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Consistency Proof via Pointwise Induction

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Abstract

We show that the consistency of the first order arithmetic $PA$ follows from the pointwise induction up to the Howard ordinal. Our proof differs from U. Schmerl [S]: We do not need Girard’s Hierarchy Comparison Theorem. A modification on ordinal assignment to proofs by Gentzen and Takeuti [T] is made so that one step reduction on proofs exactly corresponds to the stepping down $\alpha \mapsto \alpha[1]$ in ordinals. Also a generalization to theories $ID_\xi$ of finitely iterated inductive definitions is proved.

We show that the consistency of the first order arithmetic $PA$ follows from the pointwise induction up to the Howard ordinal. Our proof differs from U. Schmerl [S]: We do not need Girard’s Hierarchy Comparison Theorem.

Let $P$ be a proof of the empty sequent in $PA$ or the second order arithmetic $\Pi^1_1-CA_0$. For such a proof $P$ let $o(P)$ denote the ordinal assigned to $P$ and $r(P)$ a reduct of $P$ defined by Gentzen and Takeuti [T]. $r(P)$ is again a proof of the empty sequent and $o(r(P)) < o(P)$. Then the reduction $r: P \rightsquigarrow r(P)$ is close to but does not fit perfectly the stepping down $\alpha \mapsto \alpha[1]$ defined by Buchhoiz [B2].\footnote{This observation is also stated by M. Hamano and M. Okada [H-O].} We need to tune these functions $o$ and $r$ to stepping down in order to have $o(r(P)) = o(P)[n]$ for an $n$. For this purpose we introduce two inference rules: the padding rule and the height rule. In both rules the lowersequent is identical with the uppersequent. Let $S_u$ [$S_l$] denote the uppersequent [the lowersequent], resp. Also let $h(S)$ denote the height of a sequent (in a proof $P$).

\[
\begin{align*}
\Gamma & \vdash (pad)_b \\
\Gamma & \vdash (hgt)
\end{align*}
\]

with $h(S_u) = h(S_l) + 1$ and $o(S_l) = D_1 o(S_u)$. $D_1 a$ denotes an ordinal term defined in [B2].

Using these rules we can unwind gaps between $D$ and $r(D)$ so that $o(r(P)) = o(P)[1]$ holds.

1 Fundamental sequences

The following is the fundamental sequences given in [A] and a slight variant in Buchhoiz [B2]. Let $q$ be a natural number.

**Definition 1** (Buchhoiz [B2]) The term structure $(T(q), [\cdot])$

1. Inductive definition of the sets $PT(q)$ and $T(q)$

   \begin{align*}
   (T0) & \quad PT(q) \subseteq T(q) \\
   (T1) & \quad 0 \in T(q) \\
   (T2) & \quad a \in T(q) \& 0 \leq u \leq q \Rightarrow D_u a \in PT(q) \\
   (T3) & \quad a_0, \ldots, a_k \in PT(q)(k > 0) \Rightarrow (a_0, \ldots, a_k) \in T(q)
   \end{align*}

2. For $a_0, \ldots, a_k \in PT(q)$ and $k \in \{-1, 0\}$, we set

   \[
   (a_0, \ldots, a_k) = \begin{cases} 
   0 & k = -1 \\
   a_0 & \text{otherwise}
   \end{cases}
   \]
Remark. U. Schmerl [S] gives a proof of a variant of the Theorem 1.1 \((q = 0)\), i.e., for \(PA\) via Girard's Hierarchy Comparison Theorem. In [S] the base theory (for us \(ERA\)) contains the fast growing functions \(F_\alpha (\alpha < \psi_0 \epsilon_0)\) and/or the slow growing functions \(G_\alpha (\alpha < \psi_0 \epsilon_{\Omega+1})\) and their defining equations. Hence it seems that Schmerl's result is incomparable to ours.
2 Proof of Theorem

Fix a natural number $q$. We prove the Theorem 1.1. The Theorem 1.2 is proved similarly.
First consider the easy half: The rule $(PI)_{q+1}$ is a derived rule in $ERA + \{Con^{(n)}(ID_q) : n < \omega\}$.

Let $prov_T$ denote a standard proof predicate for a theory $T$ and "$B$" the Gödel number of an expression $B$.

This follows from the fact which is shown in [A]:

**Proposition 3** For some elementary recursive function $f$ we have

$ERA \vdash \forall a \in T_0(q + 1)\{prov_{ID_q}(f(a), "\forall n3m\widehat{a}[n]^m = 0")\}$

Next consider the other half. Let $\forall zB(z)$ be a $\Pi^0_0$ sentence. $ID_q + \forall zB(z)$ denote the theory obtained from $ID_q$ by adding extra axioms $B(t)$ for an arbitrary term $t$. It suffices to show, in $ERA + (PI)_{q+1}$, $Con(ID_q + \forall zB(z))$ under the assumption $\forall zB(z)$ is true'. Our proof is an adaption from Gentzen's and
Takeuti's reduction in [T].

First $ID_q$ is embedded in a first order theory $NID_{q+1}$. In the latter theory the universe $\omega$ of $ID_q$ is replaced by a constant $N$ and this constant is treated as if it were a $\Pi^1_1$ formula. Then as in [T] the inference rules for the constant $N$ are analysed by using a substitution rule. Also as mentioned above we introduce two new rules, the padding rule and the height rule to unwind gaps in Gentzen-Takeuti reduction. Now details follow.

The language $L$ of $ID_q$ consists of

1. function constants $0$ and the successor ',
2. arithmetic predicate constants are lower elementary recursive relations $R \in \mathcal{L}_2$ and their negations $\neg R$,
3. the least fixed points $\{P_u : 1 \leq u \leq q\}$ for a fixed positive operator form $A(X^+, Y, n)$ and
4. logical symbols $\wedge, \vee, \exists, \forall$.

The negation $\neg A$ of a formula $A$ is defined by using de Morgan’s law and the elimination of double negations. A prime formula $R(t_1, \ldots, t_n)$ or its negation $\neg R(t_1, \ldots, t_n)$ with an arithmetic predicate $R$ is an a.p.f.(arithmetic prime formula).

The axioms in $ID_q$ are axioms for function and arithmetic predicate constants, the induction axiom (IA) and axioms (P.1), (P.2) of the least fixed points $\{P_u : 1 \leq u \leq q\}$ for arbitrary formula $F$:

(IA) $F(0) \land \forall z(F(z) \lor F(z')) \supset \forall zF(z)$

(P.1) $A_u(P_u) \subseteq P_u$

(P.2) $A_u(F) \subseteq F \supset P_u \subseteq F$

where $A_u(X) = \{n : A(X, \sum_{1 \leq u \leq q} P_v, n)\}$.

The language $L_N$ of $NID_{q+1}$ consists of $L \cup \{N\} \cup \{X_i : i < \omega\}$ with a unary predicate constant $N$ and a list of unary predicates $X_i$. These unary predicates are denoted $X, Y, etc$. We sometimes write $P_0$ for the constant $N$. For a predicate constant $H \in \{P_u : u \leq q\} \cup \{X_i : i < \omega\}$, we write $t \in H$ for $H(t)$ and $t \notin H$ for $\neg H(t)$. A formula is said to be an $E$ formula if it is either an a.p.f. or a formula in one of the following shapes; $A \lor B, \exists xA$ or $t \notin H$ with $H \in \{P_u : u \leq q\} \cup \{X_i : i < \omega\}$. A formula is an $A$ formula if its negation is an $E$ formula. For a formula $A$ in $L$ let $A^N$ denote the result of restricting all quantifiers in $A$ to $N$. For each $u \leq q$ let $N_u(X, t)$ denote the formula:

$N_0(X, t) \equiv \exists \forall z(x \in X \supset x' \in X) \supset t \in X$;

$N_u(X, t) \equiv \forall \exists z(x \in X \supset x' \in X) \supset t \in X (u \neq 0)$

$NID_{q+1}$ is formulated in Tait’s calculus, i.e., one sided sequent calculus. Finite sets of formulae is called a sequent. Sequents are denoted by $\Gamma, \Delta, etc$.

Axioms in $NID_{q+1}$ are:

**Logical Axioms**

1. $\forall \exists, \neg A, A$

2. $\Gamma, A$ for a true closed a.p.f. $A$.

**Arithmetic Axioms**

1. $\Gamma, \Delta_R$

where $\Delta_R$ consists of a.p.f.'s and corresponds to the definition of a lower elementary relation $R$.
3. $\Gamma, \Delta_0$
where there exists a sequent $\Delta_1$ so that $\Delta = \Delta_0 \cup \Delta_1$ is an instance of a defining axiom for $R$ in 1 and $\Delta_1$ consists solely of false closed a.p.f.’s.

Inference rules in $NID_{q+1}$ are:

(1) $(\Lambda), (\forall), (\exists), (\neg)$, and $P_u$, $(\neg P_u)$ for $(u \leq q)$.

1. $(\Lambda), (\forall), (\exists):$ In these rules the principal formula is contained in the uppersequent. For example

$\frac{\Gamma, \exists x A(x), A(t)}{\Gamma, \exists x A(x)} (\exists)$

2. In the rule (cut)

$\frac{\Gamma, \neg A, A, \Delta}{\Gamma, \Delta} (cut)$

the cut formula $A$ is an $E$ formula.

3. $(weak)$ is the weakening:

$\frac{\Gamma}{\Delta} (weak)$

with $\Gamma \subseteq \Delta$.

4. $(P_u)$:

$\frac{\Gamma, t \in P_u, N_u(X, t)}{\Gamma, t \in P_u} (P_u)$

where $X$ is the eigenvariable, i.e., does not occur in the lowersequent.

5. $(\neg P_u)$:

$\frac{\Gamma, t \notin P_u, \neg N_u(F, t)}{\Gamma, t \notin P_u} (\neg P_u)$

for an arbitrary formula $F$ in the language $L_N$.

Lemma 1 For any sentence $A$ in $L$,

$ID_q \vdash A \Rightarrow NID_{q+1} \vdash A^N$

Proof. It suffices to show that the following sequents are provable in $NID_{q+1}$:

$t \notin P_u, t \in P_u$: This is proved by induction on $u \leq q$. By IH($=$Induction Hypothesis) we have $\neg N_u(X, t), N_u(X, t)$.

Rules $(P_u)$ and $(\neg P_u)$ yields $t \notin P_u, t \in P_u$.

$(IA)^N$: For a given formula $F(x)$ assume $a \in N$, $F(0)$ and $\forall x \in N (F(x) \supset F(x'))$. We have to show $F(a)$. Let $G(x)$ denote the formula $x \in N \wedge F(x)$. Then we see $\forall x (G(x) \supset G(x'))$ from $x \in N \supset x' \in N$. The latter follows from the rules $(\neg N) = (\neg P_0)$ and $(N)$. Also we have $G(0)$. On the other hand we have $N_0(G, a)$ by the rule $(\neg N)$ and $a \in N$. Thus we get $G(a)$ and hence $F(a)$.

$(P.1)^N$: Assume $A^N_u(P_u, a)$. We have to show $a \in P_u$. By the rule $(P_u)$ it suffices to show $N_u(X, a)$. Assume $A^N_u(X) \subseteq X$. We show $a \in X$.

Claim. $P_u \subseteq X$.

Proof of the Claim. Assume $x \in P_u$. By the rule $(\neg P_u)$ we have $N_u(X, x)$. The assumption $A^N_u(X) \subseteq X$ yields $x \in X$.

From this Claim and the positivity of $X$ in $A$ we see $A^N_u(X, a)$. Thus again by the assumption $A^N_u(X) \subseteq X$ we conclude $a \in X$.

$(P.2)^N$: For a given formula $F$ assume $A^N_u(F) \subseteq F$ and $a \in N \cap P_u$. We show $F(a)$. This follows from the rule $(\neg P_u)$.

Definition 2 The length $|A|$ of a formula $A$ in $L_N$
1. \(| A |= 0\) for a prime formula \(A\). Specifically \(| (\neg)H(t) |= 0\) for any predicate \(H\).

2. \(| QzA |= F \, | +1\) for \(Q \in \{ \forall, \exists \}\)

3. \(| A_0 \circ A_1 |= \max\{| A_i | + 1 : i = 0, 1\} \) for \(\circ \in \{ \wedge, \vee \}\).

A formula \(A \in Pos_u\), if 1. if a predicate \(P_o\) occurs positively in \(A\), then \(v \leq u\), and 2. if a predicate \(P_o\) occurs negatively in \(A\), then \(v \leq u\).

Observe that \(N_u(X, t) \in Pos_u\) and \(\neg P_u(t) \notin Pos_u \& Pos_v \subseteq Pos_u\) for \(v \leq u\).

Let \(\forall zB(z)\) denote a fixed true \(\Pi^0_1\) sentence with an a.p.f. \(B\). The system \(NID_{q+1} + \forall zB(z)\) is obtained from \(NID_{q+1}\) by adding the axiom 

\[(B) \Gamma, B(t)\]

for an arbitrary term \(t\) and three inference rules; the padding rule \((pad)_b (b \in OT(q+1))\), the height rule \((hgt)\) mentioned in the introduction and the substitution rule \((sub)_u (u \leq q)\).

\[
\frac{\Gamma(X)}{\Gamma(F)} (sub)_u
\]

where 1. \(\Gamma(X) \subseteq Pos_u\), 2. \(X\) is the eigenvariable, i.e., does not occur the lowersequent, 3. \(F\) is an arbitrary formula in \(L_N\) and 4. \(\Gamma(F)\) denotes the result of substituting \(F\) for \(X\) in \(\Gamma(X)\).

Let \(P\) be a proof (in \(NID_{q+1} + \forall zB(z)\)) and \(\Gamma\) a sequent in \(P\). We define the height \(h(\Gamma) = h(\Gamma; P)\) of \(\Gamma\) in \(P\) as follows:

1. \(h(\Gamma) = 0\) if \(\Gamma\) is either the endsequent of \(P\) or the uppersequent of a rule \((sub)\).
2. \(h(\Gamma) = h(\Delta) + 1\) if \(\Gamma\) is the uppersequent of an \((hgt)\) whose lowersequent is \(\Delta\).
3. \(h(\Gamma) = h(\Delta)\) if \(\Gamma\) is the uppersequent of a rule other than \((sub)\) and \((hgt)\) and \(\Delta\) is the lowersequent.

Again let \(P\) be a proof (in \(NID_{q+1} + \forall zB(z)\)). Let \(o\) denote an assignment of an ordinal term \(o(\Gamma) = o(\Gamma; P) \in OT(q+1)\) to each sequent \(\Gamma\) in \(P\). If the assignment \(o : \Gamma \mapsto o(\Gamma)\) enjoys the following conditions, then we say that \(o\) is an ordinal assignment for \(P\).

1. \(o(\Gamma) \neq 0\) for each axiom \(\Gamma\).
2. \(o(\Gamma) = o(\Gamma_0)\) if \(J\) is one of \((\forall), (weak), (P_u)\).
3. \(o(\Gamma) = o(\Gamma_0) = o(\Gamma_1)\) if \(J\) is \((\Lambda)\).

\(\text{N.B.}\) We require ordinals assigned to uppersequents of \(a (\Lambda)\) are equal.

4. \(o(\Gamma) = o(\Gamma_0) + b\) for some nonzero \(0 \neq b \in OT(q+1)\) if \(J\) is either \((\exists)\) or \((\forall)\).

In this case we write, e.g., \((\forall)_b\) for the rule \((\forall)\).

5. \(o(\Gamma) = o(\Gamma_0) + b\) if \(J\) is \((pad)_b\).
6. \(o(\Gamma) = o(\Gamma_0) + \Omega_{1+u}\) if \(J\) is \((\neg P_u)\).
7. \(o(\Gamma) = o(\Gamma_0) + o(\Gamma_1)\) if \(J\) is \((cut)\).
8. \(o(\Gamma) = D_{q+1}o(\Gamma_0)\) if \(J\) is \((hgt)\).
9. \(o(\Gamma) = D_u o(\Gamma_0)\) if \(J\) is \((sub)_u (u \leq q)\).

For an ordinal assignment \(o\) for a proof \(P\) we set \(o(P) = o(\Gamma_{end})\) with the endsequent \(\Gamma_{end}\) of \(P\).

\(\text{Remark}\).

1. The padding rule \((pad)_b\) is implicit in the literature, e.g., in \([B2]\).
2. The substitution rule \((sub)_u\) comes from \([T]\) but Buchholz mentions a substitution operation \(Nt \mapsto Ft\) in the proof of Lemma 4.12 in \([B2]\).

Let \(P\) be a proof in \(NID_{q+1} + \forall zB(z)\) and \(o\) an ordinal assignment for \(P\). We say that \((P, o)\) is a proof with the o.a. (=ordinal assignment) \(o\) if the following conditions are fulfilled:

\((p0)\) The endsequent of \(P\) is the empty sequent.
(p1) The final part of \( P \) is an empty \((\text{sub})_0\) followed by a nonempty series \(\{(\text{pad})_{b_i}\}_{i \leq n}\) of paddings with \(\text{dom}(b_i) \in \{\emptyset, \{0\}, \omega\}:\nabla\)

\[\begin{array}{c}
\vdots \\
(\text{sub})_0 \\
(\text{pad})_{b_0} \\
\vdots \\
(\text{pad})_{b_n}
\end{array}\]

(\(p\)) For any \((\text{cut})\) in \(P\),

\[
\frac{\Gamma, \neg A, A, \Delta}{\Gamma, \Delta} \quad \text{(cut)}
\]

\(|A| \leq h(A, \Delta; P) = h(\Gamma, \neg A; P)\).

(\(p\)) For any \((\neg P_u)\) in \(P\),

\[
\frac{\Gamma, t \notin P_u, \neg N_u(F, t)}{\Gamma, t \notin P_u} \quad (\neg P_u)
\]

\(|\neg N_u(F, t)| \leq h(\Gamma, t \notin P_u, \neg N_u(F, t); P)\).

**Proposition 4** Assume \(\text{ID}_\alpha + \forall \text{B}(x)\) is inconsistent. Then there exists a proof \(P\) with an o.a. \(o\).

**Proof.** By Lemma 1 pick a proof \(P_0\) in \(\text{NID}_{\alpha+1}\) ending with the empty sequent. \(P_0\) contains none of rules \((\text{sub}), (\text{pad}), (\text{hgt})\). Below the endsequent of \(P_0\) attach some \((\text{hgt})\)'s to enjoy the conditions \((p\)) and \((p\)).

After that attach further a \((\text{sub})_0\) and a \((\text{pad})_0\) to ensure the condition \((p\)). Let \(P\) denote the resulting proof in \(\text{NID}_{\alpha+1} + \forall \text{B}(x)\) of the empty sequent. For each sequent \(\Gamma\) in \(P_0\) set the ordinal \(o(\Gamma) = \Omega_i^{\alpha+1} \cdot n\) for some \(n < \omega\). Then the whole proof \(P\) has an ordinal \(o(P) = D_\alpha D_\alpha^\omega (\alpha \cdot k)\) for some \(k\).

Thus assuming that \(\forall \text{B}(x)\) is true, it suffices to show the following lemma.

**Lemma 2** Let \((P, o)\) be a proof with an o.a. \(o\). Then there exists a proof \((P', o') = r(P, o)\) with an o.a. \(o\) such that

\[o(P') = o(P)[1]\]

It remains to prove the Lemma 2.

Let \(P\) be a proof (not necessarily ending with the empty sequent). The main branch of \(P\) is a series \(\{\Gamma_i\}_{i \leq n}\) of sequents in \(P\) such that:

1. \(\Gamma_0\) is the endsequent of \(P\).
2. For each \(i < n\) \(\Gamma_{i+1}\) is the right uppersequent of a rule \(J_i\) so that \(\Gamma_i\) is the lowersequent of \(J_i\) and \(J_i\) is one of the rules \((\text{cut}), (\text{weak}), (\text{hgt}), (\text{sub})\) and \((\text{pad})_0\).
3. Either \(\Gamma_n\) is an axiom or \(\Gamma_n\) is the lowersequent of one of the rules \((\lor), (\exists), (\neg P_u)\) and \((\text{pad})_b\) with \(b \neq 0\).

The sequent \(\Gamma_n\) is called the top (of the main branch) of the proof \(P\).

Let \(P\) be a proof with an o.a. \(o\) and \(\Gamma\) a sequent in \(P\). The \(u\)-resolvent of \(\Gamma\) is the uppermost substitution rule \((\text{sub})_b\) below \(\Gamma\) with \(v \leq u\). Note that such a substitution rule always exists by the condition \((p1)\).

Let \(\Phi\) denote the top of the proof \(P\) with the o.a. \(o\). Put \(\alpha = o(P)\). Observe that we can assume \(\Phi\) contains no first order free variable.

**Case 1.** \(\Phi\) is the lowersequent of a rule \((p)_b\) at which the ordinal \(b\) is padded. This means that either \((p)_b = (\text{pad})_b\) with \(b \neq 0\) or \((p)_b = (\lor)_b, (\exists)_b\) with \(b > 1\): \n \n\[
\begin{array}{c}
\overline{\Phi} + b, (p)_b \\
\vdots \\
P
\end{array}
\]

**Case 1.1.** Either the top \(\Phi\) is the endsequent \(\top\), then the last rule is \((p)_b = (\text{pad})_b\) with \(b \neq 0\), and/or \(\text{dom}(b) = \omega\): \(\text{dom}(\alpha) = \text{dom}(b)\). Replace the rule \((p)_b\) by \((p)[1]_b\). Note that \(b[1] \neq 0\) if \(b > 1\).

\[
\begin{array}{c}
\overline{\Phi} + b[1], (p)[1]_b \\
\vdots \\
P'
\end{array}
\]
Case 1.2. Otherwise:

Case 1.21. $\text{dom}(b) = T_u(q+1)$: Let $I$ be the $u$-resolvent of $\Phi$ and $\Gamma$ the lowersequent of $J$. $o(\Gamma)$ is of the form $D_v a$ for some $v \leq u$ and $a$ with $\text{dom}(a) = T_u(q+1)$. We have $(D_v a)[1] = D_v a[a_1]$ with $a_1 = D_u a[1]$. Replace the $(p)_b$ by $(p)_{b[a_1]}$.

Case 1.22. $\text{dom}(b) = \{0\}$, i.e., $b = b_0 + 1$ for some $b_0$: Let $J$ denote the uppermost $(\text{sub})$ or $(\text{hgt})$ below $\Phi$ and $\Gamma$ the lowersequent of $J$. $o(\Gamma)$ is of the form $D_v(a+b_0+1)$ for some $v \leq u$ and $a$ with $\text{dom}(a) = T_u(q+1)$. We have

$$(D_v a)[1] = D_v a[a_1]$$

with $a_1 = D_u a[1]$. Replace the $(p)_{b_0+1}$ by $(p)_{b_0}$ and insert a new $(\text{pad})_c$ immediately below $J$ with $c = D_v(a+b_0)$:

$$\begin{array}{l}
\Phi \\
\Gamma + b_0 + 1
\end{array}$$

Replace the $(p)_{b_0+1}$ by $(p)_{b_0}$ and insert a new $(\text{pad})_c$ immediately below $J$ with $c = D_v(a+b_0)$:

$$\begin{array}{l}
\Phi \\
\Gamma + b_0 + 1
\end{array}$$

Case 2. $\Phi$ is an axiom and contains a true a.p.f. $A$: $\Phi = A, \Delta_0$.

$$\begin{array}{l}
A, \Delta_0 \\
a \\
\Gamma, \neg A, A, \Delta \\
a + b
\end{array}$$

where $a = o(\Gamma, \neg A), b = o(A, \Delta)(b \neq 0)$. Eliminate the false a.p.f. $\neg A$ and insert a (weak) and an appropriate (pad) as in Case 1 to get $o(P') = a[1]$.

$$\begin{array}{l}
a \\
\Gamma, \Delta (\text{weak})
\end{array}$$

Case 3. $\Phi$ is a logical axiom: $\Phi = \neg X(t), X(t), \Delta_0$. Put $X^+(t) = X(t), X^-(t) = \neg X(t), (\neg X(t))^+ = X(t), (\neg X(t))^+ = X(t)$.

$$\begin{array}{l}
a \\
\Gamma, X^+(t), X^+(t), \Delta
\end{array}$$

where $X^+(t) \in \Delta$ and $X^+$ denotes either $X$ or a formula $F$ by a (sub). As in Case 2 insert a (weak) and a (pad).

$$\begin{array}{l}
a \\
\Gamma, X^+(t) (\text{weak})
\end{array}$$

Case 4. $\Phi$ is the lowersequent of a $(\vee)_1$.

Case 5. $\Phi$ is the lowersequent of an $(\exists)_1$.

Consider the Case 4. Let $J$ denote a (cut) at which the descendent of the principal formula $A_0 \vee A_1$ of the $(\vee)_1$ vanishes.

Case 4.1. There exists an $(\text{hgt})$ or a (sub) between $\Phi$ and $J$: Let $I$ denote the uppermost one among such
where 1) $i = 0, 1$, 2) $\tilde{A}_i$ is either $A_i$ or $A_i[X := F]$ and 3) $A'_i \in \{A_i, \tilde{A}_i\}$. For the lowersequent of $I$ $o(A'_0 \lor A'_1, \Delta') = D_u(c + 1)$ for some $u \leq q + 1$ and a $c$. Lower the $(\lor)_1$ under $I$ and change +1 into $+D_u c$:

$$
\frac{A_i, A_0 \lor A_1, \Delta_0}{A_0 \lor A_1, \Delta_0} (\lor)_1 + 1
$$

where $a = o(\Gamma, A_0 \land \neg A_1), b + 1 = o(\tilde{A}_0 \lor \tilde{A}_1, \Delta)$ and $o(\Lambda) = D_{q+1}(c + 1)$ for some $c$.

Assuming $\neg \tilde{A}_i$ is an $E$ formula, let $P'$ be the following:

$$
\frac{A_i, A_0 \lor A_1, \Delta_0}{\tilde{A}_0 \lor \tilde{A}_1, \Delta_0} (\lor)_1 + 1
$$

$$
\frac{\Delta_0 \lor A_1, \Delta, \tilde{A}_i}{a + b + 1} (\lor)_2
$$

Assuming $\neg \tilde{A}_i$ is an $E$ formula, let $P'$ be the following:

$$
\frac{A_i, A_0 \lor A_1, \Delta_0}{\tilde{A}_0 \lor \tilde{A}_1, \Delta_0} (\lor)_1 + 1
$$

$$
\frac{\Delta_0 \lor A_1, \Delta, \tilde{A}_i}{a + b + 1} (\lor)_2
$$

Case 4.2. Otherwise: Let $I$ denote the uppermost $(hgt)$ below the $(cut) J$. Such an $(hgt)$ exists since $|\tilde{A}_0 \lor \tilde{A}_1| > 0$. Put $h = h(\tilde{A}_0 \lor \tilde{A}_1, \Delta) - 1$. Then $h = h(\Lambda) \geq \max(\{\tilde{A}_0, |\tilde{A}_1|\})$ for the lowersequent $\Lambda$ of the $(hgt) I$.

$$
\frac{A_i, A_0 \lor A_1, \Delta_0}{\tilde{A}_0 \lor \tilde{A}_1, \Delta_0} (\lor)_1 + 1
$$

$$
\frac{\Delta_0 \lor A_1, \Delta, \tilde{A}_i}{a + b + 1} (\lor)_2
$$

where $a = o(\Gamma, \neg \tilde{A}_0 \land \neg \tilde{A}_1), b + 1 = o(\tilde{A}_0 \lor \tilde{A}_1, \Delta)$ and $o(\Lambda) = D_{q+1}(c + 1)$ for some $c$.

Assuming $\neg \tilde{A}_i$ is an $E$ formula, let $P'$ be the following:

$$
\frac{A_i, A_0 \lor A_1, \Delta_0}{\tilde{A}_0 \lor \tilde{A}_1, \Delta_0} (\lor)_1 + 1
$$

$$
\frac{\Delta_0 \lor A_1, \Delta, \tilde{A}_i}{a + b + 1} (\lor)_2
$$

Here the subproof ending with $\neg \tilde{A}_i, \Gamma$ is obtained from the subproof of $P$ ending with the left uppersequent $\Gamma, \neg \tilde{A}_0 \lor \neg \tilde{A}_1$ of the $(cut) J$ by inversion. Observe that we still have $a = o(\neg \tilde{A}_i, \Gamma); P'$ under the same ordinal assignment since the lowersequent and the uppersequents of a rule $(\land)$ have the same assigned ordinal.

Case 6. $\Phi$ is the lowersequent of a $(\neg P_u)$. Let $J$ denote the $(cut)$ at which the descendent of the principal formula of the $(\neg P_u)$ vanishes and $I$ the $u$-resolvent of $\Phi = t \neg P_u, \Delta_0$. Here note that there is no $(sub)_u(v \leq u)$ between the $(\neg P_u)$ and $J$ by the restriction: the uppersequent of a $(sub)_v \in Pos_v$, i.e., $P_u(t) \notin Pos_v$. Therefore the $u$ resolvent $I$ is below $J$. Also by the definition there is no $(sub)_u(v \leq u)$ between $J$ and $I$. 


\[
\begin{align*}
\frac{t \notin P_u, \neg N_u(F, t), \Delta_0}{t \notin P_u, \Delta_0} & (\neg P_u), + \Omega_{u+1} \\
\frac{\Gamma, t \in P_u}{\Gamma, \Delta, \Lambda} & J, a + b \\
\frac{\Gamma, t \in P_u}{\Gamma, \Delta, \Lambda} & (sub)_u I, D_v c
\end{align*}
\]

where \( a = o(\Gamma, t \in P_u), b = o(t \notin P_u, \Delta), c = o(\Lambda) \) and \( o(t \notin P_u, \Delta_0) = b_0 + \Omega_{u+1} \) with \( b_0 = o(t \notin P_u, \neg N_u(F, t), \Lambda) \). We have \( dom(b) = dom(c) = dom(\Omega_{u+1}) = T_u(q + 1) \).

Put \( z = D_u c[1] \). Let \( P' \) be the following:

\[
\begin{align*}
\frac{\Gamma, \Lambda, \Lambda}{\Lambda, \Lambda} & D_v c[z]
\end{align*}
\]

where the subproof ending with \( \Gamma, \Lambda, \Lambda \) is obtained from the subproof in \( P \) ending with the left uppersequent \( \Gamma, t \in P_u \) of \( (cut) \) \( J \) by inversion. Note that \( N_u(F, t) \) is an \( E \) formula. We have \( o(\Gamma, N_u(X, t); P') = a = o(\Gamma, t \in P_u; P) \) and hence \( o(P_0) = o(\Lambda, P') = c[1] \). Thus \( o(N_u(F, t), \Lambda; P') = D_u c[1] = z \) and \( o(\Lambda; P') = D_v c[z] = (D_v c)[1] \) with \( D_v c = o(\Lambda; P) \). Therefore \( o(P') = o[1] \).

This completes a proof of the Lemma 2 and hence of the Theorem 1.
Remark. As in [A] we have for each $n < \omega$

$$I\Sigma_k \vdash \forall d \exists m \{(D_0D_1^{k-1}(\Omega \cdot n))[d]^m = 0\}$$

From this we can expect to sharpen the Theorem 1 for fragments, e.g., for $I\Sigma_k$ but we have no proof of the following:
Show that

$$I\Sigma_k \not\vdash \forall n \exists m \{(D_0D_1^{k-1}(\Omega \cdot n))[1]^m = 0\}$$

References


