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<thead>
<tr>
<th>項目</th>
<th>内容</th>
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</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>From the Attic</td>
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<td>Arai, Toshiyasu</td>
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From the Attic

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Abstract

We gather the following miscellaneous results in proof theory from the attic.

1. A provably well founded elementary ordering admits an elementary order preserving map.
2. A simple proof of an elementary bound for cut elimination in propositional calculus
3. Equivalents for Bar Induction, e.g., reflection schema for $\omega$ logic
4. Direct computations in an equational calculus $PRE$
5. Intuitionistic fixed point theories are conservative extensions of HA.
6. Proof theoretic strengths of classical fixed points theories
7. An equivalence between transfinite induction rule and iterated reflection schema over $\Pi_2$
8. Derivation lengths of finite rewrite rules reducing under lexicographic path orders and provably total functions in theories between $\Pi_2$ and $\Pi_3$

Each section can be read separately in principle.

1 Provably Well Founded Relations

In this section we show that if an elementary recursive relation $<$ is provably well founded in Peano Arithmetic $PA$, then there exists an elementary recursive order preserving map $f$ of $<$ into an initial segment of $\varepsilon_0$. This gives an improvement on a result by Harrington and Takeuti (cf. [24], Theorem 13.6 and [10], p.33).

We say that a binary relation $<$ is provably well founded in $PA$ if

$$PA(X) \vdash \forall n(\forall k < n X(k) \supset X(n)) \supset \forall n X(n)$$

where $PA(X)$ denotes the Peano Arithmetic with an additional unary predicate $X$. Let $<_\varepsilon$ denote a standard elementary recursive $\varepsilon_0$ well ordering and ERA the Elementary Recursive Arithmetic.

**Theorem 1.1** If $<$ is an irreflexive, transitive and provably well founded (not necessarily a total ordering) relation on $\omega$, then there exists an ordinal $\alpha < \varepsilon_0$ and an elementary recursive function $h$ so that ERA proves that

$$\forall n(n \neq n) \& \forall n, k, (n < m < k \supset n < k) \supset$$

$$\forall n, k[\forall m < h(k) \supset f(n) <_{\varepsilon_0} f(k) \& f(n) <_{\varepsilon_0} a]$$

**Proof.** Work in ERA. From the proof in [24] pp.149-154 we see that there exist an ordinal $\alpha < \varepsilon_0$ and an elementary recursive function $h$ so that $h(k)$ is an additive principal number $<_{\varepsilon_0}$ $\omega^\alpha$, i.e., $h(k) = \omega^\beta$ for some $\beta$ and

$$\forall k[\forall n < k(n \neq k \supset h(k) <_{\varepsilon_0} h(n))]$$

($<$ denotes the usual ordering on $\omega$. A definition of the function $h$ will be sketched below.)

Define

$$f(n) = \max\{h(n_0) \# \cdots \# h(n_l) : n_0 < \cdots < n_l \equiv n \& n_0, \ldots, n_{l-1} < n\}$$

Here $\max<_{\varepsilon_0}$ denotes the maximum with respect to the ordering $<_{\varepsilon_0}$ and note that $<$ is irreflexive. The function $f$ is elementary recursive and we have

**Claim 1.1** $n < k \Rightarrow f(n) <_{\varepsilon_0} f(k)$
**Proof** of Claim 1.1. Assume $n < k$. Choose a sequence $n_0, \ldots, n_l$ so that $f(n) = h(n_0) \# \cdots \# h(n_l)$, $n_0 < \cdots < n_l = n$ and $k > n_0, \ldots, n_l, k_n < n$. By transitivity of $<$ we have $n_i < k$ for any $i \leq l$. Partition the set $\{0, \ldots, l\}$ into two sets $A$ and $B$ as follows: $A = \{i \leq l : k < n_i\}$, $B = \{i \leq l : n_i < k\}$ ($<$ is irreflexive.) By (1) we have $h(n_i) <_{\epsilon_0} h(k)$ for each $i \in A$, and hence $\# \sum \{h(n_i) : i \in A\} <_{\epsilon_0} h(k)$ since $h(k)$ is additive principal. ($\# \sum \{\alpha_0, \ldots, \alpha_n\}$ denotes $\alpha_0 \# \cdots \# \alpha_n$.)

On the other hand we have, using the transitivity of $<$,

$$\# \sum \{h(n_i) : i \in B\} <_{\epsilon_0} \max \{h(k_0) \# \cdots \# h(k_{m-1}) : k_0 < \cdots < k_{m-1} < k \land k \land k_0, \ldots, k_{m-1} < k\}$$

Therefore we get $f(n) <_{\epsilon_0} f(k)$.

**Sketch of a definition of the function $h$.**

We follow notations and terminology in [24].

1) Define a **TJ proof** exactly as in [24], p.149. That is, a TJ proof may have TJ initial sequents as extra initial sequentes:

TJ initial sequent $\forall x < tX(x) \rightarrow X(t)$ ($X(t)$ is called the principal formula of the TJ initial sequent.)

Also a TJ proof ends with a sequent of the form $\rightarrow X(m_0), \ldots, X(m_n)$. We identify the $m$th numeral with the number $m$.

2) The ordinal assignment $o(P)$ for a TJ proof $P$ is defined as in [24].

3) A TJ proof is called **noncritical** if one of the reduction steps for $PA$ which lowers the ordinal applies to it. Otherwise it is called **critical**.

4) We say that a TJ proof $P'$ is the **noncritical reduct** of a noncritical TJ proof $P$ if $P'$ is obtained from $P$ by applying a reduction step for $PA$ which lowers the ordinal.

5) We call a formula in the end-piece of a TJ proof, a **principal TJ descendent** if it is a descendent of a principal formula of a TJ initial sequent. If $P$ is a critical TJ proof, then the endsequent of $P$ contains a principal TJ descendent (cf. [24], pp.151-152.)

6) Let $P$ be a critical TJ proof of $\rightarrow X(m_0), \ldots, X(m_n)$, and $k$ be a number such that $k < m_i$ for every $i \leq n$. For some $i \leq n$ the formula $X(m_i)$ in the endsequent $\rightarrow X(m_0), \ldots, X(m_n)$ is a principal TJ descendent of a TJ initial sequent $\forall x < m_i X(x) \rightarrow X(m_i)$. Then add the formula $X(k)$ to the endsequent and replace the TJ initial sequent $\forall x < m_i X(x) \rightarrow X(m_i)$ by the following proof:

$\frac{\rightarrow k < m_i \quad X(k) \rightarrow X(k)}{
\rightarrow k < m_i \quad X(k) \rightarrow X(k)}$

$\forall x < m_i X(x) \rightarrow X(m_i)$

If $P'$ is obtained from a critical $P$ and $k$ in this way, then we say that $P'$ is the **critical reduct of $P$ at $k$**.

7) Since $<$ is provably well founded, we have in the system formed from $PA(X)$ by adjoining TJ initial sequents, a proof $P(a)$ of the sequent $\rightarrow X(a)$ for a free variable $a$. Then, for each $k$, $P(k)$ denotes a TJ proof of $\rightarrow X(k)$ obtained from $P(a)$ by substituting the numeral $k$ for the variable $a$.

8) Now let us define, for each number $k$, a TJ proof $P_k$ by induction on $k$ so that for every $n$, if $X(n)$ occurs in the endsequent of $P_k$, then $k \leq n \left( \iff k < n \ or \ k = n \right)$.

**8.1** The case $\exists n < k(k < n)$: Then $P_k = P(k)$. The endsequent of $P_k$ is $\rightarrow X(k)$.

**8.2** The case $\exists n < k(k < n)$: Pick an $n_0 < k$ so that $k < n_0$ and $\forall n < k \rightarrow o(P_n_0) <_{\epsilon_0} o(P_n)$.

**8.21** If $P_n_0$ is noncritical, then $P_k$ is defined to be the noncritical reduct of $P_n_0$.

**8.22** If $P_n_0$ is critical, then $P_k$ is defined to be the critical reduct of $P_n_0$ at $k$. In **8.22** the endsequent is unchanged, while in **8.22** it is augmented with the formula $X(k)$. In any cases we have $o(P_k) <_{\epsilon_0} o(P_n_0)$.

9) Finally we set: $h(k) =_{df} \omega^o(P_k)$. Then the required condition (1) is clearly enjoyed.
2 Elementary bound for cut elimination in propositional calculus

It is well known that the length of the shortest cut free proof is bounded by an elementary function of the length of the original proof in propositional calculus, e.g., cf.[15]. In this section we give a simple proof of this fact. This yields $S^2_2(X) \neq T^2_2(X)$ as a corollary.

Let $LK_0$ denote a classical propositional calculus in a Tait calculus. To be definite $LK_0$ denotes the calculus for the propositional part in [20]. $\Gamma, \Delta$ denotes sequents, i.e., finite sets of formulae. $(\Lambda x) \Gamma, \neg A, A$ (for an atomic $A$) is the only initial sequent in $LK_0$. Inference rules are $(\land), (\lor)$ and $(\text{cut})$. A precise formulation of these rules is irrelevant to our proof. Each proof in $LK_0$ is a tree of sequents.

For a proof $P$ in $LK_0$, the depth of $P$, denoted by $dp(P)$, is defined to be the depth of the tree $P$, i.e., the length of the longest branch in the tree $P$. The length of $P$, denoted by $lh(P)$, is defined to be the total number of occurrences of inference rules in $P$. Clearly we have $lh(P) < 2^{dp(P)}$ since each inference rule is at most binary.

**Theorem 2.1** If $P_0$ is a proof of a sequent $\Gamma_0$ in $LK_0$, then there exists a cut free proof $P$ of $\Gamma_0$ so that $dp(P) \leq lh(P_0)$. Therefore $lh(P) < 2^{lh(P_0)}$.

**Proof.** First eliminate cuts in the given proof $P_0$ by a usual cut elimination procedure, e.g., in [20]. The resulting cut free proof is denoted by $P^{cf}$. We say that two inference rules $J_0$ and $J_1$ are similar if 1) these are the same type of rules, e.g., both rules are $(\land)$ and 2) their auxiliary formulae and principal formulae are the same. We denote this equivalence relation by $J_0 \simeq J_1$. For example,

$$\frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, A_0 \land A_1} J_0 \simeq \frac{\Delta, B_0 \quad \Delta, B_1}{\Delta, B_0 \land B_1} J_1$$

Then it is obvious that for each inference rule $J$ in $P^{cf}$ there exists a $J'$ in $P_0$ such that $J \simeq J'$. Hence $k \leq lh(P_0)$ with the maximum number $k$ of equivalence classes of inference rules in a branch in $P^{cf}$. Thus it suffices to show that we can collapse two similar inference rules in a branch into a single one. For example if a rule

$$\frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, A_0 \land A_1} J_0$$

is above the left uppersequent $\Delta, A_0$ of another rule

$$\frac{\Delta, A_0 \quad \Delta, A_1}{\Delta, A_0 \land A_1} J_0$$

then eliminate $J_0$ to get the sequent $\Gamma, A_0 \land A_1, A_0$ and absorb the formula $A_0 \land A_1$ into the principal formula at $J_1$. In this way we get another cut free proof $P$ such that no branch in $P$ contains a pair of similar inference rules. Therefore $dp(P) \leq lh(P_0)$ as desired. □

Let $S^2_2(X)$ denote a bounded arithmetic obtained from Buss’ $S^2_2$ in [5] by adding a unary predicate $X$ together with the equality axiom for the extra $X$ and replacing $\Sigma^1_1$-**PIND** by $\Sigma^1_2(X)$-**PIND**. $\Sigma^1_2(X)$ denotes the set of sharply bounded formulae in the language augmented by $X$. Also $T^2_2(X)$ is obtained from $S^2_2(X)$ by replacing $\Sigma^1_2(X)$-**PIND** by $\Sigma^1_2(X)$-**IND**. We show:

**Corollary 2.1** $S^2_2(X) \not\vdash X(0) \land \forall x(X(x) \lor X(x + 1)) \lor \forall x X(x)$, i.e., $S^2_2(X) \neq T^2_2(X)$.

Assume that $S^2_2(X) \vdash X(0) \land \forall x(X(x) \lor X(x + 1)) \lor \forall x X(x)$. Let $S$ denote a system arising from $S^2_2(X)$ such that 1) the language of $S$ is the same as one of $S^2_2(X)$, 2) we add initial sequents $\Gamma, X(0)$ in $S$ and 3) we add an inference rule

$$\frac{\Gamma, X(t)}{\Gamma, X(t + 1)} (prg)$$

Then we have $S \vdash X(a)$ for a variable $a$.

Let $T$ denote a propositional calculus arising from $LK_0$ such that 1) the atoms in $T$ are $X_n (n \in \omega)$, 2) we add initial sequents $\Gamma, X_0$ in $T$ and 3) we add an inference rule $(prg_n)$ for each $n \in \omega$

$$\frac{\Gamma, X_n}{\Gamma, X_{n+1}} (prg_n)$$

For each $\Sigma^1_2(X)$ sentence $A$ a propositional formula $A^*$ is associated as follows: 1) for an atomic $A$ without $X$, $A^* = X_0$ if $A$ is true, $A^* = \neg X_0$ otherwise. 2) $X(t)^* = X_n$ with the value $n$ of the closed term $t$. 3) $*$ commutes with any propositional connectives. 4) $(\exists x \leq t A(x))^* = \forall \{A(i)^* : i \leq n\}$ with the value $n$ of the
closed term $t$ and similarly for $\forall x \leq t$. For a sequent $\Gamma = \{A_0, \ldots, A_m\}$ consisting solely of $\Sigma_0^b(X)$ sentences, we set $\Gamma^* = \{A_0^*, \ldots, A_m^*\}$.

Let $\Gamma(\overline{a})$ denote a $\Sigma_0^b(X)$ sequent whose free variables are among the sequence $\overline{a} = (a_0, \ldots, a_m)$ of variables. For a sequence $\overline{n} = (n_0, \ldots, n_m)$ of natural numbers $\Gamma(\overline{n})$ denotes the result of simultaneous substitution $n_i$ for $a_i$. Then it is easy to show:

**Lemma 2.1** If $S \vdash \Gamma(\overline{a})$, then there exists a polynomial $f(\overline{a})$ such that for any $\overline{n}$ there exists a proof $P_n$ of $\Gamma(\overline{n})$ in $T$ such that

$$lh(P_n) \leq f(|\overline{n}|).$$

($|n|$ is the length of the binary expansion of the number $n$.)

This follows from the fact that for each term $t(\overline{a})$ there exists a polynomial $g$ such that $\forall \overline{n}(|t(\overline{n})| \leq g(|\overline{n}|))$. Therefore we would have a polynomial $f$ such that for each $n$ there exists a proof $P_n$ of $X_n$ in $T$ with $lh(P_n) \leq f(|n|)$. It is fairly easy to extend Theorem 2.1 to the calculus $T$. Thus we would have for a polynomial $f$ such that

for any $n$ there exists a cut free proof $P_n$ of $X_n$ in $T(dp(P_n) < f(|n|)).$ \hspace{2cm} (2)

We say that a sequent is positive if the atom $X_n$ occurs only positively in it for any $n$. Put $\vdash^k \Gamma$ iff there exists a cut free proof $P$ of $\Gamma$ in $T$ such that $dp(P) \leq k$. Also we denote $k \models \Gamma$ if $\Gamma$ is true under the truth assignment $X_n = 1$ if $n \leq k$ then true else false.

Then for any positive $\Gamma$ we have $\vdash^k \Gamma \Rightarrow k \models \Gamma$. Now (2) runs $\vdash^{f(|n|)} X_n$ and hence $\forall n(n \leq f(|n|))$. This is a contradiction.

**Remark.** Add all polynomial growth rate functions to the language. Denote the set of true $\Sigma_0^b$ sentences in this extended language by $Tr_{\Sigma_2}$. Let $S_0^b(X) + Tr_{\Sigma_2}$ denote the theory obtained from $S_0^b(X)$ by adding $Tr_{\Sigma_2}$. Then we see

$$S_0^b(X) + Tr_{\Sigma_2} \vdash X(0) \land \forall x(X(x) \supset X(x + 1)) \land \forall xX(x)$$

from the above proof. Observe that $S_0^b(X) + Tr_{\Sigma_2} \vdash \Sigma_\infty^0 - IND$ since each instance of $\Sigma_\infty^0 - IND$ is in $Tr_{\Sigma_2}$ for a bounded formula $A \in \Sigma_\infty^0$ without $X$.

### 3 Equivalents for Bar Induction

In this section we give some equivalents for Bar Induction.

$L_f$ denotes a second order language containg 1) the language of the first order arithmetic, 2) set variables $X, Y, \ldots$ and 3) unary function variables $f, g, \ldots$. $\Sigma_0^b$ denotes the set of bounded formulae in $L_f$ and $\Pi_1^b$ the set of arithmetical (=first order) formulae possibly with second order parameters. We take the theory $\Sigma_0^b - CA$ as our base theory. The theory $\Sigma_0^b - CA$ has the following axiom schemata besides the axioms for first order constants:

1. Graph Principle: $\forall x \exists! yX(j(x, y)) \supset \exists f \forall x X(j(x, f x))$

(j: a pairing function),

2. Comprehension Axiom for $\Sigma_0^b$-formulae and

3. $IA: \forall X[X(0) \land \forall n(X(n) \supset X(n + 1)) \supset \forall n X(n)]$

In this section we use signs $\supset$ and $\rightarrow$ interchangeably to denote the propositional connective 'implication'.

**Definition 3.1**

1. $BI$ denotes the axiom schema:

   - Hyp1 & Hyp2 & Hyp3 $\supset Q <>$ for $P \in \Sigma_0^b$ and an arbitrary formula $Q$ ($<>$ is the empty sequence),

   where

   Hyp1: $\forall f \exists x P(\overline{f}x) (\overline{f}x =< f0, \ldots, f(x-1)>)$

   Hyp2: $\forall c \in Seq(\overline{P}c \supset QC)$

   (Seq denotes the set of gödel numbers of finite sequences of natural numbers).

   Hyp3: $\forall c \in Seq(\forall x Q(c < x) \supset QC)$

2. For a binary relation $<, W f(<) \Leftrightarrow \forall f \exists x (f(x + 1) \not< f x)$
3. \( \text{Prf}[\prec, Q] \leftrightarrow_{df} \forall z (\forall y < xQy \supset Qz) \)

4. \( I(\prec, Q) \leftrightarrow_{df} \text{Prf}[\prec, Q] \supset \forall zQz \)

5. \( T\underline{I} \) denotes the axiom schema \( Wf(\prec) \supset I(\prec, Q) \)
   for \( \prec \in \Sigma^{0}_{0} \) and an arbitrary \( Q \).

6. \( T\underline{I}' \) denotes the axiom schema \( \forall XI(\prec, X) \supset I(\prec, Q) \)
   for \( \prec \in \Sigma^{0}_{0} \) and an arbitrary \( Q \).

7. \( Ng \leftrightarrow_{df} \forall f \forall x \exists c \in \text{Seq}(lh(c) = z \& f \in c) \leftrightarrow_{df} \forall f \forall x \exists c \in \text{Seq}(c = \tilde{f}x) \)

8. For a formula \( F(x, y) \), \( Fnec(F) \leftrightarrow_{df} \forall x \exists y F(x, y) \) and
   \( c = \tilde{F}z \leftrightarrow_{df} lh(c) = z \& \forall i < z F(i, c(i)) \) with the ith component \( c(i) \) of the sequence \( c \).

9. \( NG \) denotes the axiom schema \( Fnc(F) \supset \forall x \exists c (c = \tilde{F}x) \)
   for an arbitrary \( F \).

10. \( \forall E(\Pi^{1}_{0}) \) denotes the axiom schema \( \forall X A(X) \supset A(\{x\} F(x)) \)
    for an \( A \in \Pi^{1}_{0} \) and an arbitrary \( F \).

**Theorem 3.1 (cf. [14])** Over \( \Sigma^{0}_{0} - \text{CA} \), the following axiom schemata are mutually equivalent:

\( Ng + BI, \ Ng + TI, \ Ti' \) and \( \forall E(\Pi^{1}_{0}) \)

The theorem is seen from a series of the following propositions. Except the direction \( Ng + BI \rightarrow \forall E(\Pi^{1}_{0}) \)
these are due to Howard and Kreisel [14]. Also we learned a weaker result \( \Pi^{1}_{1} - \text{CA} + BI \rightarrow \forall E(\Pi^{1}_{0}) \) from [4], p.52.

**Remark.** We have also a second order parameter-free version of the theorem.

**Proposition 3.1**

1. \( Ng + BI \vdash TI \) (cf. [14], Theorem 5A)

2. \( TI \vdash BI \) (cf. [14], Theorem 5C)

3. \( TI' \vdash TI \)

4. \( \forall E(\Pi^{1}_{0}) \vdash BI \)

5. \( TI' + Ng \) and \( \forall E(\Pi^{1}_{0}) \vdash Ng \)

6. \( Ng + BI \vdash \Pi^{1}_{\infty} - IA \)

7. \( Ng + BI \vdash NG \)

**Proof.** 3. It suffices to show, in \( \Sigma^{0}_{0} - \text{CA}, Wf(\prec) \vdash \forall X I(\prec, X) \) for a \( \prec \in \Sigma^{0}_{0} \). This follows from

\( \forall m (m \notin X \supset \exists n \prec m ((m \notin X)) \supset \exists f \forall m (m \notin X \supset fm \prec m \& m \notin X) \)

4. As in 3, we have

\[ H_{\text{yp}1} \& H_{\text{yp}2} \& H_{\text{yp}3} \supset Q <\]

for a \( P \in \Sigma^{0}_{0} \) in \( \Sigma^{0}_{0} - \text{CA} \). Taking this formula as \( A(X) \) in \( \forall E(\Pi^{1}_{0}) \) we get any instance

\[ H_{\text{yp}1} \& H_{\text{yp}2} \& H_{\text{yp}3} \supset Q <\]

of \( BI \).

5. This follows from \( TI', \forall E(\Pi^{1}_{0}) \vdash \Pi^{1}_{\infty} - IA \).

6. \( A(0) \& \forall n A(n) \supset A(n + 1) \) we have to show \( A(a) \). Put \( Pc \equiv a \leq lh(c) \) and \( Qc \equiv A(a - lh(c)) \). By \( Ng \) we have \( H_{\text{yp}1} \). By \( BI \) we conclude \( Q <\), i.e., \( A(a) \).

7. This follows from 6.

A formula \( A(\tilde{f}) \) \( (\tilde{f} = f_{0}, \ldots , f_{n}) \) is said to be in \( \tilde{f} \) normal form if each function variable \( f_{i}, i \leq n \) occurs only of the form \( f_{i}(y) = z \) for some variables \( y, z \) in the formula \( A(\tilde{f}) \).

In a canonical way, each quantifier free formula \( R(\tilde{f}) \) is transformed into its \( \tilde{f} \) normal form \( \exists \overline{z} R_{0}(\overline{z}, \tilde{f}) \) with new variables \( \overline{z} \) and a quantifier free \( R_{0} \).

Let \( R(\tilde{f}) \) be a quantifier free formula and \( F \equiv \{x, y\} F(x, y) \) be a binary formula (abstract). Also let \( \exists \overline{z} R_{0}(\overline{z}, \tilde{f}) \)
notations the \( f \) normal form of \( R(\tilde{f}) \). Then \( R(F) \) denotes the result of replacing each \( f(x) = y \) in \( \exists \overline{z} R_{0}(\overline{z}, \tilde{f}) \) by \( F(x, y) \).
Proposition 3.2 1. For each quantifier free and $f$ normal form $R(x, f)$ there exists a $P \in \Sigma_0^2$ such that:

(a) every free variable occurring in $P$ is either a new number variable $c$ or a variable occurring in $R(x, f)$ except $x, f$.

(b) for any binary formula $F$,

$$NG \vdash Fnc(F) \rightarrow [\exists x R(x, F) \leftrightarrow \exists c(F \in c \& Pc)]$$

with $F \in c \Leftrightarrow_{df} c = \bar{F}(lh(c))$.

2. For a quantifier free and $f$ normal form $R(x, f)$, and a binary formula $F$,

$$Ng + BI \vdash Fnc(F) \& \forall f \exists x R(x, f) \rightarrow \exists x R(x, F)$$

3. For a quantifier free $R(x, f)$ and a binary formula $F$,

$$Ng + BI \vdash Fnc(F) \& \forall f \exists x R(x, f) \rightarrow \exists x R(x, F)$$

4. For a formula $A(x, y, X)$ let $F$ denote the binary formula:

$$F \equiv F(X) =_{df} \{x, y\}(y \simeq \mu y. \neg A(x, y, X))$$

with $y \simeq \mu y. \neg A \Leftrightarrow_{df} [\exists y \neg A \& y = \min\{y : \neg A\}] \lor [\forall y A \& y = 0]$.

Then for any formula $V$,

$$Ng + BI \vdash Fnc(F(V)) \text{ and}$$

$$Ng + BI \vdash \forall x \exists y \neg A(x, y, V) \rightarrow \forall y[F(V)(x, y) \rightarrow \neg A(x, y, V)]$$

that is,

$$Ng + BI \vdash \exists x A(x, F(V)(x), V) \rightarrow \exists x \forall y A(x, y, V)$$

Proof.

1. Let $P_0c$ denote a formula obtained from $R$ by replacing each $f(y) = z \ [f(y) \neq z]$ by $c(y) = z \& y < lh(c)$ $[c(y) \neq z \& y < lh(c)]$, resp. Then put $Pc \Leftrightarrow_{df} \exists x < lh(c)P_0c$. We need $NG$ to show that $Fnc(F) \& \exists x R(x, F) \rightarrow \exists c(F \in c \& Pc)$.

2. Assume $Fnc(F) \& \forall f \exists x R(x, f)$. Let $P$ denote the formula formed in 1. Then we have $Hyp1$ for this $P$.

Put

$$Qc \Leftrightarrow_{df} F \in c \rightarrow \exists d(F \in c \ast d \& P(c \ast d))$$

By $F(lh(c), x) \& Q(c \ast x < x) \rightarrow Qc$, we have $Hyp2 \& Hyp3$. Thus by $BI Q <>$ and hence $\exists d(F \in d \& Pd)$. The assertion follows from 1.

3. This follows from Proposition 3.2.2 and the definition of $\exists x R(x, F)$.

4. This follows from Proposition 3.1.6.

Lemma 3.1 $Ng + BI \vdash \forall E(\Pi_1^2)$

Proof For an $A \in \Pi_1^2$ and an arbitrary $V$ we have to show $\forall X A(X) \rightarrow A(V)$.

Step 1 First transform $A$ into a prenex normal form whose leading quantifier is $\exists$. For example assume $A(X) \leftrightarrow \exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0(X)$ with a quantifier free $A_0$. We need only logical axioms to obtain this equivalence. Hence for any formula $V$ we have $A(V) \leftrightarrow \exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0(V)$. Thus we can assume that $A$ is in prenex normal form, e.g., of the form $\exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0(X)$.

Step 2 Second transform $A$ into its Herbrand normal form. Pick new function variables $f_0, f_1$ and put $A_H \equiv A_0(x_0, f_0(x_0), x_1, f_1(x_0, x_1))$. We have logically $\exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0 \rightarrow \forall f_0 \forall f_1 \exists x_0 \exists x_1 A_H$. Put

$$F_0 =_{df} \{x_0, y_0\}(y_0 \simeq \mu y_0. \neg \exists x_1 \forall y_1 A_0)$$

$$F_1 =_{df} \{x_0, x_1, y_1\}(y_1 \simeq \mu y_1. \exists y_0(F_0(x_0, y_0) \& \neg A_0))$$
By Proposition 3.2.4, \( Ng + BI \) proves that
\[
Fnc(F_0(V)) \land Fnc(F_1(V))
\]
and
\[
\exists x_0 \exists x_1 A_0(x_0, F_0(V)(x_0), x_1, F_1(V)(x_0, x_1), V) \rightarrow \exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0(V)
\]
Hence, in \( Ng + BI \), \( \forall X A(X) \rightarrow \forall X \forall [f_0 \forall [f_1 f_2] A_H \rightarrow \exists x_0 \exists x_1 A_0(x_0, F_0(V)(x_0), x_1, F_1(V)(x_0, x_1), V) \). Thus Proposition 3.2.3 yields
\[
\exists x_0 \forall y_0 \exists x_1 \forall y_1 A_0(V) \equiv A(V).
\]
\( \square \)

Now Theorem 3.1 has been proved from these propositions and lemma.

Next we show that Bar Induction is equivalent to the reflection schema for \( \omega \) logic.

We change our language \( L_f \) to \( L_2 \): From \( L_f \) 1) remove the function variables, 2) add the \( n \)-ary predicate variables \( X^n_i \) (\( i \) \( \in \) \( \omega \)) and 3) restrict function constants to \( 0, S \) (successor). The resulting language is denoted \( L_2 \). Thus closed terms in \( L_2 \) are numerals. We understand that predicate constants corresponding to primitive recursive relations are included in \( L_2 \).

Let \( L K_2 \) denote a Tait's calculus for this second order language \( L_2 \). A second order terms is just a predicate variable \( X^n \) and hence in \( L K_2 \) the inference rule (\( \exists_2 \)) for the second order existential quantifier runs:
\[
\Gamma, F(X) \\
\Gamma, \exists Y F(Y)
\]

Also let \( RFN \) denote the reflection schema for the calculus \( L K_2 \).

**Proposition 3.3**

\[
\Sigma^0_0 - CA \vdash RFN \rightarrow \Pi^1_0 - IA
\]

**Proof.** (\( \rightarrow \)) For each \( n \) we have \( L K_2 \vdash A(0) \land \forall x (A(x) \supset A(Sx)) \supset A(n) \).

(\( \leftarrow \)) By cut elimination and the partial truth definition. \( \square \)

Let \( ACA_0 \) denote the second order arithmetic Arithmetical Comprehension Axiom with Restricted induction. Since \( ACA_0 \) is finitely axiomatizable, we get the

**Corollary 3.1** (cf. [22], Lemma 2.7)

\[
ACA_0 \vdash RFN_{ACA_0} \rightarrow \Pi^1_0 - IA
\]

Let \( X_0 \subseteq \omega \times \omega \) be a binary relation. \( L K_\omega(X_0) \) denotes the \( \omega \) logic with the relation \( X_0 \):

1. The language of \( L K_\omega(X_0) \) is obtained from \( L_2 \) by adding the binary predicate constant \( X_0 \) and removing the first order free variables. Any sequents in \( L K_\omega(X_0) \) have no first order free variable.

2. Axioms (= initial sequent) in \( L K_\omega(X_0) \) are diagrams for the relation \( X_0 \) besides the usual axioms for the constants and logical ones.

\[
\Gamma, X_0(n, m) \text{ if } X_0(n, m), \text{ and } \Gamma, \neg X_0(n, m) \text{ if } \neg X_0(n, m)
\]

3. Inference rules in \( L K_\omega(X_0) \) are those of \( L K_2 \) except the following changes. First replacing the usual rule for first order universal quantifier by the \( \omega \) rule:

\[
\frac{\{ \Gamma, A(n) : n \in \omega \} \quad (\omega)}{\Gamma, \forall x A(x)}
\]

Second restrict the rule (3) for the first order existential quantifier to:

\[
\frac{\Gamma, A(n)}{\Gamma, \exists x A(x)}
\]
By a preproof we mean an $\omega$-branching labelled tree of sequents and some data. Data should include names of axioms or inference rules and finite cut degrees. A preproof have to be locally correct with respect to the data. An $\omega$-proof is a well founded preproof.

$\omega - RFN$ denotes the schema saying that if a sequent $\Gamma$ has an $\omega$-proof in $LK_\omega(X_0)$, then the sequent $\Gamma$ is true.

**Theorem 3.2**

$$\Sigma^0_0 - CA_0 \vdash \omega - RFN \rightarrow Ng + TI$$

**Proof.** $\rightarrow$ We have $\Pi^1_\infty - IA$ and hence $Ng$. Assume $WF(\prec)$. We have to show $Prg[\prec, A] \rightarrow \forall z A(z)$. It suffices to show that there exists an $\omega$-proof of the sequent $\neg Prg[\prec, A], \forall z A(z)$. Wlog we can assume that no second order free variable (except the 'constant' $X_0$) occurs in $\prec \in \Sigma^0_0$. Thus $n < m$ has an $\omega$-proof when $n < m$ is true and similarly for the case $n \not< m$. Using this fact we can construct a preproof of the sequent $\neg Prg[\prec, A], \forall z A(z)$ which must be well founded by our assumption $WF(\prec)$.

$\leftarrow$ Again by cut elimination and the partial truth definition. Here cuts are eliminated through Mints' continuous cut elimination procedure in Mints [18]. To ensure that the resulting cut free preproof is well founded we need $\Pi^0_1 - CA$, i.e., Arithmetical Comprehension Axiom (cf. [18] or [11]). But $\Pi^0_1 - CA$ follows from $Ng + TI \iff \forall \forall(P_\Pi^0_1)$. $\Box$

$\omega - RFN$ is also equivalent to the so called $\omega$-model reflection schema over $ACA_0$.

Let $T$ be a theory in $L_\omega$. By $\omega - model - RFN_T$ we mean the following schema for an arbitrary formula $A(X)$:

$$A(X) \rightarrow \exists \text{ countable } M = (M_n : n \in \omega)(M_0 = X & M \models T & M \models A[X])$$

When $T$ consists solely of axioms for constants in $L_\omega$, we denote simply $\omega - model - RFN$.

**Proposition 3.4** (Henkin-Orey's $\omega$-completeness theorem) In $ACA_0$, there exists an $\omega$-proof of $A(X_0)$ in $LK_\omega(X_0)$ iff for any countable $\omega$ model $M \ni X_0$ $M \models A[X_0]$

**Corollary 3.2** $ACA_0 \vdash \omega - RFN \iff \omega - model - RFN$ and hence $ACA_0 \vdash \omega - model - RFN^*_{ACA_0} \iff TI'$ (cf. [22]).

## 4 Direct computations in an equational calculus $PRE$

Let $PRE$ denote the theory $PRA$ minus induction axiom. The axioms of $PRE$ are defining equations for primitive recursive functions: $0$ (zero) denotes an individual constant ans $S$ the successor function. Function constants and their defining equations are generated as follows.

1. (projection) $I^i_\omega(x_1, \ldots, x_n) = x_i (1 \leq i \leq n, n > 0)$
2. (composition) $f(\bar{z}) = h(g_1(\bar{z}), \ldots, g_m(\bar{z}))$,
   where $\bar{z}$ denotes a sequence $x_1, \ldots, x_n$ of variables.
3. (primitive recursion 1) $f(0) = k$, $f(Sy) = h(y, f(y))$
   ($k$ is a natural number).
4. (primitive recursion 2) $f(\bar{z}, 0) = g(\bar{z})$, $f(\bar{z}, Sy) = h(\bar{z}, y, f(\bar{z}, y))$

In what follows our concern is restricted to Horn clauses $E \supset e$ with an equation $e$ and a finite set $E$ of equations. Therefore it is better to consider $PRE$ as an equational theory with an extra axiom $E$. By $E \vdash e$ we mean the equation $e$ is derivable from the set $E$ in $PRE$.

For a function constant $f$, $Cl(f)$ denotes the finite set of function constants which are used to define the constant $f$ (and the successor $S$). Specifically,

1. (projection) $Cl(I^i_\omega) = \{I^i_\omega\} \cup \{S\}$
2. (composition) $Cl(f) = Cl(h) \cup \bigcup\{Cl(g_i) : 1 \leq i \leq m\} \cup \{f\}$
3. (primitive recursion 1) $Cl(f) = Cl(h) \cup \{f\}$
4. (primitive recursion 2) $Cl(f) = Cl(g) \cup Cl(h) \cup \{f\}$
For a term \( t \), \( \text{Cl}(t) = \bigcup \{ \text{Cl}(f) : f \text{ occurs in } t \} \).
For an equation \( t = s \), \( \text{Cl}(t = s) = \text{Cl}(t) \cup \text{Cl}(s) \).
For a set \( E \) of equations, \( \text{Cl}(E) = \bigcup \{ \text{Cl}(e) : e \in E \} \).

Let \( R \) denote the set of rules \( l \rightarrow r \) which are obtained from one of the defining equations \( l = r \) by replacing the equality \( = \) by the arrow \( \rightarrow \). Viewing the set \( R \) as a term-rewriting system, \( s \rightarrow_R t \) or simply \( s \rightarrow t \) denotes the relation "the term \( s \) rewrites to the term \( t \) by \( R \)" in the sense of [7]. Also \( \mathrel{\rightarrow^*} \) denotes the reflexive-transitive closure of \( \rightarrow \) and \( \mathrel{\rightarrow^*} \) the smallest congruent relation containing the relation \( R \).

The following is a folklore.

**Proposition 4.1** 1. \( \rightarrow \) is Church-Rosser, i.e., \( \mathrel{\rightarrow} \subseteq \mathrel{\rightarrow^*} \)
2. \( \rightarrow \) is terminating.
3. \( s \rightarrow t \Rightarrow \text{Cl}(s) \subseteq \text{Cl}(t) \)

Therefore we have

**Proposition 4.2** For an equation \( e \), \( \vdash e \) is decidable by computing the normal forms of the both sides of \( e \).

**Proposition 4.3** For an equation \( e \), if \( \vdash e \), then there exists a direct (in the sense of [21], p. 343) computation \( D \) of \( e \); every function constant occurring in \( D \) is in \( \text{Cl}(e) \).

This directness does not hold if we replace \( \vdash e \) by \( E \vdash e \).

**Counterexamples** (H. Friedman [9])
1. \( Sx = Sy \vdash x = y \): Apply the predecessor function \( \text{pd} \).
2. \( 0 = Sx \vdash y = z \): Apply the discriminator \( \delta \), \( \delta(y, z, 0) = y, \delta(y, z, Sx) = z \).

Our theorem below says that these are only exceptions.

**Definition 4.1** \( PRE' \) is obtained from \( PRE \) by adding the following two rules for an arbitrary equation \( e \):

\[
\begin{align*}
\frac{St_1 = St_2}{t_1 = t_2} & (S) \\
\frac{0 = St}{e} & (\delta)
\end{align*}
\]

**Theorem 4.1** For a finite set \( E \) of equations and an equation \( e \), if \( E \vdash e \), then there exists a direct computation \( D \) of \( e \) from \( E \) in \( PRE' \); every function constant occurring in \( D \) is in \( \text{Cl}(E) \cup \text{Cl}(e) \).

**Corollary 4.1** For an open formula \( A \), if \( \vdash A \), then there exists a direct derivation \( D' \) of \( A \) in \( PRE' \) and hence a weakly direct derivation \( D \) of \( A \) in \( PRE \); every function constant occurring in \( D' \) (in \( D \)) is in \( \text{Cl}(A) \cup \text{Cl}(\text{pd}) \cup \text{Cl}(\delta) \), resp.

**Proof of Corollary.** Write \( A \) in CNF \( \bigwedge \{ C : C \in \Gamma \} \) and consider each conjunct \( C \) separately. \( C \) is equivalent to \( E' \rightarrow E \) for some finite sets \( E' \) and \( E \) of equations (\( E' \rightarrow E \) denotes a sequent in Gentzen's sense). Then use the theorem and the fact: \( E' \vdash E \Rightarrow E' \vdash e \) for some \( e \in E \). \( \Box \)

We don’t know an answer to the problem raised by H. Friedman [9].

**Problem.**
1. Is \( \vdash A \) decidable for an open \( A \)?
2. Is \( e \vdash \) decidable for an equation \( e \)?

But we conjecture the following.

**Conjecture.** Let \( t_1 \) and \( t_2 \) be normal terms with respect to \( \rightarrow_R \). Then
\[
t_1 = t_2 \Leftrightarrow t_1 \equiv S^m t_0 \text{ for some term } t_0 \text{ and } t_2 \equiv S^n t_0 \text{ with } m > n \text{ or vice versa.}
\]

This means that the theory \( PRE \) can discriminate between terms only when one is 0 and the other is of the form \( S^m t \). Unless the equation \( t_1 = t_2 \) is of the form \( S^m t = u \) where the term \( u \) occurs in \( t \) and \( u \) contains a variable, the conjecture is easily seen to hold. Also if \( u \) is a variable \( x \), then, by H. Friedman [9], we have \( St(x) = x \not\in t \). That’s all what we know about the conjecture.

The rest of the section is devoted to a proof of the Theorem 4.1. Fix a finite set \( E \) of equations. \( t_1 \vdash t_2 \) denotes ambiguously the equation \( t_1 = t_2 \) or \( t_2 = t_1 \).
**Definition 4.2** 1. \(d(E)\) denotes the smallest set of equations such that
   1) \(E \subseteq d(E)\) and 2) \(d(E)\) is closed under the rules \((S), (sub)\) and \((red)\):
   \[
   \frac{e[t_1/x]}{e[t_2/x]} \quad (sub) \quad \frac{t \equiv u}{t' \equiv u} \quad (red) \quad \text{where } t \rightarrow_R t'
   \]
   2. \(t \rightarrow_E t' \iff d\) there exists a term \(t_0\) and a finite set \(\{u_i \equiv v_i : i < n\} \subseteq d(E)\) of equations such that
      \(t \equiv t_0[u_0, \ldots, u_{n-1}/x_0, \ldots, x_{n-1}]\) (simultaneous substitution) and \(t_0[v_0, \ldots, v_{n-1}/x_0, \ldots, x_{n-1}] \equiv_R t'\)
      \((\equiv_R \text{denotes the reflexive closure of } \rightarrow_R)\)
   3. \(\rightarrow_E\)[ asymmetric] denotes the reflexive-transitive [-symmetric] closure of \(\rightarrow_E\), resp.

Clearly \(e \in d(E) \Rightarrow Cl(e) \subseteq Cl(E)\) and hence we have the

**Proposition 4.4** 1. \(t \rightarrow_E t' \Rightarrow Cl(t') \subseteq Cl(t) \cup Cl(E)\)

**Lemma 4.1** Assume that \(0 = St \notin d(E)\) for any term \(t\). Then \(\rightarrow_E\) is Curch-Rosser, i.e., \(\rightarrow_E \subseteq \rightarrow_E \circ \rightarrow_E\).

**Proof of Theorem 4.1.** If \(0 = St \notin d(E)\) for any \(t\), then we get the theorem by the Lemma 4.1 and the
Proposition 4.4. Assume \(0 = St \in d(E)\) for some term \(t\). Then there exists a direct computation \(D\) of \(0 = St\) from \(E\) in \(PRE'\). By adjoining the rule \((\delta)\) we get a desired computation of \(t_1 = t_2\) from \(E\).

In what follows we assume that \(0 = St \notin d(E)\) for any term \(t\).

**Definition 4.3** 1. \(t \rightarrow_I s\) is defined inductively as follows:

   (a) \(t \rightarrow_I t\)
   (b) \(i \rightarrow_I s \Rightarrow f(i) \rightarrow_I f(s)\) where, for sequences \(i \equiv t_1, \ldots, t_n\), \(s \equiv s_1, \ldots, s_n\) of terms, \(i \rightarrow_I s \iff d\)
      \(t_i \rightarrow_I s_i\) for any \(i\).
   (c) (projection) \(t_i \rightarrow_I s \Rightarrow I^n_i(t_1, \ldots, t_n) \rightarrow_I s\)
   (d) (composition) \(i \rightarrow_I s \Rightarrow f(i) \rightarrow_I h(g_1(s), \ldots, g_m(s))\)
   (e) (primitive recursion 1) \(f(0) \rightarrow_I k; t \rightarrow_I u \Rightarrow f(St) \rightarrow_I h(u, f(u))\) if \(f(0) = k\).
   (f) (primitive recursion 2) \(i \rightarrow_I s \Rightarrow f(i, 0) \rightarrow_I g(s); \)
   \(i \rightarrow_I s \& u \rightarrow_I v \Rightarrow f(i, Su) \rightarrow_I h(s, v, f(s, v))\)

2. \(t \rightarrow_E I s \iff d\) there exist a term \(t_0\) and a sequence \(\{u_i \equiv v_i : i < n\} \subseteq d(E)\) such that
   \(t \equiv t_0[u_0, \ldots, u_{n-1}/x_0, \ldots, x_{n-1}]\) and \(s \equiv s_0[v_0, \ldots, v_{n-1}/x_0, \ldots, x_{n-1}]\).

As usual we have

1. \(\equiv_R = \equiv_I\)
2. \(\equiv_E = \equiv_E\)

3. \(\rightarrow_I\) is strongly confluent, i.e., satisfies the diamond property:
   \(\forall t, s, u \exists v(t \rightarrow_I s \& t \rightarrow_I u \Rightarrow s \rightarrow_I v \& u \rightarrow_I v)\)

Thus it suffices to show the following lemma.

**Lemma 4.2** \(\rightarrow_E\) is strongly confluent.

Define
\[
t_1 \rightarrow_{CE} t_2 \iff \\
\text{there exist a term } t_0 \text{ and a sequence } \{u_i \equiv v_i : i < n\} \subseteq d(E) \text{ such that } \\
t_1 \equiv t_0[u_0, \ldots, u_{n-1}/x_0, \ldots, x_{n-1}] \text{ and } t_2 \equiv t_0[v_0, \ldots, v_{n-1}/x_0, \ldots, x_{n-1}].
\]

Since \(d(E)\) is closed under the rule, \((sub) \rightarrow_{CE}\) is transitive. Therefore it suffices to show:

**Claim 4.1** If we have \(M \rightarrow_I M \rightarrow_{CE} N \rightarrow_I N_1\), then there exist terms \(N_2, M_2, L\) such that \(M_1 \rightarrow_{CE} N_2 \rightarrow_I L \rightarrow_I M_2 \rightarrow_{CE} N_1\).
Proof of Claim 4.1. We prove this by induction on $m + n$, where $m \lfloor n \rfloor$ denotes the depth of a derivation of $M \rightarrow M_1 \lfloor N \rightarrow N_1 \rfloor$, resp.

Case 0 $M \vdash N \in d(E)$: Then $M_1 \vdash N_1 \in d(E)$. Take $N_2 \equiv L \equiv M_2 \equiv N_1$.

Case 1 $M_1 \equiv M$: Take $N_2 \equiv N$, $L \equiv M_2 \equiv N_1$.

Case 2 $M \equiv f(\bar{t}) \rightarrow f(\bar{u}) \equiv M_1$ with $\bar{t} \rightarrow \bar{u}$:

2.1 $N \equiv f(\bar{v}) \rightarrow f(\bar{w}) \equiv N_1$ with $\bar{v} \rightarrow \bar{w}$: For each $i$ we have $u_i \rightarrow t_i \rightarrow CE v_i \rightarrow u_i$. By IH $u_i \rightarrow CE v_i \rightarrow s_i \rightarrow t'_i \rightarrow CE w_i$ for some $v'_i, s_i, t'_i$.

2.2 $N \equiv I_1^n(\bar{v}) \rightarrow w_i \equiv N_1$ with $v_i \rightarrow w_i$: As in 2.1, $M_i \equiv I_1^n(\bar{v}) \rightarrow CE I^n(\bar{u}_1, ..., u_{i-1}, v'_i, u_{i+1}, ..., u_n) \rightarrow s_i \rightarrow t'_i \rightarrow CE w_i$ for some $v'_i, s_i, t'_i$.

2.3 $N \equiv f(\bar{v}, 0) \rightarrow g(\bar{w}) \equiv N_1$ with $\bar{v} \rightarrow \bar{w}$ ($\bar{v} \equiv v_1, ..., v_{n-1}$): By IH pick $v'_i, s_i, t'_i$ for $i \neq n$ as in 2.1. Then $M_1 \equiv f(\bar{u}, u_n) \rightarrow CE f(\bar{v}, 0) \rightarrow g(i) \rightarrow g(\tilde{t}) \rightarrow CE g(\bar{w}) \equiv N_1$ by $0 \equiv t_n \in d(E)$ & $t_n \rightarrow u_n \Rightarrow 0 \equiv u_n \in d(E)$.

2.4 $N \equiv f(\bar{v}, Sv) \rightarrow h(\bar{w}, w_n, f(\bar{w}, w_n)) \equiv N_1$ with $\bar{v} \rightarrow \bar{w}$ and $v_n \rightarrow w_n$: Pick $\bar{v}', \bar{t}'$ so that $\tilde{u} \rightarrow CE \tilde{v} \rightarrow \tilde{t} \rightarrow CE \tilde{w}$.

2.1. $t_n \equiv Sv \in d(E)$: Then $u_n \equiv Sw_n \in d(E)$.

$M_1 \equiv f(\bar{u}, u_n) \rightarrow CE f(\bar{v}, Sw_n) \rightarrow h(\tilde{s}, w_n, f(\tilde{s}, w_n)) \rightarrow h(\tilde{t}, v_n, f(\tilde{v}, v_n)) \rightarrow CE h(\tilde{w}, w_n, f(\tilde{w}, w_n)) \equiv N_1$

2.5 $N \equiv f(0) \rightarrow k \equiv N$: Similar to 2.3.

2.6 $N \equiv f(Sv) \rightarrow h(\bar{w}, f(v, w_n)) \equiv v \rightarrow w$: Similar to 2.4.

Case 3 $M \equiv I_1^n(\bar{t}) \rightarrow u_i \equiv M_1$ with $t_i \rightarrow u_i$ and $N \equiv I_1^n(\bar{v}) \rightarrow w_i \equiv N_1$ with $v_i \rightarrow w_i$.

Case 4 $M \equiv f(\bar{t}, 0) \rightarrow g(\bar{u}) \equiv \bar{v} \rightarrow \bar{w}$: $N \equiv f(\bar{v}, \bar{v})$ with $0 \rightarrow CE \bar{v}$, i.e., $0 \equiv v \in d(E)$ or $v \equiv 0$. By our assumption, $v \neq Sv'$ for any $v'$. Therefore it must be the case $v \equiv 0$ and $N \equiv f(\bar{v}, 0) \rightarrow g(\bar{u})$ with $\bar{v} \rightarrow \bar{w}$. Use IH.

Case 5 $M \equiv f(\bar{t}, St) \rightarrow h(\bar{u}, u, f(\bar{u}, u))$ with $\bar{t} \rightarrow \bar{u}$ and $t \rightarrow u$: As in the Case 4 we have $N \equiv f(\bar{v}, Sv) \rightarrow h(\bar{u}, w, f(\bar{w}, w))$ with $\bar{w} \rightarrow \bar{w}$ and $v \rightarrow w$. Note that if $St \equiv CE Sv$, then $t \equiv CE v$ by the rule $(S)$.

Case 6 $M \equiv f(0) \rightarrow k \equiv k$: Similar to the Case 4.

Case 7 $M \equiv f(St) \rightarrow h(u, f(u))$ with $t \rightarrow u$: Similar to the Case 5.

This completes a proof of the Claim 4.1.

5 Intuitionistic fixed point theories

In [3] Buchholz shows that an intuitionistic fixed point theory $ID^i_n$ is conservative over Heyting Arithmetic $HA$ with respect to almost negative formulae. The proof in [3] is based on a recursive realizability interpretation of the theory $ID^i_n$. Having seen a preliminary version of [3] we can extend and strengthen this result.

Our proof runs as follows. First an extension of an intuitionistic iterated fixed point theory $ID^i_n$ is interpreted in the intuitionistic analysis $EL + AC - NF$. This is done by imitating Aczel's proof in [8] which shows that the classical fixed point theory $ID_1$ is interpretable in a second order arithmetic $\Sigma^1_2 - AC$. Then by N. Goodman's theorem [12] one can conclude our theorem. A proof of N. Goodman's theorem is based on either a combination of a realizability interpretation and a forcing or a proof theoretic analysis in G. Mints [18]. It seems that a direct analysis of $ID^i_n$ based on one of these methods is desirable.

Definition 5.1 1. $EL$ denotes the intuitionistic elementary analysis defined in [25] p.144. Function variables in $EL$ are denoted by $\alpha, \beta, \gamma, ...$.
2. The axiom schema $AC - NF$: \( \forall n \exists \alpha A(n, \alpha) \supset \exists \beta \forall n A(n, (\beta)_n) \)
   with \((\beta)_n = \lambda m \beta(j(n, m))\) and a pairing function \(j\).

3. \(L\) denotes the language of \(EL\). For a list of set parameters \(\vec{X} = X_0, X_1, \ldots, L(\vec{X})\) denotes the expanded language obtained from \(L\) by adding \(\vec{X}\).

4. \(EL(\vec{X})\) \([EL + AC - NF(\vec{X})]\) denotes the extension of \(EL\) \([EL + AC - NF]\) by expanding the language to \(L(\vec{X})\), resp.

   Each axiom schema in \(EL\) \([EL + AC - NF]\) is available for \(L(\vec{X})\) formulae in \(EL(\vec{X})\) \([EL + AC - NF(\vec{X})]\), resp.

**Lemma 5.1** For each \(n\) and each list \(\vec{X}\) of set parameters there exists a formula \(S_n(x_0, x_1, \ldots, x_n; \vec{X}, \alpha)\) in \(\Sigma^0(x_0, x_1, \ldots, x_n; \vec{X}, \alpha)\) such that for every formula \(A\) in \(\Sigma^0(x_0, x_1, \ldots, x_n; \vec{X}, \alpha)\) there is an integer \(e\) such that

\[
EL(\vec{X}) \vdash A \leftrightarrow \Sigma^0(e, x_1, \ldots, x_n; \vec{X}, \alpha)
\]

**Proof.** By formalizing the enumeration theorem. This is done in \(EL(\vec{X})\). cf. Ch. 3, Sect. 6 and 7 in [25]. □

**Definition 5.2** Let \(\vec{Y}\) be a list of set parameters and \(\mathcal{F}\) a set of formulae in \(L(\vec{Y})\). Pick an \(X \notin L(\vec{Y})\).

1. \(POS(\mathcal{F}; \vec{Y}) =_{df}\) the set of all \(L(\vec{Y}, X)\) formulae which are built up from formulae \(X(t)\) \((t: a term)\) and formulae in \(\mathcal{F}\) by means of \(\wedge, \vee, \forall m, \exists m\) \(\) (first order quantifications).

2. \(POS^*(\mathcal{F}; \vec{Y}) =_{df}\) \(\{\Phi \in POS(\mathcal{F}; \vec{Y}) : FV(\Phi) \subseteq \{x\}\}\) for a fixed number variable \(x\). \(FV(\Phi)\) denotes the set of free variables occurring in \(\Phi\). Thus no function free variable occurs in \(\Phi \in POS^*(\mathcal{F}; \vec{Y})\).

3. \(POS(\vec{Y}) = POS(\mathcal{F}_R; \vec{Y})\) and \(POS^*(\vec{Y}) = POS^*(\mathcal{F}_R; \vec{Y})\) with the set \(\mathcal{F}_R\) of all formulae in \(L(\vec{Y})\).

4. \(POS(\vec{Y}; \vec{Y}) = POS(\mathcal{A}_R; \vec{Y})\) and \(POS^*(\vec{Y}; \vec{Y}) = POS^*(\mathcal{A}_R; \vec{Y})\) with the set \(\mathcal{A}_R\) of atomic formulae \(Y_i(t)\) for \(Y_i \in \vec{Y}\).

5. \(POS = POS(\emptyset)\).

**Remark.** \(POS(\mathcal{F}; \vec{Y})\) is narrower than strictly positive formulae (with respect to \(X\)) because \(A \supset X(t) \notin POS(\mathcal{F}; \vec{Y})\) but is wider than \(POS\) in [3]. If we set \(A \supset X(t) \in POS(\mathcal{F}; \vec{Y})\), then one would need \(IP\) (Independence of Premise) for a proof of Lemma 5.3 below.

**Lemma 5.2** For each \(\Phi \in POS\) there exist a list \(\vec{Y}\) of set parameters, a \(\Phi' \in POS(\vec{Y}; \vec{Y})\) and a list \(\vec{A}\) of formulae in \(L\) such that

\[
EL(X) \vdash \Phi \leftrightarrow \Phi'[\vec{A}/\vec{Y}]
\]

where \([\vec{A}/\vec{Y}]\) denotes the simultaneous substitution.

A formula in \(L(\vec{Y})\) is said to be an \(n - \Sigma^1(\vec{Y})\) formula \((\Sigma^1)\) formula in normal form with set parameters \(\vec{Y}\) if it is of the form \(\exists \alpha \forall n R(\alpha, n, \vec{Y})\) with an open formula \(R\) in \(L(\vec{Y})\) in which no function variable except \(\alpha\) occurs.

**Lemma 5.3** For each \(\Phi \in POS(\vec{Y}; \vec{Y})\) and each \(A(x)\) in \(n - \Sigma^1(\vec{Y})\) there exists a \(C\) in \(n - \Sigma^1(\vec{Y})\) such that

\[
EL + AC - NF(\vec{Y}) \vdash \Phi[A/X] \leftrightarrow C
\]

**Proof** by induction on the length of \(\Phi\) using the facts:

\[
EL(\vec{Y}) \vdash A \vee \exists \alpha B \leftrightarrow \exists \alpha (A \vee B)
\]

\[
EL(\vec{Y}) \vdash \forall x A \vee \forall y B \leftrightarrow \exists x \forall z \forall y[(x = 0 \land A) \lor (x \neq 0 \land B)]
\]

**Lemma 5.4** For each \(\Phi \in POS^*(\vec{Y}; \vec{Y})\) there exists a formula \(P^*(\vec{Y}, x)\) in \(n - \Sigma^1(\vec{Y})\) such that

\[
EL + AC - NF(\vec{Y}) \vdash \forall x \{P^*(\vec{Y}, x) \leftrightarrow \Phi[\{x\} P^*(\vec{Y}, x)/X]\]
\]
Proof by Lemmata 5.1 and 5.3. Put \( B(u, x; Y) \equiv \exists \alpha \forall y S_0(u, y, x; Y, \alpha) \). Pick an \( n - \Sigma_1^1(Y) \) formula \( C \equiv \exists \alpha \forall y C_0(u, y, x; \alpha) \) such that \( \Phi([x] B/X] \rightarrow C \). Pick an \( \epsilon \) so that \( C_0(u, y, x; Y, \alpha) \rightarrow S_0(\epsilon, u, y, x; Y, \alpha) \). Then \( P^\Phi(Y, x) \equiv B(\epsilon, x; Y) \) is a desired one.

By Lemmata 5.2 and 5.4 we get the

Lemma 5.5 For each \( \Phi \in \text{POS} \) there exists a formula \( P^\Phi(x) \) in \( L \) such that

\[
EL + AC - NF \vdash \forall x [P^\Phi(x) \rightarrow F \Phi([x] P^\Phi(x)/X)]
\]

Let \( EL + AC - NF + \hat{D}^i_n \) denote an extension of \( EL + AC - NF \). Its language is obtained from \( L \) by adding a unary set constant \( \hat{D}^i_n \) for each \( \Phi \in \text{POS}^*(Y) \) (\( Y \): a fixed set parameter) and its axioms are those of \( EL + AC - NF \) in the expanded language plus the axiom \( (F P)^\Phi_n \):

\[
(F P)^\Phi_n \forall i < n \forall x [I^\Phi_n(x) \rightarrow \Phi(I^\Phi_n, I^\Phi_n, x)]
\]

where \( I^\Phi_n(x) \equiv I^\Phi(j(i, x)), I^\Phi_n(k, x) \equiv k < i \land I^\Phi_n(x) \) and \( \Phi \equiv \Phi(Y, x, z) \).

Theorem 5.1 \( EL + AC - NF + \hat{D}^i_n \) is a definitional extension of \( EL + AC - NF \), i.e., the set constant \( I^\phi \)

is definable in \( EL + AC - NF \), and hence,

via N. Goodman's theorem [12], \( EL + AC - NF + \hat{D}^i_n \) is a conservative extension of \( HA \) for each \( n \).

Proof. Construct \( P^\Phi_0, P^\Phi_1, \ldots, P^\Phi_{n-1} \) successively by Lemma 5.5.

6 Classical fixed point theories

Let \( L_2 \) denote the second order language obtained from the language of the first order arithmetic by adding set variables \( X, Y, \ldots \). Let \( T \supseteq ACA_0 \) denote a second order arithmetic containing \( ACA_0 \). Assume that \( T \) is \( \Pi^1_1 - \text{faithful} \), i.e., any \( \Pi^1_1 \)-consequence in \( T \) is true. Then, by [11], we have for a recursive theory \( T \)

\[
|T| =_{df} \sup \{ \alpha : T \vdash I(\prec) \text{ for some recursive well ordering } \prec \text{ of type } \alpha < \omega_{1}^{CK} \}
\]

where \( I(\prec) \) denotes the \( \Pi^1_1 \)-sentence \( \forall X \forall y S_0(x, y, X) \rightarrow \forall x X(x) \). \( Pr_0[\prec, X] \) denotes that \( X \) is progressive with respect to \( \prec \) as in Section 3.

The proof theoretic ordinal \( |T| \) of \( T \) is free from pathology, while the following alternative definition of the proof theoretic ordinal make sense relative to a vague natural well ordering \( \prec \):

\[
|T| =_{df} \sup \{ \alpha : T \vdash I(\prec, \pi_0^{-1}) \text{ for some recursive well ordering } \prec \text{ of type } \alpha \}
\]

where \( \pi_0^{-1} \) denotes the set of arithmetical formulae without set parameters and \( I(\prec, \pi_0^{-1}) \) the schema of transfinite induction of \( \prec \) applied to a formulae in \( \Pi^1_1 \).

Let \( FP - ACA_0 \) and \( FP - ACA^1 \) denote second order arithmetic in the language \( L_2 \) (without set constants \( P_A \) differing from [16]) which are obtained from \( ACA_0 \) and \( ACA \), resp. by adding the following \( \Sigma^1_{1} \) axiom:

\[
(F P) \exists X \forall x [A(X, x) \rightarrow X(x)]
\]

for each \( X \) positive arithmetical formula \( A[X, x] \) in \( L_2 \) (\( A[X, x] \) contains no free variable except \( X \) and \( x \)).

Then G. Jäger and B. Primo [16] shows that

Theorem 6.1 (G. Jäger and B. Primo [16])

1. \( |FP - ACA_0| = \varepsilon_0 \)
2. \( |FP - ACA^1| = \varepsilon_0 \)
3. \( FP - ACA_0 \) and \( \Sigma^1_1 - AC \) are proof theoretically equivalent each other.

Here note that \( |ACA_0| = \varepsilon_0 \), \( |ACA| = \varepsilon_\varepsilon_0 \) and \( |\Sigma^1_1 - AC| = \varphi_{\varepsilon_0} \). Also \( FP - ACA_0 \) is proof theoretically stronger than \( ACA_0 \), e.g., by a truth definition for arithmetical formulae in \( \Pi^1_1 \), \( FP - ACA_0 \vdash Con(ACA_0) \).

We observe that the above theorem follows from a result due to G. Kreisel [17] or [24], pp.176-177:

Theorem 6.2 Let \( T \) be a recursive, \( \Pi^1_1 \)-faithful second order arithmetic containing \( ACA_0 \).
1. (G. Kreisel[17])
\[ |T| = \sup \{ \alpha : T \vdash I(\prec) \text{ for some } \Sigma^1_1 \text{ well ordering } \prec \text{ of type } \alpha \} \]

2. Let \( T_{\Sigma^1_1} \) denote the set of true \( \Sigma^1_1 \) sentences in \( L_2 \). Then
\[ |T| = |T + T_{\Sigma^1_1}| \]

**Proof.** Assume \( T + A \vdash I(\prec) \) for a primitive recursive well ordering \( \prec \) and an \( A \in T_{\Sigma^1_1} \). Define a \( \Sigma^1_1 \) well ordering \( \prec_A \) by
\[ n \prec_A m \iff n \prec m \& A. \]

Then we have \( T \vdash I(\prec_A) \). By the Kreisel’s result, the order type of \( \prec \) is equal to the order type of \( \prec_A \leq |T| \). \( \square \)

The theorem is applied to the \( n \)th fold iterated fixed point theory \( FP_n - ACA' \). \( FP_n - ACA_0' \) is obtained from \( ACA_0 \) by adding the \( \Sigma^1_1 \) axiom \( (FP_n) \):
\[ (FP_n) \exists X_n, \ldots, X_1 \forall x \bigwedge_{1 \leq i \leq n} (x \in X_i \rightarrow A_i(X_i^+, X_1, \ldots, X_{i-1}, x)) \]

for each \( X_i \) positive formula \( A_i \) in the language \( L_2 + \{X_1, \ldots, X_i\} \).
\( FP_n - ACA' \) is obtained from \( FP_n - ACA_0' \) by adding the full induction schema \( \Pi^1_\infty - IA \).

**Corollary 6.1** For any \( n \in \omega \),
1. \( |FP_n - ACA_0'| = \epsilon_0 \)
2. \( |FP_n - ACA'| = \epsilon_0 \)

Thus the theories \( FP_n - ACA_0' \) is weak with respect to the proof theoretical ordinal \( |T| \). But these are proof theoretically much stronger than \( ACA_0 \). In the following we compute the other proof theoretical ordinal \( |FP_n - ACA_0'| \), etc.

In what follows let \( \prec \) denote a standard well ordering of type \( \Gamma_0 \) (the first strongly critical number). Ordinals \( \leq \Gamma_0 \) and their codes are identified and denoted by \( \alpha, \beta, \ldots \).

**Definition 6.1**
1. Let \( T \) be a first order theory containing \( PA \). A first order theory \( ID(T) \) (fixed point theory over \( T \)) is defined as follows: The language \( L_{ID(T)} \) of \( ID(T) \) is obtained from the language \( L_T \) of \( T \) by adding the set constants \( \{P_A : A[X^+, x] \in L_T(X), X \text{ positive } \} \).

**Axioms** \( ID(T) = T + \text{ induction schema for } L_{ID(T)}(FP) \)
\[ (FP) \forall x(x \in P_A \rightarrow A[P_A, x]) \]

2. \( ID_0 = PA \) and \( ID_{n+1} = ID(ID_n) \).
3. \( L^n = L_{ID(T)} \) and \( L^2 = L^n + \text{ second order variables } X, Y, \ldots \)
4. the norm of \( ID_n \) \((n \neq 0) \) is defined to be the following ordinal with \( |k|_A < \alpha \iff k \in I_A^{\Sigma_1} \):
\[ \inf \{ \alpha : \forall A[X, x] \in L^n(X) \forall k \in \omega [ID_n \vdash k \in P_A \Rightarrow |k|_A < \alpha] \} \]
5. \( FP_n - ACA = ACA \) for the language \( L^2 + ID_n \)

Clearly \( ID_n \) and \( FP_n - ACA_0 \) \([FP_n - ACA \text{ and } FP_n - ACA'] \) have the same arithmetical provable formulae \( \in L^0 = \Pi^1_0 \), resp. For a fixed \( A[X^+, Y, x] \), we write \( P_n \) for \( PA \), with \( A_n = A[X^+, \sum_{i<n} P_i, x] \). Thus \( ID_n \) has extra constants \( P_i \) \((i < n) \) for each \( A \).

**Definition 6.2** Let \( \Phi \) be a set of formulae.
1. \( I(< \alpha, \Phi) \) denotes the schema of transfinite induction up to each \( \beta < \alpha \) applied to a formula \( \in \Phi \).
2. A first order theory $H(\Phi)^{<\alpha}$ is defined as follows: its language $= L_0 + \text{the language of } \Phi + \{H_A : A \in \Pi_0^\infty(\Phi, X)\}$

$(A \in \Pi_0^\infty(\Phi, X) \iff A \text{ is a } \Pi_0^0 \text{ formula relative to } X \text{ formulae } \in \Phi)$

$H(\Phi)^{<\alpha} = PA$ for the language of $H(\Phi)^{<\alpha}$ $(H)$:

$$(H) \forall x \in H_A^{\beta} \Rightarrow A[H_A^{\beta}, x]$$

for each $\beta < \alpha$. $H_A^{\beta} = \sum_{\gamma < \beta} H_A^{\gamma}$

Thus $(H)$ says that $\{H_A^{\gamma} : \gamma \leq \beta\}$ forms the 'jump' hierarchy relative to formulae $\in \Phi$.

**Theorem 6.3**

1. For each $B \in L^n$ $(m < n)$,

$\hat{I}D_n \vdash B \iff H(L_m)^{<\alpha_{n-m}} + \hat{I}D_m \vdash B \iff \hat{I}D_m + I(< \alpha_{n-m+1}, L^m) \vdash B$

where $\alpha_1 = \epsilon_0$, $\alpha_{n+1} = \varphi_\alpha 0$ with the Veblen function $\varphi_\beta$.

2. $|\hat{I}D_n|_\beta = \alpha_{n+1}$

3. For each $B \in L^n$ $(m < n)$,

$$FP_n - ACA \vdash B \iff H(L_m)^{<\beta_{n-m}} + \hat{I}D_m + I(< \beta_{n-m+1}, L^m) \vdash B$$

where $\beta_1 = \epsilon_\alpha$, $\beta_{n+1} = \varphi_\beta 0$.

4. $|FP_n - ACA|_0 = \beta_{n+1}$

5. the norm of $\hat{I}D_n = \alpha_n (n \neq 0)$

6. the norm of $FP_n - ACA = \beta_n$

This is proved by using usual techniques in [16] and [8].

**Proof** of 1 and 2. An infinitary system $\hat{I}D^\infty(L^n)$ ($\hat{I}D^\infty$ over the language $L^n$) is designed as the first order part of $FP - ACA^*$ in [16], in the language $L^n$, i.e., fixed points rules in $\hat{I}D^\infty(L^n)$ are only for $P_n$ and constants $P_0, \ldots, P_{n-1} \in L^n$ are treated as set parameters in $\hat{I}D^\infty(L^n)$ Thus $\hat{I}D^\infty(L^n)$ is the first order part of $FP - ACA^*$ in [16]. Put $B_n \equiv \bigwedge_{i<n} FP_i \supset B$ for $B \in L^{n+1}$ where $FP_i$ denotes the axiom for the constant $P_i$.

**Lemma 6.1**

1. $\hat{I}D_{n+1} \vdash B \Rightarrow \hat{I}D^\infty(L^n) \vdash^{<\epsilon_\alpha} B_n$

2. $FP_{n+1} - ACA \vdash B \Rightarrow \hat{I}D^\infty(L^n) \vdash^{<\epsilon_\alpha} B_n$

For a proof we set the rank $rn(F) = 0$ if $F \in PF_n$ with respect to $P_n$. The rest is the same in [16].

**Lemma 6.2**

For an $\epsilon$-number $\alpha$,

1. $\hat{I}D^\infty(L^n) \vdash^{<\alpha} B \text{ with } B \in PF_n \Rightarrow \forall \beta < \alpha[(I_n^{<\alpha} \vdash^{<\alpha} \{\beta\} B^{\beta + \omega^{\alpha}})] \text{ where } \{\beta\} B^{\beta'} \text{ denotes the result of replacing each negative } P_n \text{ by } I_n^{<\beta} \text{ and each positive } P_n \text{ by } I_n^{<\beta'} \text{ and } (I_n^{<\alpha}) \text{ is an infinitary system whose extra rules are, for each } \beta < \alpha$,

$$\frac{\Gamma, [-] A[I_n^{<\alpha}, \sum_{i<n} P_i, s]}{\Gamma, [-] s \in I_n^{<\alpha}}$$

2. $(I_n^{<\alpha} \vdash^{<\alpha} B) \Rightarrow (I_n^{<\alpha} \vdash^{<\alpha} \omega^{\alpha} B$

3. $\hat{I}D^\infty(L^n) \vdash^{<\alpha} B_n \text{ with } B \in L^n(P_n \text{ does not occur in } B)$

$\Rightarrow \hat{I}D^\infty(L^{n-1}) \vdash^{<\alpha} B_{n-1}$

**Lemma 6.3**

$$H(X)^{<\alpha} \vdash I(< \varphi 0, X)$$

We give a sketch of a proof of this lemma below. From this lemma we see the
Lemma 6.4

\[ H(L^m)^{<\alpha_n} \vdash I(<\alpha_{n-m+1}, L^m) \quad (m \leq n, \alpha_0 = 0) \]

Thus we have shown the direction

\[ \hat{D}_m + I(<\alpha_{n-m+1}, L^m) \vdash B \Rightarrow H(L^m)^{<\alpha_n} + \hat{D}_m \vdash B \]

Next consider the direction

\[ \hat{D}_m \vdash B \Rightarrow \hat{D}_n + I(<\alpha_{n-m+1}, L^m) \vdash B \]

Assume \( \hat{D}_m \vdash B \) with \( B \in L^m, m < n \). By Lemma 6.1 we have for \( \alpha_1 = \varepsilon_0 \) \( \hat{D}_m^\infty (L^{n-1}) \vdash^{<\alpha_n} B_n \). By Lemma 6.2 we successively have

\[ \hat{D}_m^\infty (L^m) \vdash^{<\alpha_n} B_m, (I_m^{<\alpha_n}) \vdash^{<\alpha_n} B_m \text{ and } (I_m^{<\alpha_n}) \vdash^{<\alpha_n} B_m. \]

By a partial truth definition we get \( \hat{D}_m + I(<\alpha_{n-m+1}, L^m) \vdash B \).

Finally consider the direction

\[ H(L^m)^{<\alpha_n} + \hat{D}_m \vdash B \Rightarrow \hat{D}_n \vdash B \]

This follows from Lemma 6.5 below. We interpret \( H(L^m)^{<\alpha_n} + \hat{D}_m \) in \( \hat{D}_n \) as follows:

- leave \( L_m \) formulae unchanged.
- the 'jump' hierarchy \( H_A(A \in \Pi_0^1(L^m)) \) up to \( \alpha_{n-m} \) is interpreted as \( P_A^+, P_A^- \) so that \( P_A^+ = H_A, P_A^- = \neg H_A \) (simultaneously defined as fixed points over \( L^m \)). Then for each \( B \) in the language of \( H(L^m)^{<\alpha_n} + \hat{D}_m \), let \( B' \) denote the result of replacing the positive \( H_A \) by \( P_A^+ \) and negative \( H_A \) by \( P_A^- \).

Lemma 6.5 1) \( \hat{D}_n \vdash H(L^m)^{<\alpha_n} + \hat{D}_m \vdash B \Rightarrow \hat{D}_n \vdash B' \), i.e.,

\[ \hat{D}_n \vdash \forall x (x \in H^2_A \leftrightarrow \neg (x \notin H^2_A)) \text{ for each } \beta < \alpha_{n-m}. \]

2) \( \hat{D}_n \vdash I(<\alpha_{n-m+1}, L^m) \quad (m \leq n) \)

Proof by simultaneous induction on \( n - m \). We have 2) \( n \) and 2) \( n+1 \Rightarrow 1) \( m \). It remains to show 1) \( m \Rightarrow 2) \( m \).

By Lemma 6.4 and \( I(<\alpha_{n-m+1}, L^m) \in L_m \) we get 2) \( m \).\( \square \)

Thus we have proven Theorem 6.3.1 and 2.

Finally consider the norm of \( \hat{D}_n \). The upper bound \( \alpha_n \) for the norm of \( \hat{D}_n \) is obtained from Lemmata 6.1 and 6.2.

To obtain the lower bound, define a fixed point \( W = W_0 \) by

\[ \forall \beta (\beta \in W \leftrightarrow \forall \gamma < \beta (\gamma \in W)) \]

By Lemma 6.5.2), we have \( \hat{D}_n \vdash I(<\alpha_n, L^1) \). Hence by \( W \in L^1 \) and \( \hat{D}_n \vdash \forall \beta (\forall \gamma < \beta (\gamma \in W) \rightarrow \beta \in W) \), we get \( \hat{D}_n \vdash \beta \in W \) for each \( \beta < \alpha_n \).

Proof of Lemma 6.3. Put \( \lambda = \omega^\alpha \) and

\[ I^\beta_X (\gamma) \iff \forall Y \in \bigcup \text{Rec}(H^X_\delta) I(\gamma, Y) \]

where

1. \( H_\delta^X \) denotes the \( \delta^{th} \) jump of the set \( X \)
2. \( \text{Rec}(H^X_\delta) \) denotes the set of sets recursive in \( H^X_\delta \).
3. \( I(\gamma, Y) \) denotes the transfinite induction up to \( \gamma \) applied to \( Y \).

Also for each \( \alpha < \lambda \),

\[ A_\alpha^X (\gamma) \iff \forall \delta \gamma > 0 \forall \beta \forall \delta > 0 [I^\gamma_X (\delta+1)(\beta) \& \omega^\gamma (\delta + 1) \leq \alpha \rightarrow I^\omega_\delta (\varphi \gamma \beta)] \]

Then we can prove the following lemma as in [8]:
Lemma 6.6 For each $\alpha < \lambda$,
\[ H(X)^{<\lambda} \vdash \text{Pr}_g[A] \]

Lemma 6.7
\[ H(X)^{<\lambda} \vdash I(\lambda) \]
where $I(\lambda)$ denotes the schema of transfinite induction up to each ordinal $< \lambda$ and applied to any formula in the language of $H(X)^{<\lambda}$.

Proof. For $\alpha < \lambda$ let $\alpha \alpha$ denote a finite set of ordinals $\leq \alpha$ inductively generated as follows:
1. $\alpha \in \alpha \alpha$
2. If $\beta = \beta_1 + \cdots + \beta_n \in \alpha \alpha$, $\beta_1 > \cdots > \beta_n$ and $\beta_1, \ldots, \beta_n \in \alpha \alpha$.
3. If $\gamma, \delta \in \alpha \alpha$ then $\gamma, \delta \in \alpha \alpha$.

We show inductively that $\forall \beta \in \alpha \alpha H(X)^{<\lambda} \vdash I(\beta)$.
Assume that $\gamma > 0$ and $\gamma, \delta \in \alpha \alpha$, $I(\gamma)$ and $I(\delta)$. For a given formula $U$ we have to show $I(U(\gamma, \delta), U)$. Since $\lambda = \omega^\omega$ is additive principal, $\omega^\omega \cdot 2 < \omega^\omega \cdot 2 \cdot 2 = \omega^\omega \cdot 2 \leq \alpha \cdot 2 < \lambda$. Also $I(U)^{<\lambda} = H(X)^{<\lambda}$ since $U \in \bigcup_{\beta < \lambda} \text{Rec}(H_x^\beta)$ and the $\lambda$ is additive principal. Thus by Lemma 6.6 we have $\text{Pr}_g[A]^{<\lambda}$ and hence $\forall \beta [I(U)^{<\omega^\omega}(\beta) \rightarrow I(U)^{<\omega^\omega}(\gamma, \delta)]$. By $I(\delta)$ we have $I(U)^{<\omega^\omega}(\gamma, \delta)$. Thus $I(U)^{<\omega^\omega}(\gamma, \delta)$ and $I(U(\gamma, \delta), U)$. \(\square\)

Now Lemma 6.3 follows from Lemma 6.6 and 6.7.

7 Iterated reflection formulae and rules of transfinite induction

In this section we give an equivalence between transfinite induction rule and iterated reflection schema over the fragment $\text{Ir}_\Sigma_1$ of $\text{PA}$. In this section $<$ denotes a standard $\varepsilon_0$ well ordering.

Definition 7.1 1. For an additive principal number $\alpha \geq \omega$ and a set $\Phi$ of formulae, $\text{TIR}[\alpha, \Phi]$ denotes the transfinite induction rule up to $\alpha$ and applied to a formula $A \in \Phi$. Put $\text{Pr}_g[A] \Leftrightarrow \forall x(\forall y < x A(y) \supset A(x))$. Then for each $A \in \Phi$
\[ \frac{\text{Pr}_g[A]}{\forall x < \alpha A(x)} \]
is an instance of the rule $\text{TIR}[\alpha, \Phi]$.

2. For a theory $T$ containing the fragment $\text{Ir}_\Sigma_1$ let $T + \text{TIR}[\alpha, \Phi]$ denote the theory obtained from $T$ by adding the rule $\text{TIR}[\alpha, \Phi]$. Also $T + \text{TIR}^{(m)}[\alpha, \Phi]$ (for $m \in \omega$) denotes a formal system $\subseteq T + \text{TIR}[\alpha, \Phi]$ in which the rule $\text{TIR}[\alpha, \Phi]$ can be applied nestedly at most $m$ times.

For example if $T$ is $\text{TIR}^{(0)}[\alpha, \Phi] = T$ and $1)$ in $T + \text{TIR}^{(1)}[\alpha, \Phi]$ the rule $\text{TIR}[\alpha, \Phi]$ can be applied only when $T \vdash \text{Pr}_g[A] (A \in \Phi)$, etc.

3. For a theory $T \supseteq \text{Ir}_\Sigma_1$ let $C^T_\alpha(\alpha)$ denote the iterated reflection formula defined in U. Schmerl [19]. Thus in $\text{Ir}_\Sigma_1$ we have
(a) $C^T_\alpha(0) \leftrightarrow RF_{\Pi^{\alpha+1}}(T)$
(b) $C^T_\alpha(\alpha + 1) \leftrightarrow RF_{\Pi^{\alpha+1}}(T + C^T_\alpha(\alpha))$
(c) $C^T_\alpha(\lambda) \leftrightarrow \forall \alpha < \lambda C^T_\alpha(\alpha)$ for a limit $\lambda$.

4. \(\left( \begin{array}{c} \alpha \iff T + \{C^T_\alpha(\beta) : \beta < \alpha \} \end{array} \right)\) as in [19].

Proposition 7.1 Over $\text{Ir}_\Sigma_1$,
\[ \text{TIR}[^{\alpha+\alpha}, \Sigma_n] = \text{TIR}[^{\alpha+\alpha}, \Pi_{n+1}] \]

Proof. This is contained in the proof of Theorem 4.1. $\text{e)}$ in [23] \(\square\)

A formula $A(x)$ is called reflexively progressive (in $x$) with respect to a theory $T$ if
\[ T \vdash \forall x[\forall y < x Pr_T("A(y)") \supset A(x)] \]
with a canonical provability predicate $Pr_T$ for $T$ and the gödel number "$E$" of an expression $E$. 

Proposition 7.2 (cf. [19], p.337)

\[ T \vdash A(x) \Leftrightarrow A(x) \text{ is reflexively progressive with respect to } T \]

Remark. The proof of the direction \([\subseteq]\) in [19] uses Löb's theorem and the facts:

1. \( T \vdash y < z \supset Pr_T(\neg \neg y < z) \)
2. \( T \vdash < \text{ is transitive} \)

Thus any \( \Sigma \) binary relation \(<\) Proposition 7.2 holds if \( <\) is demonstrably transitive in \( T \). In other words, reflexive progressiveness is nothing but well foundedness although the name remind us the latter.

Lemma 7.1 For \( A \in \Pi_{3+1} \) and \( T \supseteq I \Sigma_1 \),

1. \( B(\alpha) \equiv C_T^T(\alpha) \supset A(\alpha) \text{ is reflexively progressive with respect to } T \)
   if \( T \vdash C_T^T(0) \supset Prg[A] \).
2. \( T \vdash Prg[A] \Rightarrow T + C_T^T(0) \vdash Prg[\forall x < \omega(1 + \alpha)A(x)] \).
3. \( T \vdash Prg[A] \Rightarrow T \vdash C_T^T(\alpha) \supset \forall x < \omega(1 + \alpha)A(x) \).

Proof.

1. Assume \( T \vdash C_T^T(0) \supset Prg[A] \). We can assume that \( \alpha \neq 0 \) since
   \( T \vdash C_T^T(0) \supset A(0) \). Then we have by \( A \in \Pi_{3+1} \),
   \[
   T \vdash \forall \beta < \alpha Pr_T(\forall \beta < \alpha A(\beta)) \& \forall \beta < \alpha A(\beta)
   \]
   By our assumption \( T \vdash C_T^T(0) \supset Prg[A] \).
2. Assume \( T \vdash Prg[A] \). Consider the case \( \alpha = 0 \). Then we have to show \( T \vdash \forall n < \omega Pr_T(\forall \beta < \omega A(x)) \). This
   follows from \( T \vdash \forall n < \omega Pr_T(\forall \beta < n A(x)) \) or better \( T \vdash \forall n < \omega Pr_T(\forall \beta < n A(x)) \). Other cases are similar.

\( \square \)

Lemma 7.2 Assume \( T \supseteq I \Sigma_n \) and \( A \) is a \( \Pi_{3+1} \)-sentence. Then

\[ T \vdash A \Rightarrow T + Thr^{(1)}[\omega^{1+\alpha}, \Pi_{3+1}] \vdash C^{I \Sigma_n + A}(\omega^\alpha) \]

with

\[
\omega^\alpha = \begin{cases} 
\omega^\alpha & \text{ if } \alpha \neq 0 \\
0 & \text{ otherwise}
\end{cases}
\]

Proof. Let \( B(\omega \beta + p) \) denote the \( \Pi_{3+1} \)-formula:

\[
\beta < \omega \beta \& p < \omega \& (\forall \Gamma \subseteq \Pi_{3+1})\{Prov_{I \Sigma_n}(p, "\neg A, \neg C_n^{I \Sigma_n}(\beta - 1), \Gamma\) \supset Tr_{\Pi_{3+1}}(" \bigvee \Gamma")\}
\]

where

1. \( \Pi_{3+1} \) = the set of gödel numbers of \( \Pi_{3+1} \)-formulae
2. \( Prov_{I \Sigma_n}(p, "\Gamma") \) is a proof predicate for \( I \Sigma_n \) which says that \( p \) is a proof of a sequent \( \Gamma \) in \( I \Sigma_n \). Here
\( I \Sigma_n \) is formulated in a Tait's truth calculus.
3. \( Tr_{\Pi_{3+1}} \) denotes a partial truth definition for \( \Pi_{3+1} \)-formulae.
4. \( \beta - 1 = \delta \) if \( \beta = n < \omega \) then \( n - 1 \) else \( \beta \), and \( C_n^{I \Sigma_n}(0) \) denotes a true formula, e.g., \( 0 = 0 \).

We assume that when \( \Gamma \subseteq \Pi_{3+1} \) and \( Prov_{I \Sigma_n}(p, "\neg A, \neg C_n^{I \Sigma_n}(\beta - 1), \Gamma\) \), every sequent in the proof \( p \) is of the form \( \neg A, \neg C_n^{I \Sigma_n}(\beta - 1), \Delta \) for some \( \Delta \subseteq \Pi_{3+1} \). This follows from a partial cut elimination which is available in \( I \Sigma_n \).

We show that \( T \vdash Prg[B] \). Argue in \( T \). We have \( A \) and \( \forall \gamma \beta \gamma p < \omega B(\omega \gamma + p) \). Hence \( C_n^{I \Sigma_n}(\beta - 1) \). By
induction on \( p < \omega \) we get \( \bigvee \Gamma \). If a \( \Sigma_{3+1} \)-formula \( \neg A, \neg C_n^{I \Sigma_n}(\beta - 1) \) is analysed by an inference rule (G),
then use the facts: \( A \) and \( C_n^{I \Sigma_n}(\beta - 1) \) are true.

\( \square \)
Theorem 7.1 For each $\alpha \geq 0$ and $0 < n, m < \omega$,

\[ I_{\Sigma_n} + TIR^{(m)[\omega^{1+\alpha}, \Pi_{n+1}]} = I_{\Sigma_n} + C_{n^{\Sigma_n}}(\omega^{\alpha \cdot m}) \]

with

\[ \omega^{\alpha \cdot m} = \begin{cases} \omega^{\alpha \cdot m} & \alpha \neq 0 \\ m - 1 & \text{otherwise} \end{cases} \]

Proof. [\subseteq] By induction on $m \geq 0$, we show, for $A \in \Pi_{n+1}$,

\[ I_{\Sigma_n} + C_{n^{\Sigma_n}}(\omega^{\alpha \cdot m}) \vdash Prq[A] \Rightarrow I_{\Sigma_n} + C_{n^{\Sigma_n}}(\omega^{\alpha \cdot (m + 1)}) \vdash \forall x < \omega^{1+\alpha} A(x) \]

where $I_{\Sigma_n} + C_{n^{\Sigma_n}}(\omega^{\alpha \cdot 0}) = I_{\Sigma_n}$. Put $T = I_{\Sigma_n} + C_{n^{\Sigma_n}}(\omega^{\alpha \cdot m})$. By Lemma 7.1 $T \vdash C_{n^{\Sigma_n}}(\omega^{\alpha \cdot m}) \supseteq \forall x < \omega^{1+\alpha} A(x)$. Also $T + C_{n^{\Sigma_n}}(\omega^{\alpha}) = I_{\Sigma_n} + C_{n^{\Sigma_n}}(\omega^{\alpha \cdot (m + 1)})$.

[\supseteq] This follows from Lemma 7.2. \hfill \Box

In what follows we concentrate on the case $n = 1$. For a limit ordinal $\lambda < \epsilon_0$, $\{\lambda[x]\}_{x \in \omega}$ denotes the fundamental sequence given in the Definition 3.7 in [23], i.e., $\omega^{\alpha+1}[x] = \omega^{\alpha \cdot (x + 1)}$.

Definition 7.2 Fast growing functions $F_\alpha$.

1. $F_\alpha$.
   
   (a) $F_\emptyset(x) = 2x + 2$
   
   (b) $F_{\omega+1}(x) = F_\omega^2(2)$
   
   (c) $F_0(0) = 2$
   
   (d) $F_\lambda(x) = F_\lambda(x\lambda)$ for a limit $\lambda$ and $x \neq 0$

2. $F_\alpha(x) \downarrow$ denotes a $\Sigma_1$ formula saying '$F_\alpha(x)$ is defined'.

3. $F_\alpha \Leftrightarrow_{df} \forall x \in \omega(F_\alpha(x) \downarrow)$ a $\Pi_2$ formula

R. Sommer [23] shows that the graph \{$(\alpha, x, y) : F_\alpha(x) = y$\} is $\Delta_0$ definable.

Definition 7.3 $Tot(T), PR(F)$ and $ER(F)$.

1. For a theory $T \supseteq I_{\Sigma_1}$, $Tot(T)$ denotes the set of provably total recursive functions in $T$.

2. For a set $F$ of functions on $\omega$, $PR(F)$ [$ER(F)$] denotes the primitive [elementary] recursive closure of $F$, resp.

Lemma 7.3 1. Each $f \in Tot(I_{\Sigma_1} + F_\alpha)$ is majorized by an $F_{\alpha+n}$ for some $n < \omega$. Thus $Tot(I_{\Sigma_1} + F_\alpha)$

2. $I_{\Sigma_1} \vdash F_\alpha \downarrow \rightarrow F_{\alpha+1} \downarrow$. Thus $Tot(I_{\Sigma_1} + F_\alpha) = PR(F_\alpha)$.

3. $I_{\Sigma_1} \vdash F_{\alpha+n} \downarrow \rightarrow RFN_{\Pi_2}(I_{\Sigma_1} + F_\alpha) \downarrow = C_{n^{\Sigma_1} + F_\alpha}(0)$

Proof. 2. It suffices to show $I_{\Sigma_1} + F_\alpha \downarrow \rightarrow \forall x F_\alpha^x(y \downarrow)$. Fix $y$ as a parameter and use $I_{\Sigma_1}$ to show $\forall x(F_\alpha^x(y) \downarrow)$ by induction on $x$.

3. [\rightarrow] by a formalization of a proof of Lemma 7.3.1 in $I_{\Sigma_1}$. [\rightarrow] follows from 2. \hfill \Box

Lemma 7.4

$I_{\Sigma_1} \vdash C_{n^{\Sigma_1}}(\alpha) \rightarrow F_{\omega(1+\alpha)}(\alpha)$

Proof. [\rightarrow] By the Lemma 7.1.3, it suffices to show $I_{\Sigma_1} \vdash Prq[A]$ with $A(x) \Leftrightarrow_{df} F_\alpha[x] \downarrow \in \Pi_2$. This follows from the Lemma 7.3.2.

[\rightarrow] Put $B(\alpha) \Leftrightarrow_{df} F_{\omega(1+\alpha)} \downarrow \rightarrow C_{n^{\Sigma_1}}(\alpha)$. We show this formula $B(\alpha)$ is reflexively progressive with respect to $I_{\Sigma_1}$. Argue in $I_{\Sigma_1}$ and assume that

\[ \forall \beta < \alpha Prq_{\Sigma_1}("B(\beta)") \& F_{\omega(1+\alpha)} \downarrow \]

Case 0. $\alpha = 0$: By the Lemma 7.3.3, $F_{\omega(1+\alpha)} \downarrow \rightarrow C_{n^{\Sigma_1}}(0)$

Case 1. $\alpha \neq 0$: Assume $\beta < \alpha \& Prq_{\Sigma_1}("C_{n^{\Sigma_1}}(\beta) \rightarrow A")$ for a $A \in \Pi_2$. By a cut, $Prq_{\Sigma_1}("F_{\omega(1+\beta)} \downarrow \rightarrow A")$.

By $\omega(1+\beta) + \omega \leq \omega(1+\alpha)$, we see $F_{\omega(1+\beta)} \downarrow$ from $F_{\omega(1+\alpha)} \downarrow$. Again by the Lemma 7.3.3, we have $RFN_{\Pi_2}(I_{\Sigma_1} + F_{\omega(1+\beta)}) \downarrow$. Thus $T_{\Pi_2}("A")$.

\hfill \Box

Observe that $\omega^{1+\alpha \cdot m} = \omega(1 + \omega^{\alpha \cdot m})$. Therefore from these lemmata and the Theorem 7.1 we see the
Theorem 7.2 For each $\alpha \geq 0$ and $0 < m < \omega$,
\[ T_{\alpha}^{(m)} = \text{df} I \Sigma 1 + TIR^{(m)}[\omega^{1+\alpha}, \Pi_2] = I \Sigma 1 + C_1^{I \Sigma 1}(\omega^\alpha m) = I \Sigma 1 + F_{\omega^{1+\alpha} m} \downarrow \]
and
\[ \text{Tot}(T_{\alpha}^{(m)}) = PR(F_{\omega^{1+\alpha} m}) \]

Corollary 7.1 For $0 \leq k, m < \omega$ with $m \neq 0$,
\[ T_{k}^{(m)} = \text{df} I \Sigma 1 + TIR^{(m)}[\omega^{1+k}, \Pi_2] = I \Sigma 1 + C_1^{I \Sigma 1}(\omega^{k \cdot m}) = I \Sigma 1 + F_{\omega^{1+k} m} \downarrow \]
and
\[ \text{Tot}(T_{k}^{(m)}) = PR(F_{\omega^{1+k} m}) \]

8 Derivation lengths of finite rewrite rules reducing under lexicographic path orders

In this section we discuss a relationship between the derivation lengths of finite rewrite rules reducing under lexicographic path orders and the provably total recursive functions in theories $T_{k}^{(m)}$ defined in Corollary 7.1. In Weiermann [26] and Buchholz [2] it is shown that

Theorem 8.1 (Weiermann [26] and Buchholz [2])
The derivation lengths of finite rewrite rules reducing under a lexicographic path order are bounded by a multiply recursive function $F_{\omega^{1+k} m}(k, m \in \omega)$.

First we introduce a variant of a slow growing function $G_n \alpha$ in [1].

Definition 8.1
1. $O \Omega, P$ and $S \alpha \in \{0,1\}$.
   (a) $P \subset O \Omega$.
   (b) $0 \in O \Omega, S 0 = 0$. [$S \alpha = 0 \Leftrightarrow \alpha < \Omega$]
   (c) $\alpha_1, \ldots, \alpha_n \in P \& \alpha_1 \geq \cdots \geq \alpha_n (n \geq 2) \Rightarrow \alpha_1 + \cdots + \alpha_n \in O \Omega$.
      [Here $\alpha \leq \beta \Leftrightarrow \alpha < \beta$ or $\alpha = \beta$.]
      $S(\alpha_1 + \cdots + \alpha_n) = \max\{S \alpha_i : 1 \leq i \leq n\} = S \alpha_1$.
   (d) $\alpha \in O \Omega = \{ \alpha \in O \Omega : \alpha \leq \Omega \} \Rightarrow \omega^\alpha \in P$. $S \omega^\alpha = S \alpha = 0$.
   (e) $\alpha \in O \Omega \Rightarrow S \alpha = 0$.
   (f) $0 < n < \omega \& \xi \in P \Omega = \{ \xi \in P : \xi < \Omega \} \Rightarrow \Omega \cdot \xi \in P$. $S \Omega \cdot \xi = 1$.

2. $K \alpha \subset P \Omega$
   (a) $K 0 = 0$
   (b) $K(\alpha_1 + \cdots + \alpha_n) = \bigcup\{K \alpha_i : 1 \leq i \leq n\}$
   (c) $K \omega^\alpha = K \alpha$
   (d) $K d \alpha = \{d \alpha\}$
   (e) $K(\Omega \cdot \xi) = K \xi$

3. $\alpha < \beta$
   (a) $\beta \neq 0 \Rightarrow 0 < \beta$
   (b) $\alpha_1 + \cdots + \alpha_n < \beta_1 + \cdots + \beta_m (\alpha_i, \beta_j \in P \& n + m > 2)$
      (i) $n < m \forall i \in n(\alpha_i = \beta_i)$ or
      (ii) $\exists l \leq \min\{n, m\}[\alpha_i < \beta_l \& \forall i < l(\alpha_i = \beta_i)]$
   (c) $\alpha \in P \Omega \Rightarrow \alpha \leq \Omega \cdot \xi$
   (d) $\alpha < d \beta \Rightarrow \omega^\alpha < d \beta$, and $d \alpha \leq \beta \Rightarrow d \alpha < \omega^\beta$
   (e) $\alpha < \beta (\leq \Omega) \Rightarrow \omega^\alpha < \omega^\beta$
   (f) $d \alpha < d \beta \Rightarrow$
      (i) $\alpha < \beta \& K \alpha < d \beta$ or
ii. $\alpha \leq K \beta$
\[
[X < \beta \Leftrightarrow \forall \alpha \in X(\alpha < \beta) \text{ and } \alpha \leq Y \Leftrightarrow \exists \beta \in Y(\alpha \leq \beta)]
\]

(g) $\Omega^n \cdot \xi < \Omega^m \cdot \zeta \Leftrightarrow$
\[
i. \ n < m \text{ or }
\]
\[
ii. \ n = m \& \xi < \zeta
\]

4. Conventions

(a) $1 = \omega^0$, $n = 1 + \ldots + 1$ for $n < \omega$.
(b) $\Omega^n \cdot 0 = 0$, $\Omega^n \cdot \xi = \xi$, $\Omega^n = \Omega^n \cdot 1$ and $\Omega = \Omega^1$.
(c) $\Omega^n \cdot (\xi_1 + \ldots + \xi_n) = \Omega^n \cdot \xi_1 + \ldots + \Omega^n \cdot \xi_n$
for $\Omega > \xi_1 \geq \ldots \geq \xi_n$, $\xi_1, \ldots, \xi_n \in P$.
(d) $\alpha \in P_{\Omega} \Leftrightarrow df \alpha < \Omega \& \alpha \in P$ or $[\alpha = \Omega^n \cdot \xi \geq \Omega$ for some $n, \xi$]
(e) $\alpha \# \beta$ denotes the natural sum.

Definition 8.2 Normal Forms

1. We write $\alpha =_{NF_0} \alpha_1 + \cdots + \alpha_n$
if $n \geq 1$, $\alpha = \alpha_1 + \cdots + \alpha_n$, $\alpha_1 \geq \cdots \geq \alpha_n$ $\forall i \leq n (\alpha_i \in P)$
2. For each $\alpha \in Od$ with $\alpha \neq 0$, $\exists n < \omega \exists (\alpha_0, \ldots, \alpha_n) \exists \!(\xi_0, \ldots, \xi_n)$ such that
\[
\alpha = \Omega^n \cdot \xi_n + \cdots + \Omega^0 \cdot \xi_0 & 0 = \alpha_0 < \cdots < \alpha_n < \omega
\]
$0 < \xi_1, \ldots, \xi_n < \Omega \& 0 \leq \xi_0 < \Omega$

In this case we write
\[
\alpha =_{\Omega-NF} \Omega^n \cdot \xi_n + \cdots + \Omega^0 \cdot \xi_0 =_{\Omega-NF} \sum_{i=0}^{n} \Omega^i \cdot \xi_i
\]
3. For each $\alpha \in Od$ with $\alpha \neq 0$,
$\exists n < \omega \exists (\alpha_0, \ldots, \alpha_n) \exists \!(\xi_0, \ldots, \xi_n) \exists \!(\beta_1, \ldots, \beta_m)$ such that
\[
\alpha = \sum_{i=1}^{n} \Omega^i \cdot \xi_i + \sum_{i=1}^{m} \beta_i \& 0 < \alpha_n < \cdots < \alpha_1 < \omega \& 0 < \xi_1, \ldots, \xi_n < \Omega
\]
$\beta_m \leq \cdots \leq \beta_1 < \Omega \& 0 \leq \xi_0 < \Omega$

In this case we write
\[
\alpha =_{NF_1} \sum_{i=1}^{n} \Omega^i \cdot \xi_i + \sum_{i=1}^{m} \beta_i =_{NF_1} \sum_{i=1}^{n} \gamma_i + \sum_{i=1}^{m} \beta_i \text{ with } \gamma_i = \Omega^i \cdot \xi_i
\]

Definition 8.3 The norm $N\alpha$ of $\alpha \in Od$

1. $N0 = 0$
2. $N\alpha = \max \{n, N\alpha_i : 1 \leq i \leq n\}$ for $\alpha =_{NF_0} \alpha_1 + \cdots + \alpha_n < \Omega$
3. $N\omega^\alpha = N\alpha + 1$
4. $N\alpha = N\alpha + 1$
5. $N\alpha = \max \{\{k-1\} \cup \{N\xi_i : i \leq n\}\}$ for $\alpha =_{\Omega-NF} \Omega^n \cdot \xi_n + \cdots + \Omega^0 \cdot \xi_0$ with $0 < k = \alpha_n < \omega$.

Definition 8.4 (cf. [2], [26]) $\alpha <_k \beta$

1. $\beta \neq 0 \Rightarrow 0 <_k \beta$ [zero]
2. $\beta =_{NF_1} \beta_1 + \cdots + \beta_m (\beta_i \in P_\Omega, m \geq 2)$:
$\exists (\alpha_1, \ldots, \alpha_m) [\alpha = \alpha_1 + \cdots + \alpha_m \& \forall i \leq m (\alpha_i \leq \beta_i) \& \exists i \leq m (\alpha_i <_k \beta_i)$
$\Rightarrow \alpha <_k \beta$ [multiset]
[Here $\alpha_i$ may be 0 and/or $\notin P_\Omega$. $\alpha \leq_\Omega \beta \Leftrightarrow \exists_\Omega \alpha \leq \beta \& \text{ or } \alpha = \beta$.]
3. $\beta \in P_\Omega \& \alpha =_{NF_1} \alpha_1 + \cdots + \alpha_n (\alpha_i \in P_\Omega, n \geq 2)$:
(a) $S\beta = 0: \forall i \leq n(\alpha_i <_k \beta) \& N\alpha < N\beta + k \Rightarrow \alpha <_k \beta$ [inaccessibility]
(b) $S\beta = 1: \forall i \leq n(\alpha_i <_k \beta) \Rightarrow \alpha <_k \beta$

4. $\alpha, \beta \in P\Omega \& 0 = S\alpha < S\beta = 1 \Rightarrow \alpha <_k \beta$ [Stufe]
5. $\alpha, \beta \in P\Omega \& S\alpha \leq S\beta = 1 \& \alpha =_{\Omega-NF}^{\Omega^n \cdot \xi} \& \beta =_{\Omega-NF}^{\Omega^n \cdot \xi}$:
   (a) $n < m$
   (b) $\alpha = \beta \& \xi <_k \zeta$

6. $\alpha <_k \beta < \Omega \Rightarrow \omega^\alpha <_k \omega^\beta$ [monotonicity]
7. $da \leq_k \beta < \Omega \Rightarrow d\alpha <_k \omega^\beta$ [subterm]
8. $\alpha <_k d\beta \& N\alpha \leq N\beta + k \Rightarrow \alpha <_k d\beta$ [inaccessibility]
9. (a) $\alpha <_k \beta \& K\alpha <_k d\beta \& N\alpha \leq N\beta + k$ [inaccessibility]
   (b) $da \leq_k K\beta$ [subterm]

Lemma 8.1
1. $N\alpha$ is a norm, i.e., the set $\{\beta \in Od : \beta < \alpha \& N\beta \leq n\}$ is finite for each $\alpha \in Od$ and $n \in \omega$.
2. The set $\{\beta \in Od : \beta <_k \alpha\}$ is finite for each $\alpha < \Omega$ and $k \in \omega$.

Definition 8.5 $G_n\alpha$ for $\alpha \in Od|\Omega$

$G_n\alpha =_{df} \max\{k \in \omega. \exists(\alpha_0, \ldots, \alpha_k)[\alpha_k <_n, \ldots, \alpha_0 = \alpha]\}$

First we show that the function $G_n\alpha$ is provably total in the fragments $T_k^{(m)}$ of $\Sigma_2$.

Definition 8.6 (cf.[2])
1. $D_k =_{df} \{(a_0, \ldots, a_i) \subset Od|\Omega : \forall j \leq l \forall \beta <_k \alpha (\alpha \in (a_0, \ldots, a_j-1))\}$
2. $W_k =_{df} \{\alpha \in Od|\Omega : \exists d \in D_k (\alpha \in d)\}$
3. $A_k(X, \alpha) \Leftrightarrow d\alpha <_k \omega^\beta$ for a unary $X$
4. $A_k(X) =_{df} \{\alpha \in Od|\Omega : A_k(X, \alpha)\}$

Note that $D_k, W_k, M_k \in \Sigma_1$ and $A_k(X, \alpha) \in \Sigma_0(X^+)$. The following lemmata are seen as in [1].

Lemma 8.2 ($W_k.1$) $I\Sigma_1 \vdash A_k(W_k) = W_k$

($W_k.2$) For each $F \in \Sigma_1 \cup \Pi_1$,
\[ I\Sigma_1 \vdash A_k(F) \subseteq F \rightarrow W_k \subseteq F \]

and
\[ I\Sigma_1 \vdash \forall \alpha \in W_k(\forall \beta <_k \alpha F(\beta) \rightarrow F(\alpha)) \rightarrow W_k \subseteq F. \]

Lemma 8.3 ($I\Sigma_1$) $\alpha, \beta \in W_k \leftrightarrow \alpha \# \beta \in W_k$

Lemma 8.4 ($I\Sigma_1$) $\beta =_{NF_k} \beta_1 + \cdots + \beta_n \& \forall i \leq n(\beta_i \in W_k) \rightarrow \beta \in W_k$

Lemma 8.5 ($I\Sigma_1$) $\alpha \in W_k \rightarrow \omega^\alpha \in W_k$

Lemma 8.6 For $\{\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n\} \subset Od|\Omega$,
\[ \sum_{i=0}^{n} \Omega^i \cdot \alpha_i <_k \sum_{i=0}^{n} \Omega^i \cdot \beta_i \Leftrightarrow (\alpha_0, \ldots, \alpha_n) <_{lex}^{\beta_0} (\beta_0, \ldots, \beta_n) \]
\[ \Leftrightarrow_{df} \exists l \leq n[\alpha_i <_k \beta_i \& \forall i (l < i \rightarrow \alpha_i = \beta_i)] \]
Lemma 8.7 For each $k, m$ with $0 \leq k, m < \omega$ & $m \neq 0$,
\[ T_k^{(m)} \vdash \forall\alpha_0, \ldots, \alpha_{k+1} \in W_n \{ d(\Omega^{2+k} \cdot (m - 1) + \sum_{i=0}^{1+k} \Omega^i \cdot \alpha_i) \in W_n \} \]

Proof by induction on $m > 0$. Argue in $T_k^{(m-1)}$. Assume $d_0, \ldots, d_{k+1} \in D_n$, $d_i = (\beta_0, \ldots, \beta_{i-1})$ with $l_i = lh(d_i)$. Show the $\Sigma_1$ formula
\[ B(j_0, \ldots, j_{k+1}) \Leftrightarrow df d(\Omega^{2+k} \cdot m + \Omega^j \cdot \alpha_j) \in W_n \]
is progressive with respect to the lexicographic order for $j_i < l_i (i \leq k + 1)$. Then the rule $TIR[\omega^{1+k}, \Pi_2] = TIR[\omega^{2+k}, \Sigma_1]$ implies the assertion.

For a proof of the progressiveness use a subsidiary induction on $\ell \alpha$ for $\alpha < d(\Omega^{2+k} \cdot m + \Omega^j \cdot \alpha_j)$ and the Lemmata 8.4, 8.5 and 8.6.

Lemma 8.8 For each $l < \omega$,
\[ T_k^{(m)} \vdash \forall\alpha \in W_n (d(\Omega^{2+k} \cdot m + \Omega^l \cdot \alpha) \in W_n) \]

Proof by metainduction on $l < \omega$.

Claim 8.1 $T_k^{(m)} \vdash d(\Omega^{2+k} \cdot m) \in W_n$.

Proof of the Claim 8.1. By induction on $\ell \alpha$, we show
\[ \alpha <_n d(\Omega^{2+k} \cdot m) \rightarrow \alpha \in W_n \]
Consider the case $\alpha = d\beta <_n d(\Omega^{2+k} \cdot m)$. Then $\beta = \Omega^{2+k} \cdot m' + \sum_{i=0}^{1+k} \Omega^i \cdot \beta_i$ for some $m' < m$, $\beta_i < \Omega$. By the Lemma 8.7 it suffices to show $\{\beta_0, \ldots, \beta_{1+k}\} \subset W_n$. This follows from $\beta_i <_n d(\Omega^{2+k} \cdot m)$ and IH. Now the lemma follows from the Claim 8.1 and the IH on $l$.

Now by a metainduction on $\ell \alpha$ we have the

Lemma 8.9 For each $\alpha < d(\Omega^{2+k} \cdot m + \Omega^\omega)$,
\[ T_k^{(m)} \vdash \alpha \in W_n \]

Next we define the lexicographic path order over a vocabulary having $m$ function symbols of the arity $2 + k$.

Let $ar(f)$ denote the arity of the function symbol $f$ when the symbol $f$ has a fixed arity.

Definition 8.7 $\mathcal{F}_{kQ}^{(m)}$

1. A set $\mathcal{F}_{kQ}^{(m)}$ of function symbols
\[ \mathcal{F}_{kQ}^{(m)} = df \{ \text{list} \} \cup \{ A_p : p < m \} \cup \{ f_q : q < Q \} \]
where list is varadic, $ar(A_p) = 2 + k$ for each $p < m$ and $ar(f_q) = 1$ for each $q < Q$.

Precedence of these symbols is given by
\[ \text{list} < A_0 < \cdots < A_{m-1} < f_0 < \cdots < f_{Q-1} \]

2. For a given countable set Var of variables, Term denotes the set of terms over $\mathcal{F}_{kQ}^{(m)} \cup \text{Var}$. Applying the symbol list to the empty sequence we produce an individual constant $0 = df \text{list}()$.

$\mathcal{G} = \mathcal{G}_{kQ}^{(m)}$ denotes the set of ground (=closed) terms in Term.
Definition 8.8 $s <_{lpot} t$ for $s, t \in \text{Term}$.
For sequences $l = (t_0, \ldots, t_{n-1})$, $\bar{s} = (s_0, \ldots, s_{l-1})$ of terms, let $<_{lpot}$ denote the multiset extension of $<_{lpot}$:

\[ \exists s_0, \ldots, s_{l-1}[\bar{s} \approx s_0 \cdots \cdot s_{l-1} \& \forall i < n(s_i <_{lpot} t_i) \& \exists i < n(s_i <_{lpot} t_i)], \]

where $\approx$ denotes the permutative congruence, * concatenation and

\[(s_0, \ldots, s_{l-1}) <_{lpot} t \iff \forall j < l(s_j <_{lpot} t). \]

Put $t \equiv g\overline{t}$, $\bar{t} = (t_0, \ldots, t_{n-1})$.
$s <_{lpot} t$ if one of the following conditions is fulfilled:

1. $s <_{lpot} t_i$ for some $t_i$.
2. $s \equiv h\overline{s}$, $\bar{s} = (s_0, \ldots, s_{l-1})$ with $h < g$: $s_j <_{lpot} t$ for each $s_j$.
3. $s \equiv g\overline{s}$:
   (a) $g = \text{list}: \bar{s} <_{lpot} \bar{t}$
   (b) $g = A_p(p < m)$:
       \[ \exists j < l = n = 2 + k[\forall i < j(s_i = t_i) \& s_j <_{lpot} t_j \& \forall i(j < i < l \rightarrow s_i <_{lpot} t)] \]

(c) $g = f_q(q < Q)$: $s_0 <_{lpot} t_0$.

Definition 8.9 The norm $|t|$ of a term $t$.

1. $|v| = 0(v \in \text{Var})$
2. $|\text{list}(t_1, \ldots, t_n)| = \max\{n \cup \{1 + |t_i| : 1 \leq i \leq n\}\}$
3. $|A_p(t_1+k, \ldots, t_0)| = \max(\{1+k, p\} \cup \{|t_i|:i<2+k\}) + 1$
4. $|f_q(t)| = \max\{1+k, m, q, |t|\} + 1$

Definition 8.10 (cf. [6]) $\pi t \in \text{Ood}$ for a ground term $t \in G$

1. $\pi\text{list}(t_1, \ldots, t_n) = \omega^{t_1} \# \cdots \# \omega^{t_n}$
2. $\pi A_p(t_1+k, \ldots, t_0) = d(\Omega^{2+k} \cdot p + \sum_{i=0}^{1+k} \Omega^{i} \cdot \pi t_i)$
3. $\pi f_q(t) = d(\Omega^{2+k} \cdot m + \Omega \cdot q + \pi t)$

Definition 8.11 (Buchholz [2]) $s <_{k} t$
Put $t \equiv g\overline{t}$, $\bar{t} = (t_0, \ldots, t_{n-1})$.
$s <_{k} t$ if one of the following conditions is fulfilled:

1. $s <_{k} t_i$ for some $t_i$.
2. $s \equiv h\overline{s}$, $\bar{s} = (s_0, \ldots, s_{l-1})$ with $h < g$: $s_j <_{k} t$ for each $s_j$ and $s \leq |t| + k$.
3. $s \equiv g\overline{s}$:
   (a) $g = \text{list}: \bar{s} <_{k} \bar{t}$ with the multiset extension $<_{k}$ of $<_{k}$ and $|s|\leq |t| + k$.
   (b) $g = A_p(p < m)$:
       \[ \exists j < l = n = 2 + k[\forall i < j(s_i = t_i) \& s_j <_{k} t_j \& \forall i(j < i < l \rightarrow s_i <_{k} t)] \]
       \[ \text{and } |s|\leq |t| + k. \]
   (c) $g = f_q(q < Q)$: $s_0 <_{k} t_0$ and $|s|\leq |t| + k$.

Lemma 8.10 1. $s <_{lpot} t \rightarrow |s\sigma| \leq |t\sigma| + |s|$ for any substitution $\sigma$.
2. $s <_{lpot} t \rightarrow s\sigma <_{|s|} t\sigma$ for any substitution $\sigma$. 

3. If a finite rewrite rule \( \mathcal{R} = \{(l, r)\} \) over \( \mathcal{F}_{kQ}^{(m)} \) is reducing under \( \prec_{I_{pQ}} \), then \( \mathcal{R} \preceq \prec_{n} \) with \( n = \max\{|r| : (l, r) \in \mathcal{R}\} \).

4. \( |t| = N \pi t \) for any ground term \( t \in \mathcal{G} \).

5. \( s \prec t \rightarrow \pi s < \pi t \) for \( s, t \in \mathcal{G} \).

6. \( |t| \leq I \rightarrow \pi t < d(\Omega^{2+k} \cdot m + \Omega \cdot Q) \) for \( t \in \mathcal{G} \).

Let \( \mathcal{R} = \{(l, r)\} \) be a finite rewrite rule over \( \mathcal{F}_{kQ}^{(m)} \) such that \( \mathcal{R} \) is reducing under \( \prec_{I_{pQ}} \). The derivation length function \( D_{\mathcal{R}} \) is defined by

\[
\begin{align*}
Dh_{\mathcal{R}}(t) &= \max\{i \in \omega : \exists(t_{0}, \ldots , t_{i})[t \equiv t_{i} \rightarrow \mathcal{R} \cdots \rightarrow \mathcal{R} t_{0}] \} \\
D_{\mathcal{R}}(n) &= \max\{dh_{\mathcal{R}}(t) : |t| \leq n\}
\end{align*}
\]

**Lemma 8.11** The derivation length function \( D_{\mathcal{R}}(n) \) is majorized by the function \( G_{n}(d(\Omega^{2+k} \cdot m + \Omega \cdot Q)) \), i.e.,

\[
\exists n_{0} \forall n \geq n_{0}[Dh_{\mathcal{R}}(n) \leq G_{n}(d(\Omega^{2+k} \cdot m + \Omega \cdot Q))]
\]

**Proof.** By Lemma 8.10 pick an \( n_{0} \) depending on \( \mathcal{R} \) so that \( Dh_{\mathcal{R}}(t) \leq G_{n_{0}}(\pi t) \). If \( n \geq n_{0} \) and \( |t| \leq n \), then by Lemma 8.10 again, \( \pi t \leq d(\Omega^{2+k} \cdot m + \Omega \cdot Q) \). Thus \( Dh_{\mathcal{R}}(t) < G_{n}(d(\Omega^{2+k} \cdot m + \Omega \cdot Q)) \) by \( <n_{0} \leq n \).

Next we show that the computation of a multiply recursive function \( F_{\omega^{1+k} \cdot m} \) \((k, m \in \omega)\) can be regarded as a derivation in a finite rewrite rule. We learnt this view from Hofbauer [13].

For a term \( t \) let \( \mathcal{S}t \) denote the term \( \text{list}(0, t_{1}, \ldots , t_{n}) \) if \( t \equiv \text{list}(t_{1}, \ldots , t_{n}) \) and \( \text{list}(t) \) otherwise. \( 0^{(m)} = \mathcal{S} \cdots \mathcal{S} 0 = \text{list}(0, \ldots , 0) \) is the \( m \)th numeral. Observe that \( |0^{(m)}| = \pi 0^{(m)} = m \).

Consider the following interpretation:

\[
0 := \text{list}(); \quad +1 := \mathcal{S}; \quad F_{\alpha}(x_{0}) := A_{p}(x_{1+k}, \ldots , x_{1}, x_{0})
\]

with \( \alpha = \omega^{1+k} \cdot p + \sum_{i=0}^{k} \omega^{i} \cdot x_{i+1} \leq p < m \) and

\[
F_{\omega^{1+k} \cdot m + (1+q)} := f_{q} (q < Q)
\]

**Definition 8.12** Grzegorczyk-Ackermann Rewrite Rule \( \mathcal{R}_{Q} \) for \( F_{\alpha} \), \( \alpha \leq \omega^{1+k} \cdot m + Q \)

1. \( F_{\alpha}(0) = 2 \)
   \[\begin{align*}
   (a) & \quad A_{p}(\bar{x}, 0) \rightarrow 2 = \mathcal{S}G \\
   (b) & \quad f_{q}(0) \rightarrow 2
   \end{align*}\]

2. \( F_{\omega^{1+k} \cdot p + \alpha + x_{i+1}}(x_{0} + 1) = A_{p}(\bar{x}, Sx_{1}, Sx_{0}) \rightarrow \\
\quad A_{p}(\bar{x}, x_{1}, A_{p}(\bar{x}, Sx_{1}, x_{0})) = F_{\omega^{1+k} \cdot p + \alpha + x_{i+1}}(F_{\omega^{1+k} \cdot p + \alpha + x_{i+1}}(x_{0})) \)

   with \( \alpha = \sum_{i=0}^{k} \omega^{i} \cdot x_{i+1} \cdot 1 \).

3. \( f_{q}(Sx) \rightarrow f_{q-1}(f_{q}(x)) \)

4. \( F_{\omega^{1+k} \cdot m + 1}(x_{0} + 1) = f_{0}(Sx_{0}) \rightarrow A_{m-1}(Sf_{0}(x_{0}), 0, f_{0}(x_{0})) 
\quad = F_{\omega^{1+k} \cdot m + 1}(F_{\omega^{1+k} \cdot m + 1}(x_{0})) \)

5. \( F_{\omega^{1+k} \cdot p}(x_{0} + 1) = A_{p}(0, Sx_{0}) \rightarrow A_{p-1}(SSx_{0}, 0, Sx_{0}) 
\quad = F_{\omega^{1+k} \cdot p + \alpha + x_{i+1}}(x_{0} + 1)(p \neq 0) \)

6. \( F_{\omega^{1+k} \cdot p + \alpha + x_{i+1}}(x_{0} + 1) = A_{p}(\bar{x}, Sx_{i+1}, 0, Sx_{0}) \rightarrow \\
\quad A_{p}(\bar{x}, x_{i+1}, SSx_{0}, 0, Sx_{0}) = F_{\omega^{1+k} \cdot p + \alpha + x_{i+1} + \omega \cdot 1}(x_{0} + 1) \)

   \((i \neq 0) \quad \text{with} \quad \alpha = \sum_{j=0}^{k} \omega^{j} \cdot x_{j} \)

7. \( F_{0}(x_{0} + 1) := A_{0}(0, Sx_{0}) \rightarrow SSA_{0}(0, x_{0}) = F_{0}(x_{0}) + 2 \)

**Definition 8.13**

1. \( \mathcal{NG} \) denotes the set of ground terms over \( 0, S, A_{p}, f_{q} \).

2. For each \( t \in \mathcal{NG} \), \( no(t) \in \omega \) is defined by

   (a) \( no(0) = 0 \)
Lemma 8.12 1. The Grzegorczyk-Ackermann rewrite rule $R_Q$ is reducing under $<_L$.

2. For each $(l.r) \in R_Q$ and each substitution $\sigma$ with $l \sigma, r \sigma \in NG$,
\[
\text{no}(r \sigma) \leq \text{no}(l \sigma) + 2
\]

3. $R_Q$ is terminating. Let $\tilde{t}$ denote the unique normal form of $t \in NG$. Then $\tilde{t}$ is a numeral and $\text{val}(t) = \text{df} \text{no}(\tilde{t})$ denotes the value of the ground term $t$.

4. For $t \in NG$,
\[
\text{val}(t) \leq \text{no}(t) + 2dh_{\mathcal{R}_Q}(t)
\]

Lemma 8.13 For each $q < \omega$

1. $F_{\omega^{1+k}m+q}$ is elementary recursive in $Dh_{\mathcal{R}_q}$.

2. $F_{\omega^{1+k}m+q}$ is majorized by the function $G_{n}(d\eta_{kmq})$ with
\[
\eta_{kmq} = \{\begin{array}{ll}
\Omega^{2+k} \cdot (m-1) + \Omega^{1+k} \cdot \omega & q = 0 \\
\Omega^{2+k} \cdot m + \Omega \cdot q + \omega & \text{otherwise}
\end{array}
\]

Proof. Case 1 $q = 0$: We have, by the Lemma 8.12
\[
F_{\omega^{1+k}m+q}(n) = F_{\omega^{1+k}(m-1)+(\omega+1)+q}(n) = \text{val}(A_{m-1}(0(n+1),0,0(n))) \leq n + 2dh_{\mathcal{R}_0}(A_{m-1}(0(n+1),0,0(n)))
\]

1. For some constant $c$ depending on $m, k, q$, $\text{val}(A_{m-1}(0(n+1),0,0(n))) \leq n + c$. Thus $F_{\omega^{1+k}m+q}(n) \leq n + 2dh_{\mathcal{R}_0}(n+c)$.

2. By the Lemma 8.10 there exists an $n_0$ such that for any $n \geq n_0$,
\[
\text{val}(A_{m-1}(0(n+1),0,0(n))) \leq G_n(d\eta_{km0})
\]

We show the following Claim which yields $\forall n \geq n_0[F_{\omega^{1+k}m+q}(n) \leq G_n(d\eta_{km0})]$:

Claim 8.2 $n + 2G_n \alpha_n < G_n(d\eta_{km0})$

Proof of the Claim 8.2. We have, by $n + 1 < \omega$ and $N d\eta_{km0} \geq 2$

$n \# \alpha_n \cdot 2 < n \, d(\Omega^{2+k} \cdot (m-1) + 1) \leq n + \Omega^{1+k} \cdot (n+1) + n$.

Also, in general, we have $G_n \alpha + G_n \beta \leq G_n(\alpha \# \beta)$. From these we see the Claim.

Case 2 $q \neq 0$: We have
\[
F_{\omega^{1+k}m+(1+q)}(n) = \text{val}(f_q(0(n))) \leq n + 2dh_{\mathcal{R}_{1+q}}(f_q(0(n)))
\]

1. For a constant $c$ depending on $m, k, q$, $\text{val}(f_q(0(n))) \leq n + c$.

2. As in the Case 1, there exists an $n_0$ such that for any $n \geq n_0$,
\[
\text{val}(f_q(0(n))) \leq G_n \alpha_n \text{ with } \alpha_n = \pi f_q(0(n)) = d(\Omega^{2+k} \cdot m + \Omega \cdot q + n).
\]

We have $n \# \alpha_n \cdot 2 < n \, d(\Omega^{2+k} \cdot m + \Omega \cdot q + \omega) = d\eta_{kmq}$. Thus for any $n \geq n_0 F_{\omega^{1+k}m+(1+q)}(n) \leq G_n(d\eta_{kmq})$. 

$\Box$
**Theorem 8.2** For each $k, m$ with $0 \leq k < \omega$, $0 < m < \omega$,

$$T_k^{(m)} = I\Sigma_1 + TI\sigma^{(m)}[\omega^{1+k}, \Pi_2] = I\Sigma_1 + C_1^{(m)}(\omega^k_m) = I\Sigma_1 + F_{\omega^{1+k}} m \downarrow$$

and

$$\text{Tot}(T_k^{(m)}) =$$

$$PR(F_{\omega^{1+k}} m) = ER\{F_{\omega^{1+k}} m+q: q < \omega\} =$$

$$PR(Dh(<\mathbb{S}_{\omega}(m))) = ER\{Dh(<\mathbb{S}_{\omega}(m)) : q < \omega\} =$$

$$PR(G_n(d(\Omega^{2+k} \cdot (m-1) + \Omega^{1+k} \cdot \omega))) = ER\{G_n(d(\Omega^{2+k} \cdot m + \Omega \cdot q)) : q < \omega\}.$$ 

Also these classes of functions are majorized by any one of the functions $F_{\omega^{1+k}} m+\omega$ and $G_n(d(\Omega^{2+k} \cdot m + \Omega \cdot \omega))$.

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**References**


[22] S. G. Simpson, $\Sigma^1_1$ and $\Pi^1_1$ transfinite induction, in: D. van Dalen et al., eds., Logic Colloquium 80 (North-Holland, Amsterdam, 1982) 239-253.


