0 \end{cases}$$

$$lv_\eta(Q^\mathcal{B}_t) := \begin{cases} 1 & \text{if } d(Q^\mathcal{B}_t) = \eta \\ 0 & \text{otherwise} \end{cases}$$

$$lv_\eta(t \prec s \land Q^\mathcal{B}_t) := \begin{cases} 1 & \text{if } d(t \prec s \land Q^\mathcal{B}_t) = \eta \\ 0 & \text{otherwise} \end{cases}$$

Note that lv_η for $\eta = \xi$ is the same as lv of the previous section (with using $Q^\mathcal{B}$ instead of $P^\mathcal{U}$ in the definition of lv of the previous section with replacing $\supset$ by $\neg$.)

We can define the fragments $S_k(AI_\xi^-)$ in the same manner as $S_k(ID_\xi))$ as follows.

Definition 7 (the subsystem $S_k(AI_\xi^-)$ of AI_ξ^-) *$S_k(AI_\xi^-)$ is AI_ξ^- except that for every abstract V in $Q^\mathcal{B}$:left and (VJ), $lv_\xi(V) \leq k$ holds.*

ID_ξ is obtained from ID_ξ^i in the previous section by changing the underlying logic from the intuitionistic to the classical. For each formula F of the language of ID_ξ, we define a formula F^* of the language of AI_ξ by substituting $Q^\mathcal{B}$ for all ocurrences of $P^\mathcal{U}$, where

$$\mathcal{B}(X, Y, c_0, c_1) := \forall y(\mathcal{U}(X, Y, c_0, y) \to Xy) \to Xc_1.$$

It is well known that by this $*$, ID_ξ is embeddable into AI_ξ (cf. [3]). Obviously $lv(F) = lv_\xi(F^*)$ holds i.e., ξ-level of a formula remains the same through the above interpretation.

Untill the end of this section, we assume that all formulas occuring in a proof figure of AI_ξ^- are of the following normal form:

Lemma 1 (the normal form of a formula in AI_ξ^-) *For arbitrary formula A of the language of AI_ξ^-, there exists a formula of the following form, called a normal formula, which is eqivalent to A (in LK);*

$$\forall \vec{x_1} \neg \cdots \forall \vec{x_n} \forall \neg \quad \forall \vec{y} D[Q^\mathcal{B} t_1 s_1, \ldots, Q^\mathcal{B} t_m s_m]$$

where $D[*_1, \ldots, *_m]$ is a context of the language of PA. and no quantifier occuring in D bounds any $*_i$ $(1 \leq i \leq m)$ and $lv_\eta(D[Q^\mathcal{B} t_1 s_1, \ldots, Q^\mathcal{B} t_m s_m]) \leq 2$ for any $\eta \preceq \xi$.

Definition 8 (normal proofs) *Let S be a sequent of normal formulas. A normal proof of S is a proof in which $\forall\neg$-left rules are used, instead of $\forall$-left rules in a proof;*

$$\frac{\Gamma - \Delta, A(t_1, \cdots, t_n)}{\forall x_1 \cdots x_n \neg A(x_1, \ldots, x_n), \Gamma - \Delta} \ \forall\neg left$$

Note that the original $\neg$-left rule may also appear in a normal proof.

Lemma 2 *Any provable sequent of normal formulas has a normal proof.*

From now on we assume any $S_k(AI_\xi^-)$-proof to be normal by virtue of the above two lemmata.

Definition 9 *For each formula A . $\eta(A) \preceq \xi$ is defined as $\eta(A) := Max\{\eta \mid lv_\eta(A) \neq 0\}$.*

Definition 10 ($g_\eta(A)$ **with** $\eta \prec \xi$)

$$g_\eta(A) := \begin{cases} g(A) & if\ \eta(A) \geq \eta \\ 0 & if\ \eta(A) < \eta \end{cases}$$

where $g(A)$ denotes the number of logical symbols in A.

We modify the notion of proof with degree $< P, d >$ of Arai [3] into $< P, \{h_\eta\}_{\eta \preceq \xi}, d >$ by introducing ξ-sort of height $\{h_\eta\}_{\eta \preceq \xi}$, as follows:

Definition 11 (A proof with ξ-sort of height $< P, \{h_\eta\}_{\eta \preceq \xi}, d >$) *A proof $< P, d >$ (with degree d) is called a proof with ξ-sort of height $< P, \{h_\eta\}_{\eta \preceq \xi}, d >$ if for each sequent S of P and each ordinal $\eta \preceq \xi$, a natural number $h_\eta(S)$ satisfying the following condition is assigned. We call h_η a η-height.*

0. $h_\eta(S) = 0$ *for every* $\eta \preceq \xi$ *if* S *is the end sequent of* P.

For the last inference I of the form

$$I \ \ \frac{S}{S'}.$$

1. $h_\eta(S) = 0$ *for every* $\eta \preceq \xi$ *if* I *is a substitution.*

2. $h_\eta(S) = h_\eta(S')$ *for every* $\eta \preceq \xi$ *if* I *is an inference except substitution, induction and cut.*

3. $\begin{cases} 1 & h_\eta(S) \geq Max\{h_\eta(S'), g_\eta(D)\} \quad for\ \eta \prec \xi \\ 2 & h_\xi(S) = Max\{h_\xi(S'), lv_\xi(D)\} \end{cases}$
   if I is a cut, where D is the cut formula of the inference I.

4. $\begin{cases} 1 & h_\eta(S) \geq Max\{h_\eta(S'), g_\eta(D)\} + 1 \quad for\ \eta \prec \xi \\ 2 & h_\xi(S) = Max\{h_\xi(S'), lv_\xi(D)\} + 1 \end{cases}$
   if I is an induction.

Definition 12 *For each sequent S of $< P, \{h_\eta\}_{\eta \preceq \xi}, d >$, $\eta(S) \preceq \xi$ is defined as*

$$\eta(S) := \begin{cases} d(I) & if\ S\ is\ the\ upper\ sequent\ of\ the\ substitution\ I \\ Max\{\eta \mid h_\eta(S) \neq 0\} & otherwise \end{cases}$$

The following is an immediate consequence from Definition 12.

Lemma 3 *For any proof with ξ-sort of height $< P, \{h_\eta\}_{\eta \preceq \xi}, d >$ and for any inference I (with a lower sequent S' and a upper sequent S) in $< P, \{h_\eta\}_{\eta \preceq \xi}, d >$,*

$$\eta(S) \geq \eta(S')$$

holds.

Notation 3 *For $i \preceq \xi$ and an ordinal diagram α, an ordinal diagram $\omega(i, n, \alpha)$ is defined inductively as follows.*

- $\omega(i, 0, \alpha) := \alpha$
- $\omega(i, n+1, \alpha) := (i, \omega(i, n, \alpha))$

Definition 13 (ordinal assignment) *Let I be an inference of the form*

$$I \quad \frac{S_1 \quad S_2}{S}.$$

Then $O(S)$ is defined as follows:

1. *When I is a cut,*

$$O(S) := \omega(\eta(S), k - h_{\eta(S)}(S), c[\omega(\eta(S_1), h_{\eta(S_1)}(S_1), O(S_1)\#O(S_2)))])$$

. *Here $k := Max\{h_{\eta(S)}(T) \mid T \text{ is above } I\}$ and $c[*] := \omega(\gamma_1, k_1, \omega(\gamma_2, k_2, \ldots, \omega(\gamma_n, k_n, *)))$, where $\{\gamma_1, \ldots, \gamma_n\} := \{\gamma \mid \eta(S) < \gamma < \eta(S_1) \text{and } h_\gamma(T) \neq 0 \text{ for some } T \text{ above } I\}$ with $\gamma_1 < \cdots < \gamma_n$ and $k_i := Max\{h_{\gamma_i}(T) \mid T \text{ is above } I\}$.* [1]

2. *When I is a logical inference,*
$O(S) := O(S_1)\#O(S_2)\#0$

3. *When I is a structural inference,*
$O(S) := O(S_1)\#O(S_2)$

4. *When I is a substitution,*
$O(S) := (d(I), O(S_1))$

Theorem 1 *The transfinite induction on $\omega(\xi, k+1, 0)$ is unprovable in $S_k(AI_\xi^-)$ for $k > 2$.*

Proof.
We refine the proof reduction process of Arai [3] to define the reduction process for $S_k(AI_\xi^-)$ ($k > 2$), and show that the well-orderness of $\omega(\xi, k+1, 0)$ implies the termination of the reduction process, hence the consistency of $S_k(AI_\xi^-)$. Then the above theorem follows from Gödel's incompleteness theorem.
(preparation)
Without loss of generality, we assume that all logical initial sequents of the form $p \to p$ where p is an atomic and that there exists no free variables which is not used as an eigenvariable.
(elimination of initial sequents in the end-piece) As usual.
(elimination of weakning) elimination of weakning known in the usual way (cf. Takeuti [14]) dose work not only for a weakinig in end-piece but also for a more general weakning with such a weakning formula D as the bundle $\mathcal{I}$ (cf. p78 of [14]) which begins with D ends with a cut formula D and no logical inference affect $\mathcal{I}$.

[1] In the case where $\eta(S) = \eta(S_1)$, $c[*]$ is $*$ and $O(S) := \omega(\eta(S), k + h_{\eta(S)}(S_1) - h_{\eta(S)}(S), O(S_1)\#O(S_2))$.

(elimination of the mathematical induction rule) As usual.
Then from sublemma 12.9 of [14], there exists a suitable cut J in the end piece of $< P, \{h_\eta\}_{\eta \preceq \xi}, d >$. Let I_1 and I_2 be boundary logical inferences whose principal formulas are ancestors of left and right cut formulas of J.
We shall demonstrate following three essential cases both for limit ordinal ξ and for successor ordinal ξ;
(Case 1) The case where the cut formula $C := A \wedge B$ with $\eta(C) \prec \xi$:
Let K (whose lower sequent is T and whose upper sequent is T_1) denotes the uppermost inference below J such that either (i) or (ii) holds;

$$\eta(T) = \eta(A) \wedge (h_{\eta(A)}(S_1) > h_{\eta(A)}(T)) \cdots (i)$$

$$\eta(T) < \eta(A) \cdots (ii)$$

where A is the auxiliary formula of I_1 and I_2
$< P, \{h_\eta\}_{\eta \preceq \xi}, d >$ is as follows:

$$\begin{array}{c}
\begin{array}{cc} \vdots\quad\vdots & \vdots \\ \dfrac{S_1^{I_1}\quad S_2^{I_1}}{S^{I_1}}\,I_1 & \dfrac{S_1^{I_2}}{S^{I_2}}\,I_2\\ \vdots & \vdots \end{array}\\
\dfrac{S_1^J \qquad\qquad S_2^J}{S^J}\,J\\ \vdots\\ \dfrac{T_1}{T}\,K\\ \vdots \end{array}
\qquad
\begin{array}{rc}
S_1^{I_1}: & \Gamma_1 \rightarrow \Delta_1, A_1 \\
S_2^{I_1}: & \Gamma_2 \rightarrow \Delta_2, B_1 \\
S^{I_1}: & \Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, A_1 \wedge B_1 \\
S_1^{I_2}: & A_3, \Pi_3 \rightarrow \Lambda_3 \\
S^{I_2}: & A_3 \wedge B_3, \Pi_3 \rightarrow \Lambda_3 \\
S_1^J: & \Gamma \rightarrow \Delta, A \wedge B \\
S_2^J: & A \wedge B, \Pi \rightarrow \Lambda \\
S^J: & \Gamma, \Pi \rightarrow \Delta, \Lambda \\
T: & \Phi \rightarrow \Psi
\end{array}$$

$< P', \{h'_\eta\}_{\eta \preceq \xi}, d' >$ is as follows, where $\tilde{I}_1$ and $\tilde{I}_2$ are weakening-right (with a weakening formula A_1) and weakening-left (with a weakening formula A_3) respectively;
;

$$\begin{array}{c}
\begin{array}{cc}
\begin{array}{c}
\begin{array}{cc} \vdots & \\ \dfrac{S_1^{I_1}}{S^{\tilde{I}_1}}\,\tilde{I}_1 & \dfrac{\vdots}{\quad}\,I_2 \\ \vdots & \vdots \end{array}\\
\dfrac{S_1^* \qquad\qquad S_2^J}{S_1^*}\,J\\ \vdots\\ \dfrac{T_1^*}{U_1}\,K
\end{array}
&
\begin{array}{c}
\begin{array}{cc} & \vdots \\ \dfrac{\vdots}{\quad}\,I_1 & \dfrac{S_1^{I_2}}{S^{\tilde{I}_2}}\,\tilde{I}_2 \\ \vdots & \vdots \end{array}\\
\dfrac{S_1^J \qquad\qquad S_2^*}{S_2^*}\,J\\ \vdots\\ \dfrac{T_1^{**}}{U_2}\,K
\end{array}
\end{array}\\
\hline
T^* \quad I\\ \vdots
\end{array}
\qquad
\begin{array}{rc}
S^{\tilde{I}_1}: & \Gamma_1 \rightarrow \Delta_1, A_1, A_1 \wedge B_1 \\
S_1^*: & \Gamma \rightarrow \Delta, A, A \wedge B \\
S^{\tilde{I}_2}: & A_3, A_3 \wedge B_3, \Pi_3 \rightarrow \Lambda_3 \\
S_2^*: & A \wedge B, A, \Pi \rightarrow \Lambda \\
U_1: & \Phi \rightarrow \Psi, A \\
U_2: & A, \Phi \rightarrow \Psi \\
T^*: & \Phi, \Phi \rightarrow \Psi, \Psi
\end{array}$$

(case 1.1): The case where (i) holds. Then for any sequent T' between S_1 and T, $\eta(T') \geq \eta(A)$ holds.
(case 1.1.1) $\eta(T_1) = \eta(T)$
$O_{P'}(T^*) <_0 O_P(T)$ is checked as usual way.
(case 1.1.2) $\eta(T_1) > \eta(T)$
special case of (case 1.2)
(Case 1.2): The case where (ii) holds. Then $\eta(T) < \eta(A) \leq \eta(T_1)$ holds. We assign

$$h'_\eta(U_1) := \begin{cases} h_\eta(T) & \text{if } \eta < \eta(T) \\ g(A) & \text{if } \eta = \eta(A) \\ 0 & \text{otherwise} \end{cases}, \text{ and } h'_\eta(T_1^*) := h'_\eta(T_1^{**}) := h_\eta(T_1) \text{ for all } \eta \preceq \xi.$$

Hence $\eta(U_1) = \eta(A)$ holds. On the other hand, there exist contexts a and b such

that $O_P(T) = \omega(\eta(T), k - h_{\eta(T)}(T), a[\omega(\eta(A), k_i, b[\alpha_1 \# \alpha_2])])$,
$O_{P'}(U_1) = \omega(\eta(U_1), m - h_{\eta(U_1)}(U_1), b[\alpha_1' \# \alpha_2]) = \omega(\eta(A), m - g(A), b[\alpha_1' \# \alpha_2])$ and
$O_{P'}(T^*) = \omega(\eta(T), k' - h_{\eta(T)}(T), a[\omega(\eta(U_1), g(A), O_{P'}(U_1) \# O_{P'}(U_2))])$
Since $\omega(\eta(A), k_i, b[\alpha_1 \# \alpha_2]) > \omega(\eta(U_1), g(A), O_{P'}(U_1) \# O_{P'}(U_2))$, $O_{P'}(T^*) <_0 O_P(T)$ holds.

(Case 2) The case where cut formula is $\forall \vec{x} \neg B(\vec{x})$:
$< P, \{h_\eta\}_{\eta \preceq \xi}, d >$ is as follows; here I_2 is $\forall\neg$left.

$$\begin{array}{c} \dfrac{S_1^{I_1}}{S^{I_1}}\, I_1 \qquad \dfrac{S_1^{I_2}}{S^{I_2}}\, I_2 \\ \vdots \qquad\qquad \vdots \\ \dfrac{S_1^J \qquad S_2^J}{S^J}\, J \\ \vdots \\ \overline{T}\, K \\ \vdots \\ \overline{\qquad} \end{array} \qquad \begin{array}{rl} S_1^{I_1}: & \Gamma_1 \rightarrow \Delta_1, \neg B(\vec{x}) \\ S^{I_1}: & \Gamma_1 \rightarrow \Delta_1, \forall \vec{x} \neg B(\vec{x}) \\ S_1^{I_2}: & \Pi_1 \rightarrow \Lambda_1, \neg B(\vec{t}) \\ S^{I_2}: & \forall \vec{x} \neg B(\vec{x}), \Pi_1 \rightarrow \Lambda_1 \\ S_1^J: & \Gamma \rightarrow \Delta, \forall \vec{x} \neg B(\vec{x}) \\ S_2^J: & \forall \vec{x} \neg B(\vec{x}), \Pi \rightarrow \Lambda \\ T: & \Phi \rightarrow \Psi \end{array}$$

$< P', \{h'_\eta\}_{\eta \preceq \xi}, d' >$ is as follows, where $\tilde{I}_1$ and $\tilde{I}_2$ are weakening-right and weakening-left (respectively) with weakening formulas $\forall \vec{x} \neg B(\vec{x})$. Note that by virtue of (preparation) and (elimination of weakening), any formula of the form $\neg B(\vec{x})$ which is an ancestor of the auxiliary formula of I_1 is a descendant of principal formulas of an inference $\neg$-right. Hence the following $S_1^{I_1}(\vec{x})$ can be obtained.

$$\begin{array}{c} \vdots \\ \dfrac{\dfrac{\dfrac{S_1^{I_1}(t)}{S^{I_1}(t)}\,\tilde{I}_1}{\vdots \atop S_1^J * (t)} \qquad \dfrac{\vdots}{\dfrac{\overline{\qquad}\, I_2}{\vdots \atop S_2^J}}}{S_1^*(t)}\, J \end{array} \qquad\qquad \begin{array}{c} \vdots \\ \dfrac{\dfrac{\overline{\qquad}\, I_1}{\vdots \atop S_1^J} \qquad \dfrac{\dfrac{S_1^{I_2}}{S^{I_2}}\,\tilde{I}_2}{\vdots \atop S_2^J *}}{S_2^*}\, J \end{array}$$

$$\begin{array}{c} \dfrac{\dfrac{\vdots}{U_1(t)}\, K \qquad \dfrac{\vdots}{U_2}\, K}{T^*}\, N \\ \vdots \\ \overline{\qquad} \end{array}$$

$$\begin{array}{rl} S_1^{I_1}(\vec{x}): & B(\vec{x}), \Gamma_1 \rightarrow \Delta_1 \\ S^{I_1}(\vec{x}): & B(\vec{x}), \Gamma_1 \rightarrow \Delta_1, \forall \vec{x} \neg B(\vec{x}) \\ S_1^J * (\vec{x}): & B(\vec{x}), \Gamma \rightarrow \Delta, \forall \vec{x} \neg B(\vec{x}) \\ S_1^*(\vec{x}): & B(\vec{x}), \Gamma, \Pi \rightarrow \Delta, \Lambda \\ S_2^J *: & \forall \vec{x} \neg B(\vec{x}), \Pi \rightarrow \Lambda, B(\vec{t}) \\ S_2^*: & \Gamma, \Pi \rightarrow \Delta, \Lambda, B(\vec{t}) \\ S_1^{I_2}: & \Pi_1 \rightarrow \Lambda_1, B(\vec{t}) \\ S^{I_2}: & \forall \vec{x} \neg B(\vec{x}), \Pi_1 \rightarrow \Lambda_1, B(\vec{t}) \\ U_1(\vec{x}): & B(\vec{x}), \Phi \rightarrow \Psi \\ U_2: & \Phi \rightarrow \Psi, B(\vec{t}) \\ T^*: & \Phi, \Phi \rightarrow \Psi, \Psi \end{array}$$

Since $lv_{\eta(B(\vec{x}))}(\forall \vec{x} \neg B(\vec{x})) > lv_{\eta(B(\vec{x}))}(B(\vec{x}))$ holds, $O(P') <_0 O(P)$ is checked as the usual way.

(Case 3)The case where the cut formula of J is $Q^{\mathcal{B}} ts$:

$< P, \{h_\eta\}_{\eta \preceq \xi}, d >$ is as follows, where K (with the lower sequent T) denotes the upper most inference below J such that $\eta(T) \preceq d(\mathcal{B}(X, Q_{\prec t}, t, s)) := i$;

Let T_1 denote such upper sequent of K that is below J.

$$
\begin{array}{l}
\vdots \\
\dfrac{S_1^{I_1}}{S^{I_1}}\, I_1 \qquad S \\
\vdots \qquad\quad\;\; \vdots \\
\dfrac{S_1^J \qquad S_2^J}{S^J}\, J \\
\qquad \vdots \\
\dfrac{T1 \qquad (T_2)}{T}\, K \\
\qquad \vdots
\end{array}
\qquad
\begin{array}{lc}
S: & t_2 \prec \xi, Qt_2s_2 \rightarrow \mathcal{B}(V, Q_{\prec t_2}, t_2, s_2) \\
S_1^{I_1}: & \Gamma_1 \rightarrow \Delta_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
S^{I_1}: & \Gamma_1 \rightarrow \Delta_1, Qt_1s_1 \\
S_1^J: & \Gamma_2 \rightarrow \Delta_2, Qts \\
S_2^J: & Qts, \Pi \rightarrow \Lambda \\
S^J: & \Gamma_2, \Pi \rightarrow \Delta_2, \Lambda_2 \\
T_1: & \Phi_1 \rightarrow \Psi_1 \\
T_2: & \Phi \rightarrow \Psi
\end{array}
$$

$$O_P(T) = \omega(\eta(T), k - h_{\eta(T)}(T), c[\omega(\eta(T_1), h_{\eta(T_1)}(T_1), O_P(T_1)\#O_P(T_2))]).$$

$< P', \{h'_\eta\}_{\eta \preceq \xi}, d' >$ is as follows, where $\tilde{I}_1$ is weakening-right with a weakening formula $Q^{\mathcal{B}}t_1s_1$;

$$
\begin{array}{l}
\vdots \\
\dfrac{S_1^{\tilde{I}_1}}{S^{\tilde{I}_1}}\, \tilde{I}_1 \qquad S \\
\vdots \qquad\quad\;\; \vdots \\
\dfrac{S_1^J * \qquad S_2^J}{S^J *}\, J \\
\qquad \vdots \\
\dfrac{\dfrac{T_1^* \qquad (T_2)}{T^*}}{\tilde{T}^*}\, sub \\
\vdots
\end{array}
$$

$$
\begin{array}{l}
\dfrac{S_1^{I_1}}{S^{I_1}}\, I_1 \\
\vdots \qquad\quad\;\; \vdots \\
\dfrac{S_1^J \qquad S_2^J}{S^J}\, J \\
\qquad \vdots \\
\dfrac{T1 \qquad (T_2)}{T}\, K \\
\qquad \vdots
\end{array}
\qquad
\begin{array}{lc}
S_1^{\tilde{I}_1}: & \Gamma_1 \rightarrow \Delta_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
S^{\tilde{I}_1}: & \Gamma_1 \rightarrow \Delta_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1), Qt_1s_1 \\
S_1^J *: & \Gamma_2 \rightarrow \Delta_2, Qts, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
S^J *: & \Gamma_2, \Pi \rightarrow \Delta_2, \Lambda_2, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
T_1^*: & \Phi_1 \rightarrow \Psi_1, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
T^*: & \Phi \rightarrow \Psi, \mathcal{B}(X, Q_{\prec t_1}, t_1, s_1) \\
\tilde{T}^*: & \Phi \rightarrow \Psi, \mathcal{B}(V, Q_{\prec t_1}, t_1, s_1)
\end{array}
$$

We assign $\{h'_\eta\}_{\eta \preceq \xi}$ as follows;

- $h'_\eta(\tilde{T}^*) := h_\eta(S)$ for all $\eta \preceq \xi$
- $h'_\eta(T_1^*) := h_\eta(T_1)$ for all $\eta \preceq \xi$.
- $h'_\eta(T^*) := \begin{cases} h_\eta(T) & if \;\; \eta \leq \eta(T) \\ 0 & \text{otherwise} \end{cases}$

$O_{P'}(\tilde{T}^*) = (i, O_{P'}(T^*))$

$O_{P'}(T^*) = \omega(\eta(T^*), l - h'_{\eta(T^*)}(T^*), d[\omega(\eta(T_1^*), h_{\eta(T_1^*)}(T_1^*), O_{P'}(T_1^*)\#O_{P'}(T_2))])$
Note that $\eta(T^*) = i$. Obviously $k = l$ from the figure of P'. And from the above assignment h', $c[*] = \omega(\gamma_1, k_1, \ldots, \omega(\gamma_s, k_s, d[*]))$ with $\gamma_s < i = \gamma_{s+1}$. Hence $O(P') <_0 O(P)$ holds.
□

The following Corollary is immediate from the above theorem and the fact that $S_k(ID^i_\xi)$ is a subsystem of $S_k(AI^-_\xi)$ under the interpretation * (cf. the paragraph after Definition 7).

Corollary 1 *The transfinite induction on $\omega(\xi, k+1, 0)$ is unprovable in $S_k(ID^i_\xi)$ for $k > 2$.*

Proof. As remarked after Definition 7, ξ-level does not change under the interpretation of an AI_ξ-formula to an ID_ξ-formula. Hence the Corollary is obvious.
□

Theorem 2 (Main Theorem)

$$|S_k(ID^i_\xi(\mathcal{U}_0))| = |S_k(ID^i_\xi)| = |S_k(AI^-_\xi)| = |\omega(\xi, k+1, 0)|_{<_0} \text{ with } k > 2.$$

Remark 2: Our system $S_k(ID_\xi)$ can be reformulated by means of the alternation complexity of quantifiers when we include $\exists$ in our language. Here, a normal formula is of the form $Q_1\vec{x_1}\tilde{Q}_1\vec{y_1}\cdots Q_n\vec{x_n}\tilde{Q}_n\vec{y_n}\ \forall\vec{y}D[P^{\mathcal{U}}t_1s_1, \ldots, P^{\mathcal{U}}t_ms_m]$, where $D[*_1, \ldots, *_m]$ is a context of the language of PA with no quantifier occuring in D bounds any $*_i$ $(1 \le i \le m)$, and $\{Q_j, \tilde{Q}_j\} = \{\forall, \exists\}$ $(j = 1, \ldots, m)$. lv is essentially the same as lv_ξ except that we measure the alternation complexity of quantifiers instead of nestedness complexity of negations; namely,

$$lv(D[P^{\mathcal{U}}t_1s_1, \ldots, P^{\mathcal{U}}t_ms_m]) := \begin{cases} 1 & \text{if all } P^{\mathcal{U}}t_is_i\ (i = 1, \ldots, m) \text{ is positive in } D \\ 2 & \text{otherwise} \end{cases}$$

Then the lv of above normal formula is $n+i$ if $\tilde{Q}_n = \forall$ and $n+1+i$ if $\tilde{Q}_n = \exists$, where $i := lv(D[P^{\mathcal{U}}t_1s_1, \ldots, P^{\mathcal{U}}t_ms_m])$. $S'_k(ID_\xi)$ is defined in the same way as the former definition of $S_k(ID_\xi)$ with using the above new notation of lv. It is easily seen that $S'_k(ID_\xi)$ is equivalent to $S_k(ID_\xi)$. In paticular $|S'_k(ID_\xi)| = |\omega(\xi, k+1, 0)|_0$ with $k > 2$.

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