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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 977: 13-22</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60820">http://hdl.handle.net/2433/60820</a></td>
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<td>Right</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
<td>publisher</td>
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JORDAN-HÖLDER TYPE THEOREM IN NORMAL INTERMEDIATE SUBFACTOR LATTICES FOR DEPTH TWO INCLUSIONS OF AFD II₁ FACTORS

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ABSTRACT. Let $N \subset M$ be a depth 2 inclusion of AFD II₁ factors with finite Jones index. Let $K$ and $L$ be normal intermediate subfactors of $N \subset M$. If $K \cap L = N$ and $M$ is generated by $K$ and $L$, then we can represent $M, K, L, N$ as $M = P \otimes R, K = Q \otimes R, L = P \otimes S,$ and $N = Q \otimes S$ for some inclusions $P \supset Q$ and $R \supset S$. Using this characterization, we shall prove Jordan-Hölder type theorem in normal intermediate subfactor lattices for depth 2 inclusions of AFD II₁ factors.

1. INTRODUCTION

Let $N \subset M$ be an irreducible inclusion of type II₁ factors with finite index. In [9], the author introduced the notion of normality for intermediate subfactors of $N \subset M$ as follows:

**Definition 1.1.** Let $K$ be an intermediate subfactor of the inclusion $N \subset M$. Let $N \subset M \subset M_1 \subset M_2$ be the Jones tower for $N \subset M$ and $K_1$ the basic extension for $K \subset M$. Then $K$ is a normal intermediate subfactor of the inclusion $N \subset M$ if $e_K \in \mathcal{Z}(N' \cap M_1)$ and $e_{K_1} \in \mathcal{Z}(M' \cap M_2)$, where $e_K$ and $e_{K_1}$ are the Jones projections for $K \subset M$ and $K_1 \subset M_1$, respectively.
With the above notation, if the depth of $N \subset M$ is 2, then $N' \cap M_1$ and $M' \cap M_2$ are a dual pair of Hopf $C^*$-algebras, and $K' \cap K_1$ is a $*$-subalgebra and a left coideal of $N' \cap M_1$ (see [1]). Then $K$ is a normal intermediate subfactor of $N \subset M$ if and only if $K' \cap K_1$ is a subHopf algebra and the left and right adjoint action of $N' \cap M_1$ leave $K' \cap K_1$ invariant (see [3]).

Watatani[10] studied intermediate subfactor lattices $\mathcal{L}(N \subset M)$ and relations between modular identity and commuting and co-commuting (nondegenerate) square conditions. The author[9] proved if the depth of $N \subset M$ is 2, then the set $\mathcal{N}(N \subset M)$ of all normal intermediate subfactors of $N \subset M$ is a sublattice of $\mathcal{L}(N \subset M)$ and a modular lattice.

Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD II$_1$ factors with finite index. Our purpose is to show Jordan-Hölder type theorem in normal intermediate subfactor lattices for $N \subset M$. To be more precise, we prove that if $M = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n = N$ and $M = B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_m = N$ are maximal chains of $\mathcal{N}(N \subset M)$, then $m = n$ and the inclusions $A_{i-1} \supset A_i$ are isomorphic to the inclusions $B_{j-1} \supset B_j$ in some order. To show this, we characterize tensor products of depth 2 inclusions of AFD II$_1$ factors with finite index as follows: Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD II$_1$ factors with finite index. Let $K$ and $L$ be normal intermediate subfactors for $N \subset M$. If $K \cap L = N$ and $M$ is generated by $K$ and $L$, then we can represent $M, K, L, N$ as $M = P \otimes R, K = Q \otimes R, L = P \otimes S$ and $N = Q \otimes S$ for some inclusions $P \supset Q$ and $R \supset S$. 
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2. A CHARACTERIZATION OF TENSOR PRODUCTS OF DEPTH 2 INCLUSIONS

Let $N \subset M$ be an irreducible, depth 2 inclusion of II$_1$ factors with $[M : N] < \infty$ and $\mathcal{N}(N \subset M)$ the all normal intermediate subfactors of $N \subset M$. Suppose that $K, L \in \mathcal{N}(N \subset M)$ and $M$ is generated by $K$ and $L$, and $N = K \cap L$. Then

$$K \subset M$$

$$\cup \quad \cup$$

$$N \subset L$$

is commuting and co-commuting (nondegenerate) square (see [6, 8]). Let $K_1 = \langle K, e_K^M \rangle$ and $L_1 = \langle L, e_L^M \rangle$ be the basic extension with the Jones projections $e_K^M$ and $e_L^M$ for $K \subset M$ and $L \subset M$, respectively. Then it is well known that

$$M \subset K_1$$

$$\cup \quad \cup$$

$$L \subset \langle L, e_K^M \rangle$$

are also nondegenerate commuting squares.

**Lemma 2.1.** With the above notation, $L \subset K_1$ and $K \subset L_1$ are irreducible, depth 2 inclusions. Moreover, $M$ and $\langle L, e_K^M \rangle$ are normal intermediate subfactors of $L \subset K_1$ and, $M$ and $\langle K, e_L^M \rangle$ are normal intermediate subfactors of $K \subset L_1$.
Since $L \subset M$ and $M \subset K_1$ are depth 2 inclusion by [9], the depth of $L \subset K_1$ is 2 by [7]. Similarly, $K \subset L_1$ is a depth 2 inclusion. It is easy to see that $L \subset K_1$ and $K \subset L_1$ are irreducible inclusions. □

Lemma 2.2. With the above notation, we have

\[ K' \cap K_1 = \langle K, e_L^M \rangle' \cap M_1 = N' \cap \langle L, e_K^M \rangle \]

\[ L' \cap L_1 = \langle L, e_K^M \rangle' \cap M_1 = N' \cap \langle K, e_L^M \rangle. \]

Proof. By Lemma 2.1 and [8], we have $[M : K] = [L : N] = [L_1 : \langle K, e_L^M \rangle]$. Therefore we have

\[ \dim_{\mathbb{C}}(K' \cap K_1) = \dim_{\mathbb{C}}(\langle K, e_L^M \rangle' \cap M_1) = \dim_{\mathbb{C}}(N' \cap \langle L, e_K^M \rangle). \]

Let $x$ be an element of $K' \cap K_1$. Since $e_L^M$ is an element of the center of $N' \cap M_1$ and $K' \cap K_1 \subset N' \cap M_1$, $x$ and $e_L^M$ are commutative and hence $x \in \langle K, e_L^M \rangle' \cap M_1$. So we have $K' \cap K_1 \subset \langle K, e_L^M \rangle' \cap M_1$. By $\dim_{\mathbb{C}}(K' \cap K_1) = \dim_{\mathbb{C}}(\langle K, e_L^M \rangle' \cap M_1)$, we have $K' \cap K_1 = \langle K, e_L^M \rangle' \cap M_1$.

Since $M_1$ is the basic extension of $K_1$ by $\langle L, e_K^M \rangle$ with the Jones projection $e_L^M$, we have $\langle L, e_K^M \rangle = \{e_L^M\}' \cap K_1$. Since $e_L^M$ is an element of the center of $N' \cap M_1 (\supset K' \cap K_1)$, if $x$ is an element of $K' \cap K_1$, then $x \in \{e_L^M\}' \cap K_1 = N' \cap \langle L, e_K^M \rangle$. And hence $K' \cap K_1 \subset N' \cap \langle L, e_K^M \rangle$. And $K' \cap K_1 = N' \cap \langle L, e_K^M \rangle$ by $\dim_{\mathbb{C}}(K' \cap K_1) = \dim_{\mathbb{C}}(N' \cap \langle L, e_K^M \rangle)$. Similarly, we have $L' \cap L_1 = \langle L, e_K^M \rangle' \cap M_1 = N' \cap \langle K, e_L^M \rangle$. □
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Theorem 2.3. Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD II$_1$ factors with $[M : N] < \infty$. If $K$ and $L$ are normal intermediate subfactors of $N \subset M$ such that $K \cap L = N$ and $M$ is generated by $K$ and $L$, then we can represent $M, N, K, L$ as $M = P \otimes R, N = Q \otimes S, K = Q \otimes R$ and $L = P \otimes S$.

Proof. $N \subset M$ has the generating property, i.e., there exists a tunnel $M = N_0 \supset N_1 \supset N_2 \supset \cdots$ such that

$$M = \bigcup_{i=1}^{\infty} (M \cap N_i) \supset N = \bigcup_{i=1}^{\infty} (N \cap N_i')$$

(see for example [4, 5]). Let

\[ A_{00} \supset A_{01} \supset A_{02} \supset \cdots \]
\[ \cup \quad \cup \quad \cup \]
\[ A_{10} \supset A_{11} \supset A_{12} \supset \cdots \]
\[ \cup \quad \cup \quad \cup \]
\[ A_{20} \supset A_{21} \supset A_{22} \supset \cdots \]
\[ \cup \quad \cup \quad \cup \]
\[ \vdots \quad \vdots \quad \vdots \]
be the commuting and co-commuting squares such that the initial commuting square is

\[
M \supset L \\
\cup \\
K \supset N
\]

and \(A_{ii} = N_i\) for \(i = 1, 2, \ldots\) as in [8]. Note that for the square

\[
A_{kl} \supset A_{k,l+1} \\
\cup \\
A_{k+1,l} \supset A_{k+1,l+1},
\]

\(A_{kl} \supset A_{k+1,l+1}\) is again irreducible, depth 2 and, \(A_{k,l+1}\) and \(A_{k+1,l}\) are normal intermediate subfactors of \(A_{kl} \supset A_{k+1,l+1}\). We put

\[
P = \bigcup_{i=1}^{\infty} (A_{00} \cap A_{0i}') \supset Q = \bigcup_{i=1}^{\infty} (A_{10} \cap A_{i0}')
\]

\[
R = \bigcup_{i=1}^{\infty} (A_{00} \cap A_{i0}') \supset S = \bigcup_{i=1}^{\infty} (A_{01} \cap A_{i0}')
\]

Then we can see \(M = P \otimes R, N = Q \otimes S, K = Q \otimes R\) and \(L = P \otimes S\) by Lemma 2.2 and [2]. □

3. JORDAN-HÖLDER TYPE THEOREM

In this section, we shall prove Jordan-Hölder type theorem for depth 2 inclusions of AFD \(\text{II}_1\) factors.
Theorem 3.1. Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD $II_1$ factor. If $K$ and $L$ are normal intermediate subfactors of $N \subset M$, then $K \subset K \vee L$ and $K \cap L \subset L$ are conjugate.

Proof. Since the set $\mathcal{N}(N \subset M)$ of all normal intermediate subfactors of $N \subset M$ is a sublattice of $\mathcal{L}(N \subset M)$, $K \vee L$ and $K \cap L$ are elements of $\mathcal{N}(N \subset M)$. Therefore $N \subset K \vee L$ and $N \subset K \cap L$ are depth 2 inclusion by [9, Theorem 4.6]. Moreover $K \cap L$ is a normal intermediate subfactor of $N \subset K \vee L$ by [9, Proposition 3.7]. So we have $K \cap L \subset K \vee L$ is depth 2 inclusion by [9, Theorem 4.6]. By theorem 2.3, there exist inclusions $P \supset Q$ and $R \supset S$ such that $K \vee L = P \otimes R, K = P \otimes S, L = Q \otimes R$ and $K \cap L = Q \otimes S$. So we can see both $K \vee L \subset K$ and $L \supset K \cap L$ are conjugate to $R \subset S$. □

Theorem 3.2. Let $N \subset M$ be an irreducible, depth 2 inclusion of AFD $II_1$ factors with $[M : N] < \infty$. Let $K, \tilde{K}, L, \tilde{L}$ be normal intermediate subfactors of $N \subset M$ with $K \supset \tilde{K}$ and $L \supset \tilde{L}$. Then the pairs $\tilde{K} \vee (K \cap L) \supset \tilde{K} \vee (K \cap \tilde{L})$ and $\tilde{L} \vee (K \cap L) \supset \tilde{L} \vee (\tilde{K} \cap L)$ are conjugate.

Proof. Since $\tilde{K} \vee (K \cap L) = (\tilde{K} \vee (K \cap L)) \vee (K \cap L)$, the pairs $\tilde{K} \vee (K \cap L) \supset \tilde{K} \vee (K \cap \tilde{L})$ and $K \cap L \supset (K \cap L) \vee (\tilde{K} \vee (K \cap L))$ are conjugate by the previous theorem. Similarly, the pair $\tilde{L} \vee (K \cap L) \supset \tilde{L} \vee (\tilde{K} \cap L)$ and $K \cap L \supset (K \cap L) \vee (\tilde{L} \vee (\tilde{K} \cap L))$ are conjugate. Since $\mathcal{N}(N \subset M)$ is a modular lattice by [9], we have $(K \cap L) \vee (\tilde{K} \vee (K \cap \tilde{L})) = ((K \cap L) \cap \tilde{L}) \vee (K \cap \tilde{L})) = (K \cap \tilde{L}) \vee (K \cap \tilde{L})$. Similarly, we have
(K \cap L) \cap (\bar{L} \lor (\bar{K} \cap L)) = (K \cap \bar{L}) \lor (K \cap \bar{L})."\ We have thus proved the theorem. □

In a lattice \( L \), a finite chain \( x = x_0 \supseteq x_1 \supseteq \cdots \supseteq x_d = y \) is maximal if \( x_i \not\supseteq x_{i+1} \) and \( x_i \supseteq a \supseteq x_{i+1} \) implies \( x = a \) or \( x_{i+1} = a \) for \( i = 1, 2, \ldots, d - 1 \).

**Theorem 3.3.** Let \( N \subset M \) be an irreducible, depth 2 inclusion of AFD \( II_1 \) factors with \( [M : N] < \infty \). If \( M = A_0 \supset A_1 \supset \cdots \supset A_n = N \) and \( M = B_0 \supset B_1 \supset \cdots \supset B_m \) are two maximal chain of \( \mathcal{N}(N \subset M) \), then \( m = n \) and the inclusions \( A_{i-1} \supset A_i \) are isomorphic to the inclusions \( B_{j-1} \supset B_j \) in some order.

**Proof.** Put

\[ A_{ij} = A_i \lor (A_{i-1} \cap B_j) \]

and

\[ B_{ji} = B_j \lor (A_i \cap B_{j-1}). \]

Then \( A_{i,j-1} \supset A_{ij} \) is isomorphic to \( B_{j,i-1} \supset B_{ji} \) by Theorem 3.2. Since \( A_0 \supset A_1 \supset \cdots \supset A_s \) is maximal chain, for any \( i (i = 1, 2, \ldots, s) \), there uniquely exists \( j \) such that \( A_{i-1} = A_{i,j-1} \supset A_{ij} = A_i \). Then \( B_{j-1} = B_{j,i-1} \supset B_{ji} = B_j \). And hence \( A_{i-1} \supset A_i \) is isomorphic to \( B_{j-1} \supset B_j \). □

**Example 3.4.** Let \( G \) be a semi direct group \( B \rtimes A \) of finite groups \( A \) and \( B \). Let

\[ M = P \rtimes_{\gamma} B \supset N = P^{(A, \gamma)} = \{ x \in P | \gamma_a(x) = x, \forall a \in A \}, \]

where \( \gamma \) is an outer action of \( G \) on \( II_1 \) factor \( P \). Then the depth of \( N \subset M \) is 2 (see for example [7]). Let \( A_0 = A \supset A_1 \supset \cdots \supset A_r = \{ e \} \) and \( B_0 = B \supset B_1 \supset \cdots \supset B_{r-1} \supset B_r = N \).
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\[ \cdots \supseteq B_i = \{e\} \text{ be normal subgroups of } G \text{ such that if } H \text{ is a normal subgroup of } G \text{ with } A_{i-1} \supseteq H \supset A_i \text{ or } B_{j-1} \supseteq H \supset B_j, \text{ then } H = A_i \text{ or } H = B_j. \]

Then \[ M = P \rtimes_{\gamma} B_0 \supset P \rtimes_{\gamma} B_1 \supset \cdots \supset P = P^{(A_r, \gamma)} \supset P^{(A_{r-1}, \gamma)} \supset \cdots P^{(A_0, \gamma)} = N \]

is a maximal chain of \( N \subset M \) by [9]. Therefore if \( M = C_0 \supset C_1 \supset \cdots C_n = N \)

a maximal chain of \( N \subset M \), then \( n = r + s \) and the inclusions \( C_{k-1} \supset C_k \)

are isomorphic to \( R \rtimes F \supset P \) or \( R \supset R^F \) for some \( \Pi_1 \) factor and some finite group \( F \).

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