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<thead>
<tr>
<th>Title</th>
<th>Consistency Proof via Pointwise Induction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1997, 976: 125-134</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-02</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60800">http://hdl.handle.net/2433/60800</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>TextVersion</td>
<td>publisher</td>
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Consistency Proof via Pointwise Induction

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Abstract

We show that the consistency of the first order arithmetic $PA$ follows from the pointwise induction up to the Howard ordinal. Our proof differs from U. Schmerl [S]: We do not need Girard's Hierarchy Comparison Theorem. A modification on ordinal assignment to proofs by Gentzen and Takeuti [T] is made so that one step reduction on proofs exactly corresponds to the stepping down $\alpha \mapsto \alpha[1]$ in ordinals. Also a generalization to theories $ID_{\varepsilon}$ of finitely iterated inductive definitions is proved.

We show that the consistency of the first order arithmetic $PA$ follows from the pointwise induction up to the Howard ordinal. Our proof differs from U. Schmerl [S]: We do not need Girard's Hierarchy Comparison Theorem.

Let $P$ be a proof of the empty sequent in $PA$ or the second order arithmetic $\Pi_{1}^{1} - CA_{0}$. For such a proof $P$ let $o(P)$ denote the ordinal assigned to $P$ and $r(P)$ a reduct of $P$ defined by Gentzen and Takeuti [T]. $r(P)$ is again a proof of the empty sequent and $o(r(P)) < o(P)$. Then the reduction $r : P \rightarrow r(P)$ is close to but does not fit perfectly the stepping down $\alpha \mapsto \alpha[n]$ defined by Buchhloz [B2].\footnote{This observation is also stated by M. Hamano and M. Okada [H-O].} We need to tune these functions $o$ and $r$ to stepping down in order to have $o(r(P)) = o(P)[n]$ for an $n$. For this purpose we introduce two inference rules: the padding rule and the height rule. In both rules the lowersequent is identical with the uppersequent. Let $S_{u}$ [$S_{l}$] denote the uppersequent [the lowersequent], resp. Also let $h(S)$ denote the height of a sequent (in a proof $P$).

\[
\frac{\Gamma}{\Pi} (pad)_b
\]

with $o(S_{l}) = o(S_{u}) + b$.

\[
\frac{\Gamma}{\Pi} (ht)
\]

with $h(S_{u}) = h(S_{l}) + 1$ and $o(S_{i}) = D_{0}o(S_{u})$. $D_{0}a$ denotes an ordinal term defined in [B2].

Using these rules we can unwind gaps between $D$ and $r(D)$ so that $o(r(P)) = o(P)[1]$ holds.

1 Fundamental sequences

The following is the fundamental sequences given in [A] and a slight variant in Buchhloz [B2]. Let $q$ be a natural number.

**Definition 1** (Buchholz [B2]) The term structure $(T(q), \subseteq)$

1. Inductive definition of the sets $PT(q)$ and $T(q)$

   (T0) $PT(q) \subseteq T(q)$

   (T1) $0 \in T(q)$

   (T2) $a \in T(q) \& 0 \leq u \leq q \Rightarrow D_{u}a \in PT(q)$

   (T3) $a_{0}, \ldots, a_{k} \in PT(q)(k > 0) \Rightarrow (a_{0}, \ldots, a_{k}) \in T(q)$

2. For $a_{0}, \ldots, a_{k} \in PT(q)$ and $k \in \{-1, 0\}$, we set

\[
(a_{0}, \ldots, a_{k}) = \begin{cases} 
0 & k = -1 \\
a_{0} & \text{otherwise}
\end{cases}
\]
3. \(a + 0 = 0 + a = a; (a_0, \ldots, a_k) + (b_0, \ldots, b_m) = (a_0, \ldots, a_k, b_0, \ldots, b_m); a \cdot 0 = 0; a \cdot (n + 1) = a \cdot n + a\)

4. \(\omega = \omega \cup \{0, 1, 1 + 1, \ldots\} \subset T(q)\) with \(1 = D_0 0\)

5. For \(u \leq q\),
   \[T_u(q) = \{(D_{u_0}a_0, \ldots, D_{u_k}a_k) : k \geq 1, a_0, \ldots, a_k \in T(q), u_0, \ldots, u_k \leq u\}\]

6. \(dom(a)\) and \(a[z]\) for \(a \in T(q)\) and \(z \in dom(a)\)
   
   (1) \(dom(0) = 0\)
   
   (2) \(dom(1) = \{0\}; 1[0] = 0\)
   
   (3) \(dom(D_{u+1}0) = T_u(q); (D_{u+1}0)[z] = z\)
   
   (4) \(a = b \in T(q)\) with \(b \neq 0\).

Proposition 1 (Buchholz [B1])

\[a, z \in OT(q), k \in dom(a) \Rightarrow a[z] \in OT(q)\]

Conventions.

1. \(\Omega_q = \omega \cup D_0 0\)

2. \(0[n] = 0; (a + 1)[n] = a \) for \(n \in \omega\)

3. \(a[n]^0 = a; a[n]^{m+1} = a[a[n]^m][n]\)

4. \(D_0^0 a = a; D_0^{k+1} a = D_u (D_0^k a)\)

\(ERA\) denotes the Elementary Recursive Arithmetic.

Let \((PI)\) denote the following inference rule:

\[A(0, p) \quad \alpha \neq 0 \land A(a[1], r(p)) \supset A(\alpha, p) \quad (PI)\]

where \(\alpha\) denotes a variable ranging over \(OT(q)\), and \(A[r]\) is an elementary recursive relation \(\in E^2 \cup \{function \in E^3\}\), resp.

For a theory \(T\) let \(Con^{(n)}(T)\) denote the iterated consistency of \(T\):

\[Con^{(0)}(T) \iff \forall x(0 = 0); Con^{(n+1)}(T) \iff Con(T + Con^{(n)}(T))\]

Now our theorems are stated as follows:

Theorem 1 For each natural number \(q\),

1. Over \(ERA\), \(\{Con^{(n)}(ID_q) : n < \omega\}\) is equivalent to \((PI)_{q+1}\).

2. Over \(ERA\), the \(1\)-consistency \(RFN_{\Sigma_1}(ID_q)\) of \(ID_q\) is equivalent to
   \[\forall n \exists m \{(D_0 D_1^{n+1} (\Omega_x \cdot n)) [n] = 0\}\].

For provably total recursive functions we have, e.g.,

Proposition 2 For each provably total recursive function \(f\) in \(PA\), there exist \(k\) and \(d\) such that \(\forall n[f(n) \leq d \cdot \mu m \{(D_0 D_1^k (\Omega_x (k + n)))[n] = 0\}]\).

This is seen from a slight modification of the proof of the theorem and so we omit a proof.

Remark. U. Schmerl [S] gives a proof of a variant of the Theorem 1.1 \((q = 0)\), i.e., for \(PA\) via Girard’s Hierarchy Comparison Theorem. In [S] the base theory (for us \(ERA\)) contains the fast growing functions \(F_a (\alpha < \epsilon_0)\) and/or the slow growing functions \(G_a (\alpha < \psi_0 \epsilon_{\Omega+1})\) and their defining equations. Hence it seems that Schmerl’s result is incomparable to ours.
2 Proof of Theorem

Fix a natural number $q$. We prove the Theorem 1.1. The Theorem 1.2 is proved similarly.

First consider the easy half: The rule $(PI)_{t+1}$ is a derived rule in $ERA + \{Con^{(n)}(ID_q) : n < \omega\}$. Let $prov_{\Sigma}$ denote a standard proof predicate for a theory $T$ and "$B$" the Gödel number of an expression $B$. This follows from the following fact which is shown in [A]:

**Proposition 3** For some elementary recursive function $f$ we have

$$ERA \vdash \forall a \in T_0(q + 1)\{prov_{ID_q}(f(a), \forall n \exists m(a[n]^m = 0))\}$$

Next consider the other half. Let $\forall zB(z)$ be a $\Pi^0_2$ sentence. $ID_q + \forall zB(z)$ denote the theory obtained from $ID_q$ by adding extra axioms $B(t)$ for an arbitrary term $t$. It suffices to show, in $ERA + \{PI\}_{t+1}$,

$Con(ID_q + \forall zB(z))$ under the assumption 'there is true'. Our proof is an adaption from Gentzen's and Takeuti's reduction in [T].

First $ID_q$ is embedded in a first order theory $NID_{q+1}$. In the latter theory the universe $\omega$ of $ID_q$ is replaced by a constant $N$ and this constant is treated as if it were a $\Pi^1_2$ formula. Then as in [T] the inference rules for the constant $N$ are analysed by using a substitution rule. Also as mentioned above we introduce two new rules, the padding rule and the height rule to unwind gaps in Gentzen-Takeuti reduction. Now details follow.

The *language* $L$ of $ID_q$ consists of

1. function constants 0 and the successor $'$,
2. arithmetic predicate constants are lower elementary recursive relations $R \in L_2^2$ and their negations $\neg R$,
3. the least fixed points $\{P_u\}_{1 \leq u \leq q}$ for a fixed positive operator form $A(X^+, Y, n)$ and
4. logical symbols $\land, \lor, \forall, \exists$.

The *negation* $\neg A$ of a formula $A$ is defined by using de Morgan's law and the elimination of double negations. A prime formula $R(t_1, \ldots, t_n)$ or its negation $\neg R(t_1, \ldots, t_n)$ with an arithmetic predicate $R$ is an a.p.f.(arithmetic prime formula).

The *axioms* in $ID_q$ are axioms for function and arithmetic predicate constants, the induction axiom $(IA)$ and axioms $(P.1), (P.2)$ of the least fixed points $\{P_u\}_{1 \leq u \leq q}$ for arbitrary formula $F$:

$(IA) \quad F(0) \land \forall z F(z) \supset F(x') \supset \forall z F(x)$

$(P.1) \quad A_u(P_u) \subseteq P_u$

$(P.2) \quad A_u(F) \subseteq F \supset P_u \subseteq F$

where $A_u(X) = \{n : A(X, \sum_{1 \leq v < u} P_v, n)\}$.

The *language* $L_N$ of $NID_{q+1}$ consists of $L \cup \{N\} \cup \{X_i : i < \omega\}$ with a unary predicate constant $N$ and a list of unary predicates $X_i$. These unary predicates are denoted $X, Y, \text{etc.}$ We sometimes write $P_0$ for the constant $N$. For a predicate constant $H \in \{P_u : u \leq q\} \cup \{X_i : i < \omega\}$, we write $t \in H$ for $H(t)$ and $t \notin H$ for $\neg H(t)$. A formula is said to be an $E$ formula if it is either an a.p.f. or a formula in one of the following shapes; $A \lor B, \exists x A$ or $t \notin H$ with $H \in \{P_u : u \leq q\} \cup \{X_i : i < \omega\}$. A formula is an $A$ formula if its negation is an $E$ formula. For a formula $A$ in $L$ let $A^N$ denote the result of restricting all quantifiers in $A$ to $N$. For each $u \leq q$ let $N_u(X, t)$ denote the formula:

$$N_0(X, t) \equiv 0 \in X \land \forall z(x \in X \supset x' \in X) \supset t \in X;$$

$$N_u(X, t) \equiv A_u^N(X) \subseteq X \supset t \in X (u \neq 0)$$

$NID_{q+1}$ is formulated in Tait's calculus, i.e., one sided sequent calculus. Finite sets of formulae is called a *sequent*. Sequents are denoted by $\Gamma, \Delta, \text{etc.}$

**Axioms** in $NID_{q+1}$ are:

**Logical Axiom** $\Gamma, \neg A, A$

where $A$ is either an a.p.f. or a formula of the shape $t \in X$.

**Arithmetic Axiom**

1. $\Gamma, \Delta_R$

   where $\Delta_R$ consists of a.p.f.'s and corresponds to the definition of a lower elementary relation $R$.

2. $\Gamma, A$ for a true closed a.p.f. $A$. 

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3. $\Gamma, \Delta_0$
where there exists a sequent $\Delta_1$ so that $\Delta = \Delta_0 \cup \Delta_1$ is an instance of a defining axiom for $R$ in 1 and $\Delta_1$ consists solely of false closed $a.p.f.$s.

**Inference rules in $NID_{q+1}$ are:**

- $(\Lambda), (\forall), (\exists), (\text{cut}), (weak)$ and $(P_u), (\neg P_u)$ for $(u \leq q)$.

1. $(\Lambda), (\forall), (\exists):$ In these rules the principal formula is contained in the uppersequent. For example

\[
\frac{\Gamma, \exists x A(x), A(t)}{\Gamma, \exists x A(x)} (\exists)
\]

2. In the rule $(\text{cut})$

\[
\frac{\Gamma, \neg A, A, \Delta}{\Gamma, \Delta} (\text{cut})
\]

the cut formula $A$ is an $E$ formula.

3. $(weak)$ is the weakening:

\[
\frac{\Gamma}{\Delta} (weak)
\]

with $\Gamma \subseteq \Delta$.

4. $(P_u)$:

\[
\frac{\Gamma, t \in P_u, N_u(X, t)}{\Gamma, t \in P_u} (P_u)
\]

where $X$ is the eigenvariable, i.e., does not occur in the lowersequent.

5. $(\neg P_u)$:

\[
\frac{\Gamma, t \notin P_u, \neg N_u(F, t)}{\Gamma, t \notin P_u} (\neg P_u)
\]

for an arbitrary formula $F$ in the language $L_N$.

**Lemma 1** For any sentence $A$ in $L$,

\[ ID_q \vdash A \Rightarrow NID_{q+1} \vdash A^N \]

**Proof.** It suffices to show that the following sequents are provable in $NID_{q+1}$:

- $t \notin P_u, t \in P_u$: This is proved by induction on $u \leq q$. By IH (Induction Hypothesis) we have $\neg N_u(X, t), N_u(X, t)$. Rules $(P_u)$ and $(\neg P_u)$ yields $t \notin P_u, t \in P_u$.

$(IA)^N$: For a given formula $F(x)$ assume $a \in N$, $F(0)$ and $\forall x \in N (F(x) \supset F(x'))$. We have to show $F(a)$. Let $G(x)$ denote the formula $x \in N \land F(x)$. Then we see $\forall x (G(x) \supset G(x'))$ from $x \in N \supset x' \in N$. The latter follows from the rules $(\neg N) = (\neg P_0)$ and $(N)$. Also we have $G(0)$. On the other hand we have $N_0(G, a)$ by the rule $(\neg N)$ and $a \in N$. Thus we get $G(a)$ and hence $F(a)$.

$(P.1)^N$: Assume $A^N_u(P_u, a)$. We have to show $a \in P_u$. By the rule $(P_u)$ it suffices to show $N_u(X, a)$. Assume $A^N_u(X) \subseteq X$. We show $a \in X$.

**Claim.** $P_u \subseteq X$

**Proof of the Claim.** Assume $x \in P_u$. By the rule $(\neg P_u)$ we have $N_u(X, x)$. The assumption $A^N_u(X) \subseteq X$ yields $x \in X$. From this Claim and the positivity of $X$ in $A$ we see $A^N_u(X, a)$. Thus again by the assumption $A^N_u(X) \subseteq X$ we conclude $a \in X$.

$(P.2)^N$: For a given formula $F$ assume $A^N_u(F) \subseteq F$ and $a \in N \cap P_u$. We show $F(a)$. This follows from the rule $(\neg P_u)$.

**Definition 2** The length $|A|$ of a formula $A$ in $L_N$
1. $|A| = 0$ for a prime formula $A$. Specifically $|\neg H(t)| = 0$ for any predicate $H$.

2. $QA A \models F + 1$ for $Q \in \{\forall, \exists\}$

3. $A_{0} \circ A_{1} = \max\{|A_{i}| + 1 : i = 0, 1\}$ for $\circ \in \{\land, \lor\}$.

A formula $A \in \text{Pos}_u$ if 1. a predicate $P_v$ occurs positively in $A$, then $v \leq u$, and 2. if a predicate $P_v$ occurs negatively in $A$, then $v < u$.

Observe that $N_{u}(X, t) \in \text{Pos}_u$ and $\neg P_{u}(t) \notin \text{Pos}_u$ & $\text{Pos}_v \subseteq \text{Pos}_u$ for $v \leq u$.

Let $\forall xB(x)$ denote a fixed true $\Pi^0_1$ sentence with an a.p.f. $B$. The system $NID_{q+1} + \forall xB(x)$ is obtained from $NID_{q+1}$ by adding the axiom

$$(B) \Gamma, B(t)$$

for an arbitrary term $t$ and three inference rules; the padding rule $(pad)_b (b \in \text{OT}(q + 1))$, the height rule $(hgt)$ mentioned in the introduction and the substitution rule $(sub)_u (u \leq q)$.

$$\frac{\Gamma(X)}{\Gamma(F)} (sub)_u$$

where 1. $\Gamma(X) \subseteq \text{Pos}_u$, 2. $X$ is the eigenvariable, i.e., does not occur the lowersequent, 3. $F$ is an arbitrary formula in $L_N$ and 4. $\Gamma(F)$ denotes the result of substituting $F$ for $X$ in $\Gamma(X)$.

Let $P$ be a proof (in $NID_{q+1} + \forall xB(x)$) and $\Gamma$ a sequent in $P$. We define the height $h(\Gamma) = h(\Gamma; P)$ of $\Gamma$ in $P$ as follows:

1. $h(\Gamma) = 0$ if $\Gamma$ is either the endsequent of $P$ or the uppersequent of a rule $(sub)$.

2. $h(\Gamma) = h(\Delta) + 1$ if $\Gamma$ is the uppersequent of an (hgt) whose lowersequent is $\Delta$.

3. $h(\Gamma) = h(\Delta)$ if $\Gamma$ is the uppersequent of a rule other than $(sub)$ and $(hgt)$ and $\Delta$ is the lowersequent.

Again let $P$ be a proof (in $NID_{q+1} + \forall xB(x)$). Let $o$ denote an assignment of an ordinal term $o(\Gamma) = o(\Gamma; P) \in \text{OT}(q + 1)$ to each sequent $\Gamma$ in $P$. If the assignment $o : \Gamma \mapsto o(\Gamma)$ enjoys the following conditions, then we say that $o$ is an ordinal assignment for $P$.

1. $o(\Gamma) \neq 0$ for each axiom $\Gamma$.

Assume that $\Gamma$ is the lowersequent of a rule $J$ and $\Gamma_0$ and $\Gamma_1$ denote the uppersequents of $J$.

2. $o(\Gamma) = o(\Gamma_0)$ if $J$ is one of $(\forall), (weak), (P_u)$.

3. $o(\Gamma) = o(\Gamma_0) = o(\Gamma_1)$ if $J$ is $(\land)$.

N.B. We require ordinals assigned to uppersequents of a $(\land)$ are equal.

4. $o(\Gamma) = o(\Gamma_0) + b$ for some nonzero $0 \neq b \in \text{OT}(q + 1)$ if $J$ is either $(\exists)$ or $(\forall)$.

In this case we write, e.g., $(\forall)_b$ for the rule $(\forall)$.

5. $o(\Gamma) = o(\Gamma_0) + b$ if $J$ is $(pad)_b$.

6. $o(\Gamma) = o(\Gamma_0) + \Omega_{1+u}$ if $J$ is $(\neg P_u)$.

7. $o(\Gamma) = o(\Gamma_0) + o(\Gamma_1)$ if $J$ is $(cut)$.

8. $o(\Gamma) = D_{q+1} o(\Gamma_0)$ if $J$ is $(hgt)$.

9. $o(\Gamma) = D_o o(\Gamma_0)$ if $J$ is $(sub)_u (u \leq q)$.

For an ordinal assignment $o$ for a proof $P$ we set $o(P) = o(\Gamma_{end})$ with the endsequent $\Gamma_{end}$ of $P$.

Remark.

1. The padding rule $(pad)_b$ is implicit in the literature, e.g., in [B2].

2. The substitution rule $(sub)_u$ comes from [T] but Buchholz mentions a substitution operation $Nt \rightarrow Ft$ in the proof of Lemma 4.12 in [B2].

Let $P$ be a proof in $NID_{q+1} + \forall xB(x)$ and $o$ an ordinal assignment for $P$. We say that $(P, o)$ is a proof with the o.a. (ordinal assignment) $o$ if the following conditions are fulfilled:

(p0) The endsequent of $P$ is the empty sequent.
The final part of \( P \) is an empty \((\text{sub})_0\) followed by a nonempty series \{\((\text{pad})_{b_i}\)\}_{i \leq n} of paddings with \( \text{dom}(b_i) \in \{\emptyset, \{0\}, \omega\} \):

\[
\begin{array}{c}
\vdots \\
(\text{sub})_0 \\
(\text{pad})_{b_0} \\
\vdots \\
(\text{pad})_{b_n}
\end{array}
\]

For any \((\text{cut})\) in \( P \),

\[
\frac{\Gamma, \neg A, A, \Delta}{\Gamma, \Delta} \quad \text{(cut)}
\]

\(| A | \leq h(A, \Delta; P) = h(\Gamma, \neg A; P)\).

For any \((\neg P_u)\) in \( P \),

\[
\frac{\Gamma, t \notin P_u, \neg N_u(F, t)}{\Gamma, t \notin P_u} \quad \text{(\neg P_u)}
\]

\(| \neg N_u(F, t) | \leq h(\Gamma, t \notin P_u; \neg N_u(F, t); P)\).

**Proposition 4** Assume \( ID_{q} + \forall B(x) \) is inconsistent. Then there exists a proof \( P \) with an a.a. o.

**Proof.** By Lemma 1 pick a proof \( P_0 \) in \( NID_{q+1} \) ending with the empty sequent. \( P_0 \) contains none of rules \((\text{sub}), (\text{pad}), (\text{hgt})\). Below the endsequent of \( P_0 \) attach some \((\text{hgt})\)’s to enjoy the conditions \((\text{p2})\) and \((\text{p3})\). After that attach further a \((\text{sub})_0\) and a \((\text{pad})_0\) to ensure the condition \((\text{p1})\). Let \( P \) denote the resulting proof in \( NID_{q+1} + \forall B(x) \) of the empty sequent. For each sequent \( \Gamma \) in \( P_0 \) set the ordinal \( o(\Gamma) = \Omega_{q+1} \cdot n \) for some \( n < \omega \). Then the whole proof \( P \) has an ordinal \( o(P) = D_0 D_{q+1}(\Omega \cdot k) \) for some \( k \).

Thus assuming that \( \forall B(x) \) is true, it suffices to show the following lemma.

**Lemma 2** Let \((P, o)\) be a proof with an a.a. o. Then there exists a proof \((P', o) = r(P, o)\) with an a.a. o such that

\[ o(P') = o(P)[1] \]

It remains to prove the Lemma 2.

Let \( P \) be a proof (not necessarily ending with the empty sequent). The main branch of \( P \) is a series \{\( \Gamma_i \)\}_{i \leq n} of sequents in \( P \) such that:

1. \( \Gamma_0 \) is the endsequent of \( P \).
2. For each \( i < n \) \( \Gamma_{i+1} \) is the right uppersequent of a rule \( J_i \) so that \( \Gamma_i \) is the lowersequent of \( J_i \) and \( J_i \) is one of the rules \((\text{cut}), (\text{weak}), (\text{hgt}), (\text{sub})\) and \((\text{pad})_0\).
3. Either \( \Gamma_n \) is an axiom or \( \Gamma_n \) is the lowersequent of one of the rules \((\lor), (\exists), (\neg P)\) and \((\text{pad})_b\) with \( b \neq 0 \).

The sequent \( \Gamma_n \) is called the top (of the main branch) of the proof \( P \).

Let \( P \) be a proof with an o.a. \( P \) and \( \Gamma \) a sequent in \( P \). The u-resolvent of \( \Gamma \) is the uppermost substitution rule \((\text{sub})_b\) below \( \Gamma \) with \( v \leq u \). Note that such a substitution rule always exists by the condition \((\text{p1})\).

Let \( \Phi \) denote the top of the proof \( P \) with the o.a. \( o \). Put \( \alpha = o(P) \). Observe that we can assume \( \Phi \) contains no first order free variable.

**Case 1.** \( \Phi \) is the lowersequent of a rule \((p)\) at which the ordinal \( b \) is padded. This means that either \((p)_b = (\text{pad})_b\) with \( b \neq 0 \) or \((p)_b = (\lor)_b, (\exists)_b\) with \( b > 1 \).

\[
\begin{array}{c}
\overline{\Phi} + b, (p)_b \\
\vdots \\
P
\end{array}
\]

**Case 1.1.** Either the top \( \Phi \) is the endsequent \( \), then the last rule is

\((p)_b = (\text{pad})_b\) with \( b \neq 0 \), and/or \( \text{dom}(b) = \omega: \text{dom}(\alpha) = \text{dom}(b) \). Replace the rule \((p)_b\) by \((p)_{b[1]}\). Note that \( b[1] \neq 0 \) if \( b > 1 \).

\[
\begin{array}{c}
\overline{\Phi} + b[1], (p)_{b[1]} \\
\vdots \\
P'
\end{array}
\]
Case 1.2. Otherwise:

Case 1.2.1. \( \text{dom}(b) = T_u(q+1) \): Let \( I \) be the \( u \)-resolvent of \( \Phi \) and \( \Gamma \) the lowersequent of \( J \). \( o(\Gamma) \) is of the form \( D_v a \) for some \( v \leq u \) and \( a \) with \( \text{dom}(a) = T_u(q+1) \). We have \( (D_v a)[1] = D_v a[a_1] \) with \( a_1 = D_u a[1] \). Replace the \((p)\) by \((p)[a_1]\).

Case 1.2.2. \( \text{dom}(b) = \{0\} \), i.e., \( b = b_0 + 1 \) for some \( b_0 \): Let \( J \) denote the uppermost \((\text{sub})\) or \((\text{hgt})\) below \( \Phi \) and \( \Gamma \) the lowersequent of \( J \). \( o(\Gamma) \) is of the form \( D_v a \) for some \( v \leq u \) and \( a \) with \( \text{dom}(a) = T_u(q+1) \).

Case 2. \( \Phi \) is an axiom and contains a true a.p.f. \( A: \Phi = A, \Delta_0 \).

\[
\begin{array}{c}
\Phi \vdash a \\
\Gamma \vdash b \\
\Gamma, \neg A \vdash A, \Delta \\
\Gamma, \Delta \vdash a + b \\
\end{array}
\]

where \( a = o(\Gamma, \neg A), b = o(A, \Delta) \) \((b \neq 0)\). Eliminate the false a.p.f. \( \neg A \) and insert a \((\text{weak})\) and an appropriate \((\text{pad})\) as in Case 1 to get \( o(P') = \alpha[1] \).

Case 3. \( \Phi \) is a logical axiom: \( \Phi = \neg X(t), X(t), \Delta_0 \). Put

\[
\begin{array}{c}
X^+(t) = X(t), X^-(t) = \neg X(t), (\neg X(t))^+ = X(t), (\neg X(t))^- = X(t). \\
\end{array}
\]

\[
\begin{array}{c}
\Gamma, X^\pm(t) \vdash X^\pm(t), \Delta \\
\Gamma, \Delta \\
\end{array}
\]

where \( X^\pm(t) \in \Delta \) and \( X \) denotes either \( X \) or a formula \( F \) by a \((\text{sub})\). As in Case 2 insert a \((\text{weak})\) and a \((\text{pad})\).

Case 4. \( \Phi \) is the lowersequent of a \((\vee)_1 \).

Case 5. \( \Phi \) is the lowersequent of an \((\exists)_1 \).

Consider the Case 4. Let \( J \) denote a \((\text{cut})\) at which the descendent of the principal formula \( A_0 \vee A_1 \) of the \((\vee)_1 \) vanishes.

Case 4.1. There exists an \((\text{hgt})\) or a \((\text{sub})\) between \( \Phi \) and \( J \): Let \( I \) denote the uppermost one among such
\begin{align*}
\frac{A_i, A_0 \lor A_1, \Delta_0}{A_0 \lor A_i, \Delta_0} \quad (\lor)_1 + 1 \\
\frac{A'_0 \lor A'_1, \Delta'}{I} \\
\frac{\tilde{A}_0 \land \tilde{A}_1 \lor \tilde{A}_0 \lor \tilde{A}_1, \Delta}{\Gamma, \Delta} \quad J \\
\end{align*}

where 1) \( i = 0,1 \), 2) \( \tilde{A}_i \) is either \( A_i \) or \( \neg A_i \) \((X := F)\) and 3) \( A'_i \in \{A_i, \tilde{A}_i\}\). For the lowersequent of \( I \)
\( o(A'_0 \lor A'_1, \Delta') = D_u(c+1) \) for some \( u \leq q+1 \) and a \( c\).

Lower the \((\lor)_1\) under \( I \) and change +1 into \(+D_u c\):
\begin{align*}
\frac{A_i, A_0 \lor A_1, \Delta_0}{A_0 \lor A_i, \Delta_0} \\
\frac{A'_0 \lor A'_1, \Delta', \Delta'}{I} \quad (\lor)_{D_u c} \\
\frac{\tilde{A}_0 \land \tilde{A}_1 \lor \tilde{A}_0 \lor \tilde{A}_1, \Delta}{\Gamma, \Delta} \quad J, a+b+1 \\
\end{align*}

\begin{align*}
\frac{\Gamma, \tilde{A}_0 \land \tilde{A}_1 \lor \tilde{A}_0 \lor \tilde{A}_1, \Delta}{\Gamma, \Delta} \quad (hgt) I, D_{u+1}(c+1) \\
\end{align*}

Case 4.2. Otherwise: Let \( I \) denote the uppermost \((hgt)\) below the \((cut)\) \( J \). Such an \((hgt)\) exists since \(| \tilde{A}_0 \lor \tilde{A}_1 | > 0 \). Put \( h = h(\tilde{A}_0 \lor \tilde{A}_1, \Delta) - 1 \). Then \( h = h(\Lambda) \geq \max(| \tilde{A}_0 |, | \tilde{A}_1 |) \) for the lowersequent \( \Lambda \) of the \((hgt) I \).
\begin{align*}
\frac{A_i, A_0 \lor A_1, \Delta_0}{A_0 \lor A_i, \Delta_0} \quad (\lor)_1 + 1 \\
\frac{\tilde{A}_0 \land \tilde{A}_1 \lor \tilde{A}_0 \lor \tilde{A}_1, \Delta}{\Gamma, \Delta} \quad J, a+b+1 \\
\end{align*}

\begin{align*}
\frac{\Gamma, \tilde{A}_0 \land \tilde{A}_1 \lor \tilde{A}_0 \lor \tilde{A}_1, \Delta}{\Gamma, \Delta} \quad (hgt) I, D_{u+1}(c+1) \\
\end{align*}

where \( a = o(\Gamma, \neg \tilde{A}_0 \land \neg \tilde{A}_1), b + 1 = o(\tilde{A}_0 \lor \tilde{A}_1, \Delta) \) and \( o(\Lambda) = D_{u+1}(c+1) \) for some \( c \).

Assuming \( \neg \tilde{A}_i \) is an \( E \) formula, let \( P' \) be the following:
\begin{align*}
\frac{A_i, A_0 \lor A_1, \Delta_0}{A_0 \lor A_i, \Delta_0} \\
\frac{\tilde{A}_0 \land \tilde{A}_1 \lor \tilde{A}_0 \lor \tilde{A}_1, \Delta}{\Gamma, \Delta, \tilde{A}_i} \\
\frac{\Gamma, \tilde{A}_i, \tilde{A}_i, \Delta}{a+b} \quad (weak, a) \\
\frac{\Gamma, \tilde{A}_i, \tilde{A}_i, \Delta}{a+b} \quad (pad) b, a+b \\
\frac{\Lambda, \tilde{A}_i}{D_{u+1} c} \quad (cut) \quad (D_{u+1} c) \cdot 2 \\
\end{align*}

Here the subproof ending with \( \neg \tilde{A}_i, \Gamma \) is obtained from the subproof of \( P \) ending with the left uppersequent \( \Gamma, \neg \tilde{A}_0 \lor \neg \tilde{A}_1 \) of the \((cut) J \) by inversion. Observe that we still have \( a = o(\neg \tilde{A}_i, \Gamma; P') \) under the same ordinal assignment since the lowersequent and the uppersequents of a rule \((\land)\) have the same assigned ordinal.

Case 6. \( \Phi \) is the lowersequent of a \((\neg P_u)\).

Let \( J \) denote the \((cut) \) at which the descendent of the principal formula of the \((\neg P_u) \) vanishes and \( I \) the \( u\)-resolvent of \( \Phi = \neg P_u, \Delta_0 \). Here note that there is no \( (sub)_v (v \leq u) \) between the \((\neg P_u) \) and \( J \) by the restriction: the uppersequent of a \((sub)_v \subseteq Pos_v, \) i.e., \( \neg P_u(t) \notin Pos_v \). Therefore the \( u \) resolvent \( I \) is below \( J \). Also by the definition there is no \( (sub)_v (v \leq u) \) between \( J \) and \( I \).
$\frac{t \notin P_u, \neg N_u(F,t), \Delta_0}{t \notin P_u, \Delta_0}$ (\(\neg P_u\)), \(\Omega_{u+1}\)\]

\[
\frac{\Gamma, t \in P_u \quad t \notin P_u, \Delta}{\Gamma, \Delta, a + b}
\]

\[J, a + b\]

\[a\]

\[b\]

\[c\]

\[\Delta (sub)_u I, D_v c\]

where \(a = o(\Gamma, t \in P_u), b = o(t \notin P_u, \Delta), c = o(\Lambda)\) and \(o(t \notin P_u, \Delta_0) = b_0 + \Omega_{u+1}\) with \(b_0 = o(t \notin P_u, \neg N_u(F,t), \Delta_0)\). We have \(\text{dom}(b) = \text{dom}(c) = \text{dom}(\Omega_{u+1}) = T_u(q + 1)\).

Put \(z = D_v c[1]\). Let \(P'\) be the following:

\[
\frac{\Gamma, N_u(X,t)}{N_u(X,t), \Gamma, \Delta} (\text{weak})
\]

\[
\frac{N_u(X,t), \Gamma, \Delta}{\Gamma, N_u(X,t), \Delta} (pad)_s[1], b[1]
\]

\[
\frac{N_u(X,t), \Lambda}{\tilde{\Lambda}} (\text{sub})_u, z = D_v c[1]
\]

\[
\frac{t \notin P_u, \Delta_0 \neg N_u(F,t), \Delta_0}{t \notin P_u, \Delta_0} (\text{cut}), b_0 + z
\]

\[
\frac{\Gamma, t \in P_u \quad t \notin P_u, \Delta, \Lambda}{\Gamma, \Delta, \Lambda, a + b}
\]

\[J, a + b[z]
\]

\[c[z]\]

\[\Lambda, \Lambda, D_v c[z]
\]

where the subproof ending with \(\Gamma, N_u(X,t)\) is obtained from the subproof in \(P\) ending with the left uppersequent \(\Gamma, t \in P_u\) of the (cut) \(J\) by inversion. Note that \(N_u(F,t)\) is an \(E\) formula. We have \(o(\Gamma, N_u(X,t); P') = a = o(\Gamma, t \in P_u; P)\) and hence \(o(P_b) = o(N_u(X,t), \Lambda; P') = c[1]\). Thus \(o(N_u(F,t), \Lambda; P') = D_v c[1] = z\) and \(o(\Lambda; P') = D_v c[z] = (D_v c)[1]\) with \(D_v c = o(\Lambda; P)\). Therefore \(o(P') = o[1]\).

This completes a proof of the Lemma 2 and hence of the Theorem 1.
Remark. As in [A] we have for each $n < \omega$

$$I\Sigma_k \vdash \forall d \exists m \{(D_0 D_1^{k-1}(\Omega \cdot n))[d]^m = 0\}$$

From this we can expect to sharpen the Theorem 1 for fragments, e.g., for $I\Sigma_k$ but we have no proof of the following:
Show that

$$I\Sigma_k \not\vdash \forall n \exists m \{(D_0 D_1^{k-1}(\Omega \cdot n))[1]^m = 0\}$$

References


