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A Hierarchy of the Fragments of the System of Inductive Definition (Preliminary Report)

Masahiro Hamano\textsuperscript{*} and Mitsuhiro Okada\textsuperscript{†}
Department of Philosophy
Keio University, Tokyo

1 introduction

Gentzen [7] proved the consistency of $PA$ (Peano Arithmetic) by using the transfinite induction up to the first epsilon number $\epsilon_0$. Here $\epsilon_0$ is $\lim_k \omega_k$, where $\omega_0 = 0$ and $\omega_{k+1} = \omega^\omega$. Later in [8] he proved that the accessibility (i.e., transfinite induction) proof up to any ordinal less than $\epsilon_0$, e.g., $\omega_k$ for any natural number $k$, is provable in $PA$.

In his [8] the nestedness complexity of implications used in the accessibility proof increases by one while the accessibility of one higher $\omega$-tower $\omega_{k+1}$ is proved from the accessibility of $\omega_k$. Hence by considering Gentzen's work [7, 8] a natural question arises: does the hierarchy of $\omega$-towers, $\{\omega_k\}_{k=1,2,\ldots}$, correspond exactly to a certain hierarchy of fragments of $PA$?

Mints [10] answered this question by estimating the least upper bounds of accessibility ordinals for the fragments of $PA$, where the fragments are defined by means of the number of alternations of quantifiers, using one quantifier system developed in his former paper [9]. (Shirai [13] also gave a similar result by means of the number of quantifiers.)

The purpose of our paper is to investigate in a similar correspondence (between the hierarchy of critical ordinals and the hierarchy of fragment systems) for the system of $\xi$-iterated Inductive Definition $ID_\xi$ [6]. We first analyze in Section 2 Arai's optimal accessibility proof for $ID_\xi$ ([3]) to obtain a hierarchy of accessible ordinals for the fragments of intuitionistic $ID_\xi$, where the fragments are defined in terms of the nestedness complexity of implications. Then we show in Section 3 the least upper bounds of accessible ordinals (i.e., the critical ordinals) for those fragments, by analyzing Takeuti-Arai's consistency proofs of $ID_\xi$ ([3]). In fact, for the upper bounds proof we use the fragments of classical $ID_\xi$ in terms of the nestedness complexity of classical negations. Since the fragments of $ID_\xi$ obtained by means of the number of alternations of quantifiers (in a prenex normal form) are also characterized by the nestedness complexity of negations with the help of universal quantifiers (by representing an existential quantifier $\exists$ by means of $\neg\forall\neg$), our result for $ID_\xi$ corresponds to Mints' ([10]) for $PA$.

\textsuperscript{*}慶應義塾大学哲学科 日本学術振興会特別研究員 亀野 正浩 hamano@abelard.flet.mita.keio.ac.jp
\textsuperscript{†}慶應義塾大学哲学科 周田 光弘 mitsu@abelard.flet.mita.keio.ac.jp
2 Provability of transfinite inductions on $\omega(\xi, k, 0)$ in subsystems of $S_k(ID^I_\xi(U_0))$

Let $(I, \prec)$ be the well ordered system whose order type is ordinal $\xi + 1$. Arai [1] proved the well ordering of Takeuti's system of ordinal diagram $O(\xi + 1, 1)$ in the system $ID^I_\xi$ (the intuitionistic system of $\xi$-times iterated inductive definition).

In this chapter we introduce a hierarchy of fragments $S_k(ID^I_\xi)$ of $ID^I_\xi$ based on the nestedness complexity of implications, and observe Arai’s well ordering proof of [1] on these fragments.

Now we recall the definitions of $ID^I_\xi(U)$ and $ID^I_\xi$ of Feferman [6].

Definition 1 (System $ID^I_\xi(U)$ and $ID^I_\xi$, cf. Feferman [6])

For any positive operator form $U$, $ID^I_\xi(U)$ is obtained from $PA$ by adding the following axiom schemata.

\[(P_\xi, 1) \quad \forall x < \xi (A(P^\xi_x, P^\xi_x, x) \subseteq P^\xi_x)\]

\[(P_\xi, 2) \quad \forall x < \xi (\forall(V, P^\xi_x, x) \subseteq V \supset P^\xi_x \subseteq V)\]

\[(TI) \quad \text{Prog}[I, \prec, V] \rightarrow (I \subseteq V)\]

where $P^\xi_{\alpha a} := \{x, y|(x < a \land P^\xi x y)\}$

$ID^I := \bigcup\{ID^I_\xi | \ U \text{ is a positive operator form}\}$

The starting point of Arai’s well ordering proof is to define the notion of accessibility with respect to $<_i$ for $i < \xi$ (cf. §26 [14]) by using the set constants $A_i$ which is definable in $ID^I_{\xi}(U_0)$ with the following $U_0$;

\[(A.1) \quad \forall i < \xi \text{Prog}[F_i, <_i, A_i] \quad \forall i < \xi \text{Prog}[F_i, <_i, A_i]\]

\[(A.2) \quad \forall i < \xi (\forall(x, y, z) \rightarrow A_i \subseteq V) \quad \forall i < \xi (\forall(x, y, z) \rightarrow A_i \subseteq V)\]

where $U_0$ is a $X$-positive operator form defined as $U_0(\forall, \forall, i, \mu) := T(i, \nu, Y) \land \forall x \prec \mu(\forall(x, y, z) \rightarrow \beta(y) \rightarrow \beta(z))$, and $F_i(\mu) := \forall i < \forall x \prec \mu i A_j(\nu)$ (the intended meaning of $F_i(\mu)$ is that $\mu$ is an $i$-fan (cf. Definition 26.16 [14])).

Remember that $ID^I_{\xi}(U)$ has the mathematical induction of the following form;

\[(VJ) \quad \forall(V(0) \rightarrow \forall(x \rightarrow V(x'))) \rightarrow V(t)\]

The above $ID^I_{\xi}(U_0)$ is the specific subsystem of the system $ID^I_{\xi}$ of Inductive Definition in which the induction schemata are used only for the accessibility predicate $A_i$ of ordinals.

We consider the subsystem $S_k(ID^I_{\xi}(U_0))$ of $ID^I_{\xi}(U_0)$ where each abstract $V$ in $(A.2)_{\xi}, (TI)_{\xi}$ and $(VJ)$ is restricted to that of level $lv(V) \leq k$; where $lv(V)$ is defined by the definition below.

We introduce the notion of level of $A (lv(A))$ for a formula $A$ to express, roughly speaking, the implicational complexity of $A$. We assume that the language contains only $\forall$, $\exists$, $\cup$ and $\land$ for the logical connectives in this section.

We first recall the degree $d$ of a formula in the language of $ID^I_{\xi}(U)$ defined in Arai [3], which intends to indicate how many times inductive definition is applied.

Definition 2 (cf. Def 2.4 in Arai [3])

\[d(t = s) = 0 \text{ for all term } t, s \text{ and predicate variable } X.\]

\[d(P^\xi x t s) = \begin{cases} i + 1 & \text{if } t \text{ is a closed term whose value is } i < \xi. \\ \xi & \text{otherwise} \end{cases}\]
Definition 3 (level lv(A) of formula A in the language of $ID^j(U)$) For the formula A in the language of $ID^j(U)$, the level lv(A) of the formula A is defined inductively as follows:

\[
\begin{align*}
    lv(P) &:= 0 \text{ for any atom of the language of PA.} \\
    lv(A \land B) &:= \max\{lv(A), lv(B)\} \\
    lv(\forall x A) &:= \left\{ \begin{array}{ll}
    \max\{2, lv(A)\} & \text{if } lv(A) \geq 1 \\
    0 & \text{if } lv(A) = 0
    \end{array} \right. \\
    lv(A \lor B) &:= \left\{ \begin{array}{ll}
    \max\{lv(A) + 1, lv(B)\} & \text{if } lv(A) \geq 1 \\
    0 & \text{if } lv(A) = 0
    \end{array} \right. \\
    lv(P^d t) &:= \left\{ \begin{array}{ll}
    1 & \text{if } d(P^d t) = \xi \\
    0 & \text{otherwise}
    \end{array} \right. \\
    lv(t < s \land P^d t) &:= \left\{ \begin{array}{ll}
    1 & \text{if } d(P^d t) = \xi \\
    0 & \text{otherwise}
    \end{array} \right.
\end{align*}
\]

The subsystems $S_k(ID^j(U))$ and $S_k(ID^j(U))$ of $ID^j(U)$ and $ID^j(U)$ are defined in terms of level lv as follows;

Definition 4 (the subsystem $S_k(ID^j(U))$ of $ID^j(U)$) $S_k(ID^j(U))$ is $ID^j(U)$ except that for every abstract $V$ in $(A.2)_k$, $(T1)_k$ and $(VJ)$, $lv(V) \leq k$ holds.

$S_k(ID^j(U)) := \bigcup \{S_k(ID^j(U)) \mid U \text{ is a positive operator form} \}$

The following notation is introduced;

Notation 1 Let $TI[\alpha, \gamma, \mu]$ denote the schema defined as $TI[\alpha, \gamma, \mu] := \alpha(\mu) \land (Prog[\alpha, \gamma, V] \land V \gamma(\mu, \nu) \land \alpha(\nu) \land V(\nu))$. And $TI[\alpha, \gamma, \mu]_Q$ is the result of $TI[\alpha, \gamma, \mu]$ by substituting $Q$ for $V$.

Notation 2 $\omega(\xi, 0, \alpha) := \alpha$ and $\omega(\xi, n + 1, \alpha) := (\xi, \omega(\xi, n, \alpha))$.

Then by checking Arai's well ordering proof of $O(\xi + 1, 1)$ [1] carefully, Proposition 1 is easily observed.

Proposition 1 For a formula $Q$ with $lv(Q) \leq 2$ and $k > 2$, $TI[F_0, <_0, \omega(\xi, k, 0)]$ is provable in $S_k(ID^j(U_0))$. Namely, the ordinal $\omega(\xi, k, 0)$ is accessible in $S_k(ID^j(U_0))$ with respect to $<_0$.

Proof. We follow Arai's [1].

We only consider the case in which $\xi$ is a limit. (See Remark after Proposition 2 for the successor $\xi$ case.) Let $\bigcap_k A_k := \{i\}_{\forall k < \xi} A_k(\mu)$. In Lemma 3 of [1] $(T1)_k$ is used with the abstract $\{i\} Prog[F_i, <_i, \bigcap_k A_k] := \{i\} \forall x(F_i(x) \land \forall y <_i x(F_i(y) \rightarrow \bigcap_k A_k(x)))$, here $lv(Prog[F_i, <_i, \bigcap_k A_k(\mu)]) = 3$. Let $A := \bigcap_i A_i$ and $R(\nu) := \forall \mu <_\xi (i, \nu)(F_i(\mu) \rightarrow \bar{A}(\mu))$. In Lemma 4 of [1] $(A.2)_k$ is used with the abstract $\{i\} R(\xi) := \forall \mu <_\xi (i, x)(F_i(\mu) \rightarrow \bar{A}(\mu))$ (with $lv(R(\xi)) = 2$) and $(T1)_k$ is used with the abstract $\{i\} R_0(\xi) := \forall \mu <_\xi (i, 0)(F_i(\mu) \rightarrow \bar{A}(\mu))$ (with $lv(R_0(\xi)) = 2$).

Then in Lemma 5 of [1] it is shown that $TI[F_0, <_\xi, (\xi, 0)]_Q$ is provable in $ID^j(U_0)$ for each unary predicate $Q(x)$ in $ID^j(U)$; In the case where $lim(\xi), (A.2)_k$ are used.
with the abstract \( \{x\}(x \prec \xi (i,0) \rightarrow Q(x)) \) for all \( i < \xi \) (with level \( lv(Q) \)). In the case where \( \text{Suc}(\xi) \), \((A.2)_\xi\) is used with the abstract \( \{x\}(x \prec (\xi,0) \rightarrow Q(x)) \) (with level \( lv(Q) \)).

Hence until now it is observed that

\[(I) \quad S_{\text{Max}}(3, lv(Q))(ID^*_\xi(U_0)) \vdash TI[F_\xi, \prec \xi, (\xi,0)]_Q.\]

From \((I)\) it is derived in the way familiar by Gentzen [8] that

\[(II) \quad S_{k+3}(ID^*_\xi(U_0)) \vdash TI[F_\xi, \prec \xi, \omega(\xi, k+3,0)]_Q \text{ with } lv(Q) \leq 2 \text{ and } k \geq 0.\]

Let us observe the proof of \((II)\). In Lemma 7 of [1] it is shown that \( \text{Prog}[F_\xi, \prec \xi, Q] \rightarrow \text{Prog}[F_\xi, \prec \xi, s[Q]] \), where \( s[Q] \) is a jump operator defined as \( s[Q](\mu) := \forall \rho(F_\xi(\rho) \rightarrow \forall \nu < \xi \rho(F_\xi(\nu) \rightarrow Q(\nu))) \rightarrow \forall \nu < \xi \rho + (\xi,\mu)^{\xi}(F_\xi(\nu) \rightarrow Q(\nu))) \), where \( \lambda \nu. \mu + \nu^\xi \) is a primitive recursive function which is a generalization of \( \lambda \nu. \nu + \omega^\mu \) of Gentzen [8] and defined in [1] as follows;

- If \( \mu = 0 \), then \( \mu + \nu^\xi = \nu + \nu^\xi = \nu \)
- Suppose \( \mu \neq 0 \) and \( \nu \neq 0 \) and \( \mu \equiv \mu_1 \# \cdots \# \mu_m \) with \( \mu_1 \geq \xi \cdots \geq \xi \mu_m \neq 0 \)
  \( \nu = \nu_1 \# \cdots \# \nu_n \) with \( \nu_1 \geq \xi \cdots \geq \nu_n \neq 0 \)

Let \( l \) be the number such that \( 0 < l < m \) and \( \mu_l \leq \xi \nu_1 < \xi \mu_{l+1} \),

then \( \mu + \nu^\xi := \mu_1 \# \cdots \# \mu_l \nu_1 \# \cdots \# \nu_n \)

Note that \( lv(s^n[Q]) = n + \text{Max}(2, lv(Q)) \) with \( n \geq 1 \), where \( s^n[Q] := s[s'[\cdots s[Q] \cdots ]] \).

Let us sketch the proof of \( \text{Prog}[F_\xi, \prec \xi, Q] \rightarrow \text{Prog}[F_\xi, \prec \xi, s[Q]] \) due to Gentzen [8], where a mathematical induction of the level \( \leq lv(Q) \) is used;

Assume

\[ \text{Prog}[F_\xi, \prec \xi, Q] \quad \ldots (1) \]

\[ F_\xi(x) \land \forall y < \xi x(F_\xi(y) \rightarrow s[Q](y)) \quad \ldots (2) \]

We have to show \( s[Q](x) \). So assume further

\[ F_\xi(\rho) \quad \ldots (3) \]

\[ \forall \nu < \xi \rho(F_\xi(\nu) \rightarrow Q(\nu)) \quad \ldots (4) \]

\[ \nu < \xi \rho(\xi, x)^\xi \land F_\xi(\nu) \quad \ldots (5) \]

Under the above assumptions \((1) \sim (5)\), we have to show \( Q(\nu) \).

Consider the case where \( x \neq 0 \). Since \( \nu < \xi \rho(\xi, x)^\xi \), there exists primitive recursive functions \( f \) and \( g \) such that \( \nu < \xi \rho(\xi, f(x, \nu, \rho)) : g(x, \nu, \rho) \) with \( f(x, \nu, \rho) \leq \xi x \) and \( F_\xi(f(x, \nu, \rho)) \). From \((2)\), \( s[Q](f(x, \nu, \rho)) \). Then a universal instantiation

with \( \rho(\xi, f(x, \nu, \rho))^\xi \cdot n \) (note that \( \rho(\xi, f(x, \nu, \rho))^\xi \cdot n < \xi \rho(\xi, x)^\xi \)) for an arbitrary \( n \) allows the following:

\[ F_\xi(\rho(\xi, f(x, \nu, \rho))^\xi \cdot n) \rightarrow \forall \eta < \xi \rho(\xi, f(x, \nu, \rho))^\xi \cdot n(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta < \xi (\rho(\xi, f(x, \nu, \rho))^\xi \cdot n) \Rightarrow \exists \xi(\xi, f(x, \nu, \rho))^\xi \cdot n(F_\xi(\eta) \rightarrow Q(\eta)) \ldots (6) \]

From \( F_\xi(\rho(\xi, f(x, \nu, \rho))^\xi \cdot n) \) (from \((5)\)) and the property of \( \text{Suc} \), the following holds;

\[ \forall \eta < \xi \rho(\xi, f(x, \nu, \rho))^\xi \cdot n(F_\xi(\eta) \rightarrow Q(\eta)) \rightarrow \forall \eta < \xi \rho(\xi, f(x, \nu, \rho))^\xi \cdot \text{Suc}(n)(F_\xi(\eta) \rightarrow Q(\eta)) \ldots (7) \]

Then mathematical induction with abstract \( \{n\}(\forall \eta < \xi \rho(\xi, f(x, \nu, \rho))^\xi \cdot n(F_\xi(\eta) \rightarrow Q(\eta))) \), whose level is \( \text{Max}(2, lv(Q)) \), implies (with \((4)\)) \( \forall \eta < \xi \rho(\xi, f(x, \nu, \rho))^\xi \cdot g(x, \nu, \rho)(F_\xi(\eta) \rightarrow Q(\eta)) \). Hence from \((5)\), \( Q(\nu) \) holds.

Consider the case where \( x = 0 \). For each formula \( Q \), \( s[Q] \) denotes the formula of the following form; \( s[Q](\mu) := \forall \nu \rho(F_\xi(\rho) \rightarrow \forall \nu < \xi \rho(F_\xi(\nu) \rightarrow Q(\nu)) \rightarrow \forall \nu < \xi \rho(x, \nu)^{\xi}(F_\xi(\nu) \rightarrow Q(\nu))). \) Then we can prove without \((A.1)_\xi\) \((A.2)_\xi\) \( TI_\xi \) and the mathematical induction that \( \text{Prog}[F_\xi, \prec \xi, Q] \rightarrow \text{Prog}[F_\xi, \prec \xi, s[Q]]. \) As is shown
above, in Lemma 7 of [1] all the mathematical inductions used are restricted to those of level $\leq \text{Max}(2, lv(Q))$.

From now we assume $lv(Q) \leq 2$. With the help of $\text{Prog}[F, <_\xi, Q] \rightarrow \text{Prog}[F, <_\xi, s(Q)]$ and $\text{Prog}[F, <_\xi, s(Q)] \rightarrow \text{Prog}[F, <_\xi, s^2(Q)]$, in which proof all mathematical inductions are restricted to those of level $\leq 3$, (I) implies the following (II)₀;

$$(II)_0 \quad S_2(ID_\xi(U_0)) \vdash TI[F, <_\xi, \omega(\xi, 3, 0)]_Q$$

By replying this method, the above (II) is obtained.

Then following Arai [1], the next proposition is derived from (II).

$S_{k+3}(ID_\xi(U_0)) \vdash TI[F, <_0, \omega(\xi, k + 3, 0)]_Q$ with $lv(Q) \leq 2$ and $k \geq 0$.

Hence the proposition holds.

Using the above, Proposition 2 follows;

**Proposition 2** For $k > 2$, the ordinal up to $\omega(\xi, k + 1, 0)$ is accessible in $S_k(ID_\xi(U_0))$ with respect to $<_0$.

**Remark 1:**
From the case in which $\xi$ is a successor ordinal, the transfinite induction formula

$\{i\} \text{Prog}[F, <_i, t_k, A_i]$ at the beginning of the proof of Proposition 1 above is replaced by $\{i\} \text{Prog}[F, <_i, A_i]$, which has level 2, instead of 3. Hence, the Propositions 1 and 2 hold for $k > 1$.

### 3 Unprovability of the transfinite induction up to $\omega(\xi, k + 1, 0)$ in system $S_k(\text{AI}_\xi^-)$

Our aim in this chapter is to prove the estimation we have observed in previous chapter is sharp one;

$S_k(ID_\xi) \nvdash TI[F, <_0, \omega(\xi, k + 1, 0)]$ for $k > 2$

On the whole segment of $ID_\xi = \bigcup S_n(ID_\xi)$, Arai [3] proves that $ID_\xi \nvdash TI[F, <_0, O(\xi + 1, 1)]$. Note that $O(\xi + 1, 1) := \bigcup_0 \omega(\xi, k, 0)$. He shows that the consistency of $ID_\xi$ is provable using transfinite induction up to $O(\xi + 1, 1)$ by the proof reduction method which is originally due to Gentzen-Takeuti. In this section we modify his consistency proof in more delicate manner and prove the following by the cut elimination (proof reduction) method;

$TI[F, <_0, \omega(\xi, k + 1, 0)] \vdash \text{Cons}(S_k(ID_\xi))$ for $k > 2$

Our crucial point is to introduce a $\eta$-height $h_\eta$ for each $\eta \leq \xi$ (Definition 11) and consider a ordinal assignment to a proof $< P, \{h_\eta\}_{\eta \leq \xi}, d >$ with $\xi$-sort of height (Definition 13).

For the Gentzen-Takeuti cut elimination procedure to work, Arai [3] formalises his system $\text{AI}_\xi^-$ of $\xi$-times iterated inductive definition in the form of iterated comprehension axiom by using second order free variables. System $\text{AI}_\xi^-$ is defined by adding the following principles based on $PA$. 
Definition 5 (System $AI_\xi^-$, cf. Arai [3])

For any arithmetical form $B$, the following axioms schemata are added.

\[ (Q^B : \text{right}) \quad \Gamma \vdash \Delta, B(X, Q^B_{x}, t, s) \quad \text{where } Q^B_{x,t} := \{x,y)(x \prec t \land Q^Bxy\} \]

\[ (Q^B : \text{left}) \quad t \prec \xi, Q^Bts \rightarrow B(V, Q^B_{x}, t, s) \]

We assume that the language contains only $\forall, \land$ and $\land$ for the logical connectives. Then, the definition of $lv$ in the previous section is modified as follows;

Definition 6 ($\eta$-level $lv_{\eta}(A)$ of a formula $A$ with $\eta \preceq \xi$) For the formula $A$ in the language of $AI_\xi^-$ and an ordinal $\eta \preceq \xi$, the $\eta$-level $lv_{\eta}(A)$ of the formula $A$ is defined inductively as follows, where $d$ is defined in Definition 2 of previous section with using $Q_\eta$ instead of $\muU$ and $d(Xt) := 0$ (for $X$ a predicate variable):

\[
\begin{align*}
lv_{\eta}(P) &:= 0 \text{ for any atom of } L_{PA}, \\
LV_{\eta}(A \land B) &:= \max\{lv_{\eta}(A), lv_{\eta}(B)\} \\
LV_{\eta}(\forall x A) &:= \begin{cases} 
\max\{2, lv_{\eta}(A)\} & \text{if } lv_{\eta}(A) \geq 1 \\
0 & \text{if } lv_{\eta}(A) = 0 
\end{cases} \\
LV_{\eta}(\neg A) &:= \begin{cases} 
LV_{\eta}(A) + 1 & \text{if } LV_{\eta}(A) \geq 1 \\
0 & \text{if } LV_{\eta}(A) = 0 
\end{cases} \\
LV_{\eta}(Q^B) &:= \begin{cases} 
1 & \text{if } \eta = LV_{\eta}(Q^B) \\
0 & \text{otherwise} 
\end{cases} \\
LV_{\eta}(t \prec s \land Q^B) &:= \begin{cases} 
1 & \text{if } \eta \prec \xi \land Q^B = \eta \\
0 & \text{otherwise} 
\end{cases}
\end{align*}
\]

Note that $lv_\eta$ for $\eta = \xi$ is the same as $lv$ of the previous section (with using $Q^B$ instead of $\muU$, in the definition of $lv$ of the previous section with replacing $\supset$ by $\land$).

We can define the fragments $Sk(AI_\xi^-)$ in the same manner as $Sk(ID_\xi)$ as follows.

Definition 7 (the subsystem $Sk(AI_\xi^-)$ of $AI_\xi^-$) $Sk(AI_\xi^-)$ is $AI_\xi^-$ except that for every abstract $V$ in $Q_\eta^B$: left and $(VJ)$, $lv_{\xi}(V) \leq k$ holds.

$ID_\xi$ is obtained from $ID_\xi'$ in the previous section by changing the underlying logic from the intuitionistic to the classical. For each formula $F$ of the language of $ID_\xi$, we define a formula $F^*$ of the language of $AI_\xi$ by substituting $Q^B$ for all occurrences of $\muU$, where

\[ B(X, Y, c_0, c_1) := \forall y(U(X, Y, c_0, y) \land Xy) \land Xc_1. \]

It is well known that by this $\ast$, $ID_\xi$ is embeddable into $AI_\xi$ (cf. [3]). Obviously $lv(F) = lv_{\xi}(F^*)$ holds i.e., $\xi$-level of a formula remains the same through the above interpretation.

Until the end of this section, we assume that all formulas occuring in a proof figure of $AI_\xi^-$ are of the following normal form:

Lemma 1 (the normal form of a formula in $AI_\xi^-$) For arbitrary formula $A$ of the language of $AI_\xi^-$, there exists a formula of the following form, called a normal formula, which is equivalent to $A$ (in $L_{KA}$):

\[
\forall \bar{x}_1 \cdots \forall \bar{x}_m \forall \bar{y} \forall \bar{D} [Q^B_{t_1 s_1}, \ldots, Q^B_{t_m s_m}]
\]

where $D[\ast_1, \ldots, \ast_m]$ is a context of the language of $PA$, and no quantifier occurring in $D$ bounds any $\ast_i$ (1 $\leq i \leq m$) and $lv_{\eta}(D[Q^B_{t_1 s_1}, \ldots, Q^B_{t_m s_m}]) \leq 2$ for any $\eta \preceq \xi$. 

Definition 8 (normal proofs)  Let $S$ be a sequent of normal formulas. A normal proof of $S$ is a proof in which $\forall -$left rules are used, instead of $\forall$-left rules in a proof;

$$
\forall x_1 \cdots x_n \rightarrow A(x_1, \ldots, x_n), \Gamma \rightarrow \Delta \quad \forall$-left

Note that the original $\forall$-left rule may also appear in a normal proof.

Lemma 2  Any provable sequent of normal formulas has a normal proof.

From now on we assume any $S_k(A_1^2)$-proof to be normal by virtue of the above two lemmata.

Definition 9  For each formula $A$, $\eta(A) \leq \xi$ is defined as $\eta(A) := \text{Max}\{\eta \mid lv_\eta(A) \neq 0\}$.

Definition 10 ($g_\eta(A)$ with $\eta < \xi$)

$$
g_\eta(A) := \begin{cases} g(A) & \text{if } \eta(A) \geq \eta \\ 0 & \text{if } \eta(A) < \eta \end{cases}
$$

where $g(A)$ denotes the number of logical symbols in $A$.

We modify the notion of proof with degree $< P, d >$ of Arai [3] into $< P, \{h_\eta\}_{\eta \leq \xi}, d >$ by introducing $\xi$-sort of height $\{h_\eta\}_{\eta \leq \xi}$, as follows:

Definition 11 (A proof with $\xi$-sort of height $< P, \{h_\eta\}_{\eta \leq \xi}, d >$) A proof $< P, d >$ (with degree $d$) is called a proof with $\xi$-sort of height $< P, \{h_\eta\}_{\eta \leq \xi}, d >$ if for each sequent $S$ of $P$ and each ordinal $\eta \leq \xi$, a natural number $h_\eta(S)$ satisfying the following condition is assigned. We call $h_\eta$ a $\eta$-height.

0. $h_\eta(S) = 0$ for every $\eta \leq \xi$ if $S$ is the end sequent of $P$.

For the last inference $I$ of the form

$$
I \quad \frac{S}{S'}
$$

1. $h_\eta(S) = 0$ for every $\eta \leq \xi$ if $I$ is a substitution.

2. $h_\eta(S) = h_\eta(S')$ for every $\eta \leq \xi$ if $I$ is an inference except substitution, induction and cut.

3. $\begin{cases} 1 \quad h_\eta(S) \geq \text{Max}\{h_\eta(S'), g_\eta(D)\} & \text{for } \eta < \xi \\ 2 \quad h_\xi(S) = \text{Max}\{h_\xi(S'), lv_\xi(D)\} & \text{if } I \text{ is a cut, where } D \text{ is the cut formula of the inference } I. \end{cases}$

4. $\begin{cases} 1 \quad h_\eta(S) \geq \text{Max}\{h_\eta(S'), g_\eta(D)\} + 1 & \text{for } \eta < \xi \\ 2 \quad h_\xi(S) = \text{Max}\{h_\xi(S'), lv_\xi(D)\} + 1 & \text{if } I \text{ is an induction.} \end{cases}$

Definition 12  For each sequent $S$ of $< P, \{h_\eta\}_{\eta \leq \xi}, d >$, $\eta(S) \leq \xi$ is defined as $\eta(S) := \begin{cases} d(I) & \text{if } S \text{ is the upper sequent of the substitution } I \\ \text{Max}\{\eta \mid h_\eta(S) \neq 0\} & \text{otherwise} \end{cases}$

The following is an immediate consequence from Definition 12.
Lemma 3 For any proof with $\xi$-sort of height $P, \{h_\eta\}_{\eta \leq \xi}, d >$ and for any inference $I$ (with a lower sequent $S'$ and a upper sequent $S$) in $P, \{h_\eta\}_{\eta \leq \xi}, d >$,

$$\eta(S) \geq \eta(S')$$

holds.

Notation 3 For $i \leq \xi$ and an ordinal diagram $\alpha$, an ordinal diagram $\omega(i, n, \alpha)$ is defined inductively as follows.

- $\omega(i, 0, \alpha) := \alpha$
- $\omega(i, n + 1, \alpha) := (i, \omega(i, n, \alpha))$

Definition 13 (ordinal assignment) Let $I$ be an inference of the form

\[
\begin{array}{ccc}
S_1 & \text{S}_2 & S \\
& (S_1) & (S_2) \\
& (S_1) & (S_2) & (S_1) \\
\end{array}
\]

Then $O(S)$ is defined as follows:

1. When $I$ is a cut,

$$O(S) := \omega(\eta(S), k - h_\eta(S_1)(S), \omega(\eta(S_1), h_\eta(S_1)(S_1), O(S_1) \# O(S_2))))$$

Here $k := \text{Max}\{h_\eta(T) | T \text{ is above } I\}$ and $c[\bullet] := \omega(\gamma_1, k_1, \omega(\gamma_2, k_2, \ldots, \omega(\gamma_n, k_n, \bullet)))$,

\[
\begin{align*}
\{\gamma_1, \ldots, \gamma_n\} := & \{\gamma | \eta(S) < \gamma < \eta(S_1) \text{ and } h_\eta(T) \neq 0 \text{ for some } T \text{ above } I\}
\end{align*}
\]

with $\gamma_1 < \ldots < \gamma_\eta$ and $k_i := \text{Max}\{h_\eta(T) | T \text{ is above } I\}$.  

2. When $I$ is a logical inference, $O(S) := O(S_1) \# O(S_2) \# 0$

3. When $I$ is a structural inference, $O(S) := O(S_1) \# O(S_2)$

4. When $I$ is a substitution, $O(S) := (d(I), O(S_1))$

Theorem 1 The transfinite induction on $\omega(\xi, k + 1, 0)$ is unprovable in $S_\xi(\text{AI}_\xi)$ for $k > 2$.

Proof.

We refine the proof reduction process of Arai [3] to define the reduction process for $S_\xi(\text{AI}_\xi)$ ($k > 2$), and show that the well-orderness of $\omega(\xi, k + 1, 0)$ implies the termination of the reduction process, hence the consistency of $S_\xi(\text{AI}_\xi)$. Then the above theorem follows from Gödel's incompleteness theorem.

(preparation)

Without loss of generality, we assume that all logical initial sequents of the form $p \rightarrow p$ where $p$ is an atomic and that there exists no free variables which is not used as an eigenvariable.

(elimination of initial sequents in the end-piece) As usual.

(elimination of weakening) elimination of weakening known in the usual way (cf. Takeuti [14]) dose work not only for a weakening in end-piece but also for a more general weakening with such a weakening formula $D$ as the bundle $I$ (cf. p78 of [14]) which begins with $D$ ends with a cut formula $D$ and no logical inference affect $I$.

\[1\text{In the case where } \eta(S) = \eta(S_1), c[\bullet] \text{ is } \bullet \text{ and } O(S) := \omega(\eta(S), k - h_\eta(S_1)(S_1) - h_\eta(S_1)(S_1), O(S_1) \# O(S_2)).\]
(elimination of the mathematical induction rule) As usual.

Then from sublemma 12.9 of [14], there exists a suitable cut \( J \) in the end piece of \( < P, \{h_\eta\}_{\eta \leq \xi}, d > \). Let \( I_1 \) and \( I_2 \) be boundary logical inferences whose principal formulas are ancestors of left and right cut formulas of \( J \).

We shall demonstrate following three essential cases both for limit ordinal \( \xi \) and for successor ordinal \( \xi \):

(Case 1) The case where the cut formula \( C := A \land B \) with \( \eta(C) < \xi \):

Let \( K \) (whose lower sequent is \( T \) and whose upper sequent is \( T_1 \)) denotes the uppermost inference below \( J \) such that either (i) or (ii) holds:

\[
\eta(T) = \eta(A) \land (h_{\eta(A)}(S_1) > h_{\eta(A)}(T)) \cdots (i)
\]

\[
\eta(T) < \eta(A) \cdots (ii)
\]

where \( A \) is the auxiliary formula of \( I_1 \) and \( I_2 \)

\( < P, \{h_\eta\}_{\eta \leq \xi}, d > \) is as follows:

\[
\begin{array}{cccc}
S^{l_1}_1 & S^{l_2}_1 & I_1 & S^{l_2}_2 \\
S^{l_1}_2 & I_2 & S^{l_2}_1 \\
S^{l_1}_j & J \\
S^{l_2}_j \\
T_1 \\
\hline
T_1 \quad K \\
\hline
T''
\end{array}
\]

\( < P', \{h'_\eta\}_{\eta \leq \xi}, d' > \) is as follows, where \( I_1 \) and \( I_2 \) are weakening-right (with a weakening formula \( A_1 \)) and weakening-left (with a weakening formula \( A_3 \)) respectively:

\[
\begin{array}{cccc}
S^{l_1}_1 & S^{l_2}_1 & I_1 & S^{l_2}_2 \\
S^{l_1}_2 & I_2 & S^{l_2}_1 \\
S^{l_1}_j & J \\
S^{l_2}_j \\
T_1 \quad K \\
\hline
T''
\end{array}
\]

(case 1.1): The case where (i) holds. Then for any sequent \( T'' \) between \( S_1 \) and \( T \), \( \eta(T'') \geq \eta(A) \) holds.

(case 1.1.1) \( \eta(T_1) = \eta(T) \)

\( O_{P'}(T'') \prec O_P(T) \) is checked as usual way.

(case 1.1.2) \( \eta(T_1) > \eta(T) \)

special case of (case 1.2)

(Case 1.2): The case where (ii) holds. Then \( \eta(T) < \eta(A) \leq \eta(T_1) \) holds. We assign

\[
h'_\eta(U_1) := \begin{cases} 
  h_\eta(T) & \text{if } \eta < \eta(T) \\
  g(A) & \text{if } \eta = \eta(A) \\
  0 & \text{otherwise}
\end{cases}
\]

Hence \( \eta(U_1) = \eta(A) \) holds. On the other hand, there exist contexts \( a \) and \( b \) such
that \( O_p(T) = \omega(\eta(T), k - h_{n(T)}(T), a[\omega(\eta(A), k, b[\alpha_1 & \alpha_2]])] \),
\( O_p(U_1) = \omega(\eta(U_1), m - h_{n(U_1)}(U_1), b[\alpha_1 & \alpha_2]) = \omega(\eta(A), m - g(A), b[\alpha_1 & \alpha_2]) \) and
\( O_p(T^*) = \omega(\eta(T), k' - h_{n(T)}(T), a[\omega(\eta(U_1), g(A), O_p(U_1) \& O_p(U_2))]) \)
Since \( \omega(\eta(A), k, b[\alpha_1 & \alpha_2]) > \omega(\eta(U_1), g(A), O_p(U_1) \& O_p(U_2)) \), \( O_p(T^*) <_o O_p(T) \) holds.

(Case 2) The case where cut formula is \( \forall \bar{z} \neg B(\bar{z}) \):
\(< P, \{ h_n \}_{n \leq i}, d > \) is as follows; here \( I_2 \) is \( \forall \)-left.

\[
\begin{array}{cccccccc}
S_{11}^{I_1} & S_{12}^{I_1} & I_1 & S_{12}^{I_2} & I_2 & \cdots & \cdots & \cdots \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
S_{21}^{J} & S_{22}^{J} & J & S_{11}^{J} & S_{12}^{J} & & & \\
\end{array}
\]

Since \( \nu_\eta(B(\bar{z}))(\forall \bar{z} \neg B(\bar{z})) > \nu_\eta(B(x))(B(x)) \) holds, \( O(P') <_o O(P) \) is checked as the usual way.

(Case 3) The case where the cut formula of \( J \) is \( Q\bar{z}B(\bar{z}) \):

\(< P, \{ h_n \}_{n \leq i}, d > \) is as follows, where \( K \) (with the lower sequent \( T \)) denotes the upper most inference below \( J \) such that \( \eta(T) \leq d(B(X, Q, t, s)) := i \);
Let $T_1$ denote such upper sequent of $K$ that is below $J$.

\[
\begin{array}{cccccc}
S_1^f & I_1 & S \\
S_1^j & S_2^j & J \\
T_1 & (T_2) & K
\end{array}
\]

\[
\begin{array}{cccccc}
S : & t_2 < \xi, Q_t_2 s_2 \rightarrow B(V, Q_{< t_1}, t_2, s_2) \\
S_{1^f}^f : & \Gamma_1 \rightarrow \Delta_1, B(X, Q_{< t_1}, t_1, s_1) \\
S_{1^f}^j : & \Gamma_1 \rightarrow \Delta_1, Q_t_1 s_1 \\
S_1^j : & \Gamma_2 \rightarrow \Delta_2, Q_t s \\
S_2^j : & Q_t s, \Pi \rightarrow \Lambda \\
S_{1^*}^j : & \Gamma_2, \Pi \rightarrow \Delta_2, \Delta_2 \\
T_1 : & \Phi_1 \rightarrow \Psi_1 \\
T_2 : & \Phi \rightarrow \Psi
\end{array}
\]

$O_P(T) = \omega(\eta(T), k - h_\eta(T_1)(T), c[\omega(\eta(T_1), h_\eta(T_1)(T_1), O_P(T_1) \# O_P(T_2)])].$

$< P', \{h'_\eta \}_{\eta \leq \xi}, \Delta' >$ is as follows, where $I_1$ is weakening-right with a weakening formula $Q^{B_{t_1,s_1}}$:

\[
\begin{array}{cccc}
S_1^f & I_1 & S \\
S_1^j & S_2^j & J \\
T_1 & (T_2) & T^* & \text{sub}
\end{array}
\]

\[
\begin{array}{cccc}
S_1^f & I_1 \\
S_1^j & S_1^j & J \\
T_1 & (T_2) & K
\end{array}
\]

\[
\begin{array}{cccccc}
S_{1^f}^f : & \Gamma_1 \rightarrow \Delta_1, B(X, Q_{< t_1}, t_1, s_1) \\
S_{1^f}^j : & \Gamma_1 \rightarrow \Delta_1, B(X, Q_{< t_1}, t_1, s_1) \\
S_1^j : & \Gamma_2 \rightarrow \Delta_2, Q_t s, B(X, Q_{< t_1}, t_1, s_1) \\
S_{1^*}^j : & \Gamma_2, \Pi \rightarrow \Delta_2, \Delta_2, B(X, Q_{< t_1}, t_1, s_1) \\
T_1^* : & \Phi_1 \rightarrow \Psi_1, B(X, Q_{< t_1}, t_1, s_1) \\
T^* : & \Phi \rightarrow \Psi, B(X, Q_{< t_1}, t_1, s_1)
\end{array}
\]

We assign $\{h'_\eta \}_{\eta \leq \xi}$ as follows:

- $h'_\eta(T^*) := h_\eta(S)$ for all $\eta \leq \xi$
- $h'_\eta(T_1) := h_\eta(T_1)$ for all $\eta \leq \xi$.
- $h'_\eta(T^*) := \begin{cases} 
  h_\eta(T) & \text{if } \eta \leq \eta(T) \\
  0 & \text{otherwise}
\end{cases}$

$O_P(T^*) = (i, O_P(T^*))$
\[ O_P(T^*) = \omega(\eta(T^*), l - h_{\eta(T^*)}(T^*), d[\omega(\eta(T^*), h_{\eta(T^*)}(T^*), O_P(T^*), O_P(T))] ) \]

Note that \( \eta(T^*) = 1 \). Obviously \( k = l \) from the figure of \( P^* \). And from the above assignment \( h^*, c[\star] = \omega(\gamma_1, k_1, \ldots, \omega(\gamma_s, k_s, d[\star])) \) with \( \gamma_s < i = \gamma_{s+1} \). Hence \( O(P^*) <_0 O(P) \) holds.

The following Corollary is immediate from the above theorem and the fact that \( S_k(ID^*_\xi) \) is a subsystem of \( S_k(AL^*_\xi) \) under the interpretation * (cf. the paragraph after Definition 7).

**Corollary 1** The transfinite induction on \( \omega(\xi, k + 1, 0) \) is unprovable in \( S_k(ID^*_\xi) \) for \( k > 2 \).

**Proof.** As remarked after Definition 7, \( \xi \)-level does not change under the interpretation of an \( AL^*_\xi \)-formula to an \( ID^*_\xi \)-formula. Hence the Corollary is obvious.

**Theorem 2** (Main Theorem)

\[ |S_k(ID^*_\xi)(L_\omega)| = |S_k(ID^*_\xi)| = |S_k(AL^*_\xi)| = |\omega(\xi, k + 1, 0)|_{<_0} \text{ with } k > 2. \]

**Remark 2:** Our system \( S_k(ID^*_\xi) \) can be reformulated by means of the alternation complexity of quantifiers when we include \( \exists \) in our language. Here, a normal formula is of the form \( Q_1x_1^1 \cdots Q_nx_n^{2\xi} \forall \exists D[P^{\star_1}t_{\star_1}^1, \ldots, P^{\star_m}t_{\star_m}^m] \), where \( D[\star_1, \ldots, \star_m] \) is a context of the language of \( PA \) with no quantifier occurring in \( D \) bounds any \( \star_i \) \( (1 \leq i \leq m) \), and \( \{Q_j, Q_j^\prime\} = \{\forall, \exists\} \) \( (j = 1, \ldots, m) \). \( lv \) is essentially the same as \( I_v \) except that we measure the alternation complexity of quantifiers instead of nestedness complexity of negations; namely,

\[ lv(D[P^{\star_1}t_{\star_1}^1, \ldots, P^{\star_m}t_{\star_m}^m]) := \begin{cases} 1 & \text{if all } P^{\star_i}t_{\star_i}^i (i = 1, \ldots, m) \text{ is positive in } D \\ 2 & \text{otherwise} \end{cases} \]

Then the \( lv \) of above normal formula is \( n + i \) if \( \bar{Q}_n = \forall \) and \( n + 1 + i \) if \( \bar{Q}_n = \exists \), where \( i := lv(D[P^{\star_1}t_{\star_1}^1, \ldots, P^{\star_m}t_{\star_m}^m]) \). \( S'_k(ID^*_\xi) \) is defined in the same way as the former definition of \( S_k(ID^*_\xi) \) with using the above new notation of \( lv \). It is easily seen that \( S'_k(ID^*_\xi) \) is equivalent to \( S_k(ID^*_\xi) \). In particular \( |S'_k(ID^*_\xi)| = |\omega(\xi, k + 1, 0)|_{<_0} \) with \( k > 2 \).

**References**


