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GENERALIZED MINIMAX THEOREMS FOR MULTI-VALUED FUNCTIONS*

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1. Introduction

Minimax theorems are concerned with various fields in mathematics, operational research, and economics. Among many benefits of minimax theorems, most important result is as follows. The saddle point theorem in usual game theory insists that

> a real-valued payoff function possesses a saddle point if and only if the minimax value and the maximin value of the function are coincident;

and accordingly (scalar-valued) minimax theorems say:

> the minimax and maximin values are coincident under certain conditions.

A point (strategy pair) $(x_0, y_0) \in X \times Y$ is said to be a saddle point of f if

$$f(x_0, y) \leq f(x_0, y_0) \leq f(x, y_0)$$

for all $x \in X$, $y \in Y$. We know the minimax value is greater than or equal to the maximin value in general, and hence the insistence of minimax theorems is coincident with the following: the minimax value is less than or equal to the maximin value under some appropriate conditions. These results hold for real-valued functions, but it is not always true in the case of vector-valued payoff functions.

In the decade from 1983, some researchers have studied vector-valued minimax theorems. The common topic is whether or not games with multiple noncomparable criteria have an acceptable theory similar to standard results for scalar games, in particular, what type of minimax equation or inequality holds. In 1983, Nieuwenhuis gave his pioneer idea [15] to this area, and then Corley and Ferro presented important results; [4] and [5, 6, 7]. The author has separately researched such minimax problems in general setting and proved minimax theorems, existence theorems for saddle points, and saddle point theorems in [17, 18, 19, 21, 22, 23, 24]. These results have been approached by vector optimization method.

These papers suggest interesting answers for the following questions: If we give reasonable definitions for minimax values and maximin values of a vector-valued function, what type of minimax equation or inequality holds? Also, if we give a suitable definition for saddle points of the vector-valued function, under what conditions do

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there exist such saddle points? What relationship holds among such minimax values and maximin values and saddle values? Moreover, this kind of research is continued for more general payoff functions, especially multi-valued functions (or set-valued maps) up to now; see [8, 14, 16].

On the other hand, it is also well-known that the convexity and continuity of real-valued functions play very important roles in the area of nonlinear optimization as well as in various fields of mathematics. Such situation remains to vector-valued minimax theory as well as vector optimization. In [21, 23, 24], some types of cone-convexity and (cone-)semicontinuity are introduced, and then vector-valued minimax theorems are proved for vector-valued functions which satisfy these properties.

It is, however, unfortunate that those generalizations for such relaxations and modifications into multi-valued version are incomplete, in particular, with respect to relaxations of continuity. In this paper, we consider a certain relaxation of continuity for vector-valued and multi-valued functions, which corresponds to a generalization of ordinary lower (upper) semicontinuity into vector-valued and multi-valued versions. One of such relaxations was also done in [13]. For vector-valued functions with this generalized lower semicontinuity, we prove existence theorems for generalized saddle points (cone saddle points) of vector-valued functions, and then show some results of [24]. Furthermore, we observe those of loose saddle points for multi-valued functions. For this end, we also need results on cone-convexity and cone-semicontinuity for multi-valued function; see [8, 10, 12] and [9].

2. Saddle and Loose Saddle Points

Let Z be an ordered real topological vector space (ordered t.v.s. for short), as a range space of functions, with the vector ordering $\leq_C$ induced by a convex cone C, that is, for $x, y \in Z$, $x \leq_C y$ if $y - x \in C$. Throughout the paper, the convex cone C is assumed to be solid, that is, its topological interior **int** C is nonempty; and to be pointed, that is, $C \cap (-C) = \{0\}$. For C, an element x_0 of a subset A of Z is said to be a C-minimal point of A (or an efficient point of A with respect to C) if $\{x \in A \mid x \leq_C x_0, x \neq x_0\} = \emptyset$, which is equivalent to $A \cap (x_0 - C) = \{x_0\}$. We denote the set of all C-minimal points of A by $\mathbf{Min}A$. Also, C^0-minimal [resp., C-maximal, C^0-maximal] set of A is defined similarly, and denoted by $\mathbf{Min_w}A$ [resp. $\mathbf{Max}A$, $\mathbf{Max_w}A$], where $C^0 := (\mathbf{int}\ C) \cup \{0\}$. These C^0-minimality and C^0-maximality are weaker than C-minimality and C-maximality, respectively; see [26].

Under the previous notation, we give definitions for generalized saddle point of a vector-valued function and a set-valued map. Let $f : X \times Y \to Z$ and $F : X \times Y \rightsquigarrow Z$ be a vector-valued function and a set-valued map, respectively.

Definition 1. A point (x_0, y_0) is said to be: (i) a C-saddle point of f with respect to $X \times Y$ if

$$f(x_0, y_0) \in \mathbf{Max} f(x_0, Y) \cap \mathbf{Min} f(X, y_0);$$

(ii) a weak C-saddle point of f with respect to $X \times Y$ if

$$f(x_0, y_0) \in \mathbf{Max_w} f(x_0, Y) \cap \mathbf{Min_w} f(X, y_0);$$

(iii) a C-saddle point of F with respect to $X \times Y$ if

$$F(x_0, y_0) \cap \mathbf{Max} F(x_0, Y) \cap \mathbf{Min} F(X, y_0) \neq \emptyset;$$

(iv) a C-loose saddle point of F with respect to $X \times Y$ if

$$F(x_0, y_0) \cap \mathbf{Max} F(x_0, Y) \neq \emptyset$$

and

$$F(x_0, y_0) \cap \mathbf{Min} F(X, y_0) \neq \emptyset.$$

We note that any C-saddle point of f is a weak C-saddle point of f and that any C-saddle point of F is a C-loose saddle point of

F obviously. Also, in the case $C^0 = C$, the conditions (i) and (ii) are coincident. When F is single-valued, every loose saddle point is also a saddle point. Let $f : Z \to \boldsymbol{R}$ be strictly monotonic, that is, $f(a) < f(b)$ whenever $a \leq_C b$ and $a \neq b$, then any loose saddle point (resp. saddle point) of $f \circ F$ in $X \times Y$ is also a loose saddle point (resp. saddle point) of F in $X \times Y$; see Lemma 3.2 of [14]. We have given three types of existence theorem of weak C-saddle points for vector-valued functions, and we consider that of C-loose saddle point for set-valued maps via scalarization method. For this end, we observe cone-convexity and cone-continuity of vector-valued function, and moreover cone-convexity and cone-semicontinuity of set-valued map in the following section.

3. Convexity and Semicontinuity

First, we introduce various types of convexity of vector-valued functions. Some of these convexities are collected in [20, 22, 23]; called cone-convexity. In particular, we explicate some relationships among vector-valued versions of quasi-convexity which correspond to generalizations of ordinary quasi-convexity for real-valued functions. We have the following Table 1 about the relationship among the cone-convexities, and we also observe on their extensions to set-valued functions.

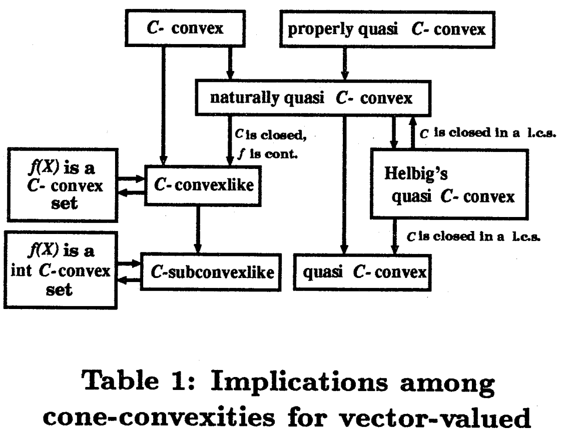

Table 1: Implications among cone-convexities for vector-valued functions.

Based on the notation, terminology, and results in [12], we use the following six kinds of classification for set-relationship:

Definition 2. For nonempty sets $A, B \subset Z$ and a convex cone C in Z, we denote

- $A \Cap C \supset B$ by $A \leq_C^{(\mathrm{i})} B$;
- $A \cap (B \sqcap C) \neq \emptyset$ by $A \leq_C^{(\mathrm{ii})} B$;
- $A \uplus C \supset B$ by $A \leq_C^{(\mathrm{iii})} B$;
- $(A \Cap C) \cap B \neq \emptyset$ by $A \leq_C^{(\mathrm{iv})} B$;
- $A \subset B \sqcup C$ by $A \leq_C^{(\mathrm{v})} B$;
- $(A \uplus C) \cap B \neq \emptyset$ by $A \leq_C^{(\mathrm{vi})} B$,

where

$$A \Cap C := \bigcap_{a \in A} (a + C), A \uplus C := \bigcup_{a \in A} (a + C)$$

and

$$B \sqcap C := \bigcap_{b \in B} (b - C) = B \Cap (-C),$$

$$B \sqcup C := \bigcup_{b \in B} (b - C) = B \uplus (-C).$$

It is easy to see that $A \Cap C \subset A \uplus C$ and $B \sqcap C \subset B \sqcup C$, and also that $A \uplus B = A + B$ and $A \sqcup B = A - B$.

As shown in Fig.3 in the last page, all implications among the set-relations are easily verified.

Proposition 1. For nonempty sets $A, B \subset Z$ and a convex cone C in Z, the following statements hold:

- $A \leq_C^{(\mathrm{i})} B$ implies $A \leq_C^{(\mathrm{ii})} B$;
- $A \leq_C^{(\mathrm{ii})} B$ implies $A \leq_C^{(\mathrm{iii})} B$;
- $A \leq_C^{(\mathrm{iii})} B$ implies $A \leq_C^{(\mathrm{vi})} B$;
- $A \leq_C^{(\mathrm{i})} B$ implies $A \leq_C^{(\mathrm{iv})} B$;
- $A \leq_C^{(\mathrm{iv})} B$ implies $A \leq_C^{(\mathrm{v})} B$;
- $A \leq_C^{(\mathrm{v})} B$ implies $A \leq_C^{(\mathrm{vi})} B$.

Using the six kinds of relationships between two nonempty sets, we consider some different concepts with respect to six different set-relations $\leq_C^{(k)}$ (k = i, ..., vi) for each convexity of set-valued map as generalizations of those of vector-valued function. We can categorize such generalized convexities into five class, that is, convexity, convexlikeness, quasiconvexity, properly quasiconvexity, naturally quasiconvexity, but we concentrate upon convexity, properly quasiconvexity, and quasiconvexity in this paper; see [12] for others.

Definition 3. For each k = i, ..., vi, a set-valued map $F : X \rightsquigarrow Z$ is said to be **type (k) convex** if for every $x_1, x_2 \in$ **Dom**F and $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(k)} \lambda F(x_1) + (1-\lambda)F(x_2).$$

Proposition 2. For a set-valued map $F : X \rightsquigarrow Z$, the following relationships hold:

$$\begin{array}{ccc} \text{type (i) convex} & \rightarrow & \text{type (iv) convex} \\ \downarrow & & \downarrow \\ \text{type (ii) convex} & & \text{type (v) convex} \\ \downarrow & & \downarrow \\ \text{type (iii) convex} & \rightarrow & \text{type (vi) convex} \end{array}$$

Table 2: Implications among type (k) convexity.

The set **Graph**$(F) + (\{\theta_X\} \times C)$ is said to be the epigraph of set-valued map F, and then we have the following result on the epigraph convexity.

Proposition 3. A set-valued map $F : X \rightsquigarrow Z$ is type (iii) convex if and only if its epigraph is convex.

Remark 1. In [10], four notions of convexity of set-valued map are defined, which are included in Definition 3.

Next, we proceed to definitions for properly quasiconvexity of set-valued map.

Definition 4. For each k = i, ..., vi, a set-valued map $F : X \rightsquigarrow Z$ is said to be **type (k) properly quasiconvex** if for every $x_1, x_2 \in$ **Dom**F and $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(k)} F(x_1)$$

or

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(k)} F(x_2).$$

Proposition 4. For a set-valued map $F : X \rightsquigarrow Z$, the relationships shown in Table 3 hold among type (k) properly quasiconvexity (p-qconvex for short in the table).

$$\begin{array}{ccc} \text{type (i) p-qconvex} & \rightarrow & \text{type (iv) p-qconvex} \\ \downarrow & & \downarrow \\ \text{type (ii) p-qconvex} & & \text{type (v) p-qconvex} \\ \downarrow & & \downarrow \\ \text{type (iii) p-qconvex} & \rightarrow & \text{type (vi) p-qconvex} \end{array}$$

Table 3: Implications among type (k) properly quasiconvexity.

Thirdly, we consider some generalizations of quasiconvexity. For a set-valued map $F : X \to Z$ and $x_1, x_2 \in$ **Dom**F, we denote, respectively, the dominated set from below by sets $F(x_1)$ and $F(x_2)$ and the set of points dominating sets $F(x_1)$ and $F(x_2)$ simultaneously from above by

$$C_L(F(x_1), F(x_2)) = (F(x_1) \uplus C) \cap (F(x_2) \uplus C),$$

and

$$C_U(F(x_1), F(x_2)) = (F(x_1) \Cap C) \cap (F(x_2) \Cap C).$$

By using such two sets and the six different set-relations $\leq_C^{(k)}$ (k = i, ..., vi), we generalize quasi C-convexity of vector-valued function, but types (iv)–(vi) generalizations are meaningless since the following conditions are trivial in the cases.

Definition 5. For each $k = \text{i, ii, iii}$, a set-valued map $F : X \rightsquigarrow Z$ is said to be

- **type (k)-lower quasiconvex** if for every $x_1, x_2 \in \mathbf{Dom}F$ and $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(k)} C_L(F(x_1), F(x_2));$$

- **type (k)-upper quasiconvex** if for every $x_1, x_2 \in \mathbf{Dom}F$ and $\lambda \in (0,1)$,

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{(k)} C_U(F(x_1), F(x_2)).$$

Definition 6. A set-valued map $F : X \rightsquigarrow Z$ is said to be

- **type (-1) level-set convex** if for every $z \in Z$,

$$F^{-1}(z-C) := \{x \in X \mid F(x) \cap (z-C) \neq \emptyset\}$$

is convex or empty;

- **type ($+1$) level-set convex** if for every $z \in Z$,

$$F^{+1}(z-C) := \{x \in X \mid F(x) \subset (z-C)\}$$

is convex or empty.

In [14, 16], the notion of type (-1) level-set convexity is used in existence theorems for loose saddle points. By Proposition 1 and simple demonstration, we have the following interesting implications among quasiconvexities above, including the level-set convexity.

Proposition 5. For a set-valued map $F : X \rightsquigarrow Z$, the following relationships hold:

		type (i)-lower quasiconvex	$\rightarrow$	type (i)-upper quasiconvex	$\leftrightarrow$	type ($+1$) level-set convex
		$\downarrow$		$\downarrow$		
		type (ii)-lower quasiconvex	$\rightarrow$	type (ii)-upper quasiconvex		
		$\downarrow$		$\downarrow$		
type (-1) level-set convex	$\leftrightarrow$	type (iii)-lower quasiconvex	$\rightarrow$	type (iii)-upper quasiconvex		

Table 4: Implications among type (k) lower and upper quasiconvexities and level-set convexities.

With respect to set-relation (iii), type (iii) lower quasiconvexity, equivalently type (-1) level-set convexity, is a weaker notion than type (iii) convexity and type (iii) properly quasiconvexity, respectively; see [12].

Now, we generalize a vector-valued version of lower semicontinuity, which is a generalization of the ordinary lower semicontinuity on real-valued functions, that is, the notion of classical upper semicontinuity of set-valued map is generalized to cone-upper semicontinuity. Hence, simultaneous semicontinuity of a real-valued function and a set-valued map is transmitted to the supremum type marginal function associated with them.

With respect to researches on upper and lower semicontinuities for set-valued maps, there are an extensive bibliography in [2]. Let X and Y be two topological spaces. A set-valued map $F : X \rightsquigarrow Y$ is said to be upper semicontinuous (**u.s.c.** for short) at x_0 if for any open set V with $F(x_0) \subset V$, there exists a neighborhood U of x_0 such that

$$F(x) \subset V \quad \text{for all} \quad x \in U.$$

In [25], some modifications for this notion are given as follows: a set-valued map $F : X \rightsquigarrow Y$ is said to be weak upper semicontinuous (**wusc** for short) at x_0 if for any open set V with **cl** $F(x_0) \subset V$, there exists a neighborhood U of x_0 such that

$$F(x) \subset V \quad \text{for all} \quad x \in U;$$

moreover if Y is a t.v.s., a set-valued map $F : X \rightsquigarrow Y$ is said to be equally weak upper semicontinuous (**ewusc** for short) at x_0 if for any open neighborhood G of the origin of Y, there exists a neighborhood U of x_0 such that

$$F(x) \subset F(x_0) + G \quad \text{for all} \quad x \in U.$$

These notions are slightly different, and the following relation holds:

$$\text{u.s.c.} \longrightarrow \text{wusc} \longrightarrow \text{ewusc}.$$

Moreover, we introduce three types of cone-upper semicontinuity of set-valued map which extend ordinary u.s.c. and its modifications above, respectively and which are also generalizations of real-valued lower semicontinuity.

Definition 7. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. A set-valued map $F : X \rightsquigarrow Y$ is said to be:

(u1) C-upper semicontinuous at x_0 (C-**usc**) if for any open neighborhood V of $F(x_0)$, there exists an open neighborhood U of x_0 such that $F(x) \subset V + C$ for all $x \in U \cap \mathbf{Dom}F$ ([13, Def.7.1(p.33)]);

(u2) C-weak upper semicontinuous at x_0 (C-**wusc**) if for any open neighborhood V of **cl** $F(x_0)$, there exists an open neighborhood U of x_0 such that $F(x) \subset V + C$ for all $x \in U \cap \mathbf{Dom}F$;

(u3) C-equally weak upper semicontinuous at x_0 (C-**ewusc**) if for any open neighborhood G of $\theta_Y \in Y$, there exists an open neighborhood U of x_0 such that $F(x) \subset F(x_0) + G + C$ for all $x \in U \cap \mathbf{Dom}F$,

where $\mathbf{Dom}F := \{x \in X \mid F(x) \neq \emptyset\}$.

Ordinary upper semicontinuity of set-valued map implies cone-upper semicontinuity. Such types of cone-upper semicontinuity are also regarded as extensions of ordinary lower semicontinuity for real-valued functions, or extensions of a vector-valued version (called cone-lower semicontinuity) of the lower semicontinuity; see [24, Def.2.1]. In fact, whenever $Y = \boldsymbol{R}$, $C = \boldsymbol{R}_+$, and F is an ordinary (singleton) function, all types of C-upper semicontinuity above are the same as the ordinary lower semicontinuity.

In the case of cone-semicontinuity of vector-valued function, the corresponding notions to three different definitions above are coincident with each other ([24, Def.2.1]), but such notions for set-valued map are not always coincident.

Proposition 6. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. In the above definition, **(u1)** $\Rightarrow$ **(u2)** $\Rightarrow$ **(u3)**. Moreover, if $F(x_0)$ is closed then **(u2)** $\Rightarrow$ **(u1)**. Also, if **cl** $F(x_0)$ is compact in Y, then **(u3)** $\Rightarrow$ **(u2)**.

Next, we introduce cone-semicontinuity for lower semicontinuity of set-valued map. Let X and Y be two topological spaces. A set-valued map $F : X \rightsquigarrow Y$ is said to be lower semicontinuous (**l.s.c.** for short) at x_0 if

(i) for any open set V with $F(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$;

equivalently

(ii) for any $y_0 \in F(x_0)$ and any open neighborhood V of y_0, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$;

moreover if Y is a t.v.s., we have the following equivalent condition

(iii) for any $y_0 \in F(x_0)$ and any open neighborhood G of $\theta_Y \in Y$, there exists a neighborhood U of x_0 such that $F(x) \cap y_0 + G \neq \emptyset$ for all $x \in U$.

If Y is a t.v.s., we can provide the following modification of lower semicontinuity, which is stronger than lower semicontinuity. A set-valued map $F : X \rightsquigarrow Y$ is said to be equally lower semicontinuous (**elsc** for short) at x_0 if for any open neighborhood G of $\theta_Y \in Y$, there exists a neighborhood U of x_0 such that $F(x_0) \subset F(x) + G$ for all $x \in U$. If a set-valued map $F : X \rightsquigarrow Y$ is elsc at x_0 then it is also l.s.c. at the point. Conversely, if F is l.s.c. at x_0 and **cl** $F(x_0)$ is a compact set, then it is elsc at the point; see [25] for detail.

Definition 8. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. A set-valued map $F : X \rightsquigarrow Y$ is said to be:

(l1) C-equally lower semicontinuous at x_0 (C-**elsc**) if for any open neighborhood G of $\theta_Y \in Y$, there exists an open neighborhood U of x_0 such that $F(x_0) \subset F(x) + G - C$ for all $x \in U \cap \mathbf{Dom}F$;

(l2) C-lower semicontinuous at x_0 (C-**lsc**) if for any $y_0 \in F(x_0)$ and any neighborhood G of $\theta_Y \in Y$, there exists a neighborhood U of x_0 with $F(x) \cap (y_0 + G + C) \neq \emptyset$ for any $x \in U \cap \mathbf{Dom}F$.

Two types of cone-lower semicontinuities of set-valued map above generalize equally lower semicontinuity and lower semicontinuity of set-valued map which are proposed in [25]. Of course, ordinary lower semicontinuity of set-valued map implies C-lower semicontinuity. Also, conditions **(u3)** and **(l1)** are precisely dual concepts in the sense of complementary notions by exchanging $(F(x_0), C)$ and $(F(x), -C)$, respectively. For more detail research on cone-semicontinuity, see a forthcoming paper [9].

Now, we consider the composition of a real-valued function and a set-valued map, $\varphi : X \rightsquigarrow \boldsymbol{R}$ defined by $\varphi(x) := f \circ F(x) = \cup_{y \in F(x)} \{f(y)\}$, and consider $C = \boldsymbol{R}_+$ or $C = \boldsymbol{R}_-$; then the marginal functions are denoted by $g(x) = \sup \varphi(x)$ and $h(x) = \inf \varphi(x)$. A real-valued function $f : Y \to \boldsymbol{R}$ is called monotonically u.s.c. (resp. monotonically l.s.c.) if for any set $V \subset \boldsymbol{R}$ and $\varepsilon > 0$, $f^{-1}(V + (-\varepsilon, \varepsilon) + \boldsymbol{R}_-)$ is open and $f^{-1}(V) + G \subset f^{-1}(V + (-\varepsilon, \varepsilon) + \boldsymbol{R}_-)$ for some open neighborhood G of θ_Y (resp. by replacing $\boldsymbol{R}_-$ by $\boldsymbol{R}_+$).

Proposition 7. Let X and Y be a topological space and an ordered topological vector space with a convex cone C, respectively. If $F : X \rightsquigarrow Y$ and $f : Y \to \boldsymbol{R}$ have the following semicontinuity, then we have the following eight statements on semicontinuity for φ, $\sup \varphi$, and $\inf \varphi$:

	F	f	φ	$\sup \varphi$	$\inf \varphi$
(1-1)	usc	usc	$\boldsymbol{R}_-$-ewusc	usc	
(1-2)	ewusc	mono. usc	$\boldsymbol{R}_-$-ewusc	usc	
(2-1)	usc	lsc	$\boldsymbol{R}_+$-ewusc		lsc
(2-2)	ewusc	mono. lsc	$\boldsymbol{R}_+$-ewusc		lsc
(3)	elsc	mono. usc	$\boldsymbol{R}_-$-elsc	lsc	
(4)	elsc	mono. lsc	$\boldsymbol{R}_+$-elsc		usc
(5)	lsc	usc	$\boldsymbol{R}_-$-lsc		usc
(6)	lsc	lsc	$\boldsymbol{R}_+$-lsc	lsc	

Based on these results, we prove existence theorems for generalized saddle points of multi-valued functions.

Theorem 1. [Th.3.3 in [14]] Let X, Y be two locally convex spaces over reals, and Z an ordered topological vector space with a convex cone C, respectively. Assume that the following conditions hold:

(i) A and B are nonempty compact convex sets in X and Y, respectively;

(ii) F is continuous compact-valued;

(iii) there exists a strictly monotonic continuous single-valued map f from Z to $\boldsymbol{R}$ such that $f \circ F(x,y)$ is type (-1) level-set convex in x for any fixed y and that $-f \circ F(x,y)$ is type (-1) level-set convex in y for any fixed x.

Then, there exists at least one loose saddle point of F on $A \times B$.

The proof of Theorem 1 is based on applying the Browder fixed point theorem [3] to the following map:

$$T(x_0, y_0) = \left\{ (x,y) \in A \times B \;\middle|\; \begin{array}{l} \min f \circ F(A, y_0) \in f \circ F(x, y_0) \\ \max f \circ F(x_0, B) \in f \circ F(x_0, y) \end{array} \right\}$$

for each $(x_0, y_0) \in A \times B$.

Theorem 2. [Th.3.1 in [16]] Let X, Y be two Hausdorff topological vector spaces over reals, and Z an ordered Hausdorff topological vector space with a convex cone C, respectively. Assume that the following conditions hold:

(i) A and B are nonempty compact convex sets in X and Y, respectively;

(ii) F is compact-valued and upper semicontinuous such that, for each fixed $x \in A$, $y \mapsto F(x,y)$ is lower semicontinuous on B and, for each fixed $y \in B$, $x \mapsto F(x,y)$ is lower semicontinuous on A;

(iii) there exists a strictly monotonic continuous single-valued map f from Z to $\boldsymbol{R}$ such that, $f \circ F(x,y)$ is type (-1) level-set convex in x for any fixed y and that $-f \circ F(x,y)$ is type (-1) level-set convex in y for any fixed x.

Then, there exists at least one loose saddle point of F on $A \times B$.

These results can be improved by using several kinds of cone-convexity and cone-semicontinuity for set-valued maps.

4. Minimax Theorems

In a few of the author's papers, he has proposed some minimax theorems for vector-valued functions. Their results are based on both existence theorems of saddle points and a saddle point theorem of a vector-valued function, which is a corollary of existence for vector-valued minimax and maximin sets. His vector-valued minimax theorems consists of three types: topological space type, topological vector space type, and locally convex space type.

They are similar statements to the ordinary minimax theorems for real-valued functions. In fact, vector-valued minimax theorems tell us that there exist some minimax strategy and maximin strategy of f such that their values are ordered by $\leq_C$ and dominated each other whenever f has a weak C-saddle point. As illustrated in Fig. 1,

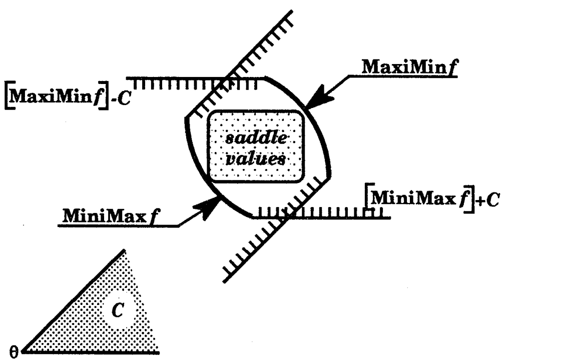

Fig.1:Minimax inequality among minimax values, maximin values, and saddle values (type I).

first type minimax theorem means that minimax values and maximin values of f are entirely contained in the set of maximin values of f minus the pointed convex cone C and in the set of minimax values of f plus the pointed convex cone C, respectively. Also, as illustrated in Fig. 2,

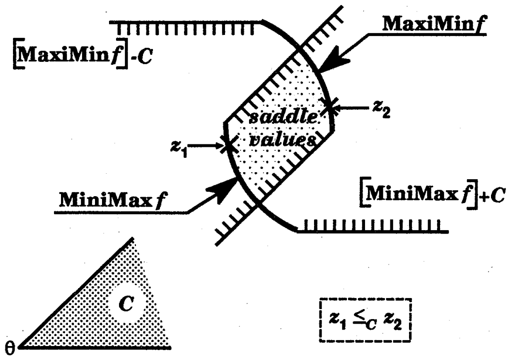

Fig.2:Minimax inequality among minimax values, maximin values, and saddle values (types II, III).

second and third type minimax theorems mean that there exist some minimax values and maximin values of f such that both vectors are ordered by $\leq_C$ and dominated each other.

With respect to multi-valued functions, such minimax theorem is open problems, but different results are expected, which will be found in forthcoming papers.

References

[1] Aubin J-P and Frankowska H (1990) Set-Valued Analysis. Birkhäuser, Boston.

[2] Borisovich YG, Gel'man BD, Myshkis AD, and Obukhovskii VV (1984) Multivalued Mappings. J Soviet Math 24:719–791.

[3] Browder FE (1984) Coincidence Theorems, Minimax Theorems, and Variational Inequalities. Contemporary Mathematics 26:67–80.

[4] Corley HW (1985) Games with Vector Payoffs. J Optim Theory Appl 47:491–498.

[5] Ferro F (1982) Minimax Type Theorems for n-Valued Functions. Annali di Matematica Pura ed Applicata, 32:113–130.

[6] Ferro F (1989) A Minimax Theorem for Vector-Valued Functions. J Optim Theory Appl 60:19–31.

[7] Ferro F (1991) A Minimax Theorem for Vector-Valued Functions, Part 2. J Optim Theory Appl 68:35–48.

[8] Ha TXD (1996) Cones Admiting Strictly Positive Functionals and Scalarization of Some Vector Optimization Problems. (to appear in J Optim Theory Appl)

[9] Kimura Y, Tanaka K, and Tanaka T (1996) On Semicontinuity of Marginal Functions $\sup_{y\in F(x)} f(y)$ and $\inf_{y\in F(x)} f(y)$, 「非線形解析学と凸解析学の研究」研究集会, 京都大学数理解析研究所; 予稿集, 5–6, to appear in RIMS Kokyuroku.

[10] Kuroiwa D (1996) Convexity for Set-Valued Maps. Appl Math Lett 9:97–101.

[11] Kuroiwa D (1996) Convexity for Set-Valued Maps and Optimization. Doctoral thesis, Niigata University, Niigata.

[12] Kuroiwa D, Tanaka T, and Ha TXD (1996) On Cone Convexity of Set-Valued Maps. to appear in the proceedintgs of the second world congress of Nonlinear Analysts held in Athens, Greece, July 10–17, 1996.

[13] Luc DT (1989) Theory of Vector Optimization. Lecture Notes in Economics and Mathematical Systems, Vol.319, Springer-Verlag, Berlin.

[14] Luc DT, Vargas C (1992) A Saddlepoint Theorem for Set-Valued Maps. Nonlinear Analysis: Theory, Methods & Appl 18:1–7.

[15] Nieuwenhuis JW (1983) Some Minimax Theorems in Vector-Valued Functions. J Optim Theory Appl 40:463–475.

[16] Tan KK, Yu J, Yuan XZ (1996) Existence Theorems for Saddle Points of Vector-Valued Maps. J Optim Theory Appl 89:731–747.

[17] Tanaka T (1988) Some Minimax Problems of Vector-Valued Functions. J Optim Theory Appl 59:505–524.

[18] Tanaka T (1989) Existence Theorems for Cone Saddle Points of Vector-Valued Functions in Infinite-Dimensional Spaces. J Optim Theory Appl 62:127–138.

[19] Tanaka T (1990) A Characterization of Generalized Saddle Points for Vector-Valued Functions via Scalarization. Nihonkai Mathematical Journal 1:209–227.

[20] Tanaka T (1990) Cone-Convexity of Vector-Valued Functions. Science Reports of Hirosaki University 37:170–177.

[21] Tanaka T (1991) Two Types of Minimax Theorems for Vector-Valued Functions. J Optim Theory Appl 68:321–334.

[22] Tanaka T (1992) Minimax Theorems in Vector Optimization. Ph.D. thesis (Doctor of Science), Niigata University, Niigata.

[23] Tanaka T (1994) Generalized Quasiconvexities, Cone Saddle Points, and Minimax Theorem for Vector-Valued Functions. J Optim Theory Appl 81:355–377.

[24] Tanaka T (1996) Generalized Semicontinuity and Existence Theorems for Cone Saddle Points. (to appear in J Appl Math Optim)

[25] Tanaka T and Seino T (1996) On a Theoretically Conformable Duality for Semicontinuity of Set-Valued Mappings. 京都大学数理解析研究所講究録, RIMS Kokyuroku 939:1–10.

[26] Yu PL (1974) Cone Convexity, Cone Extreme Points, and Nondominated Solutions in Decision Problems with Multiobjectives. J Optim Theory Appl 14:319–377.

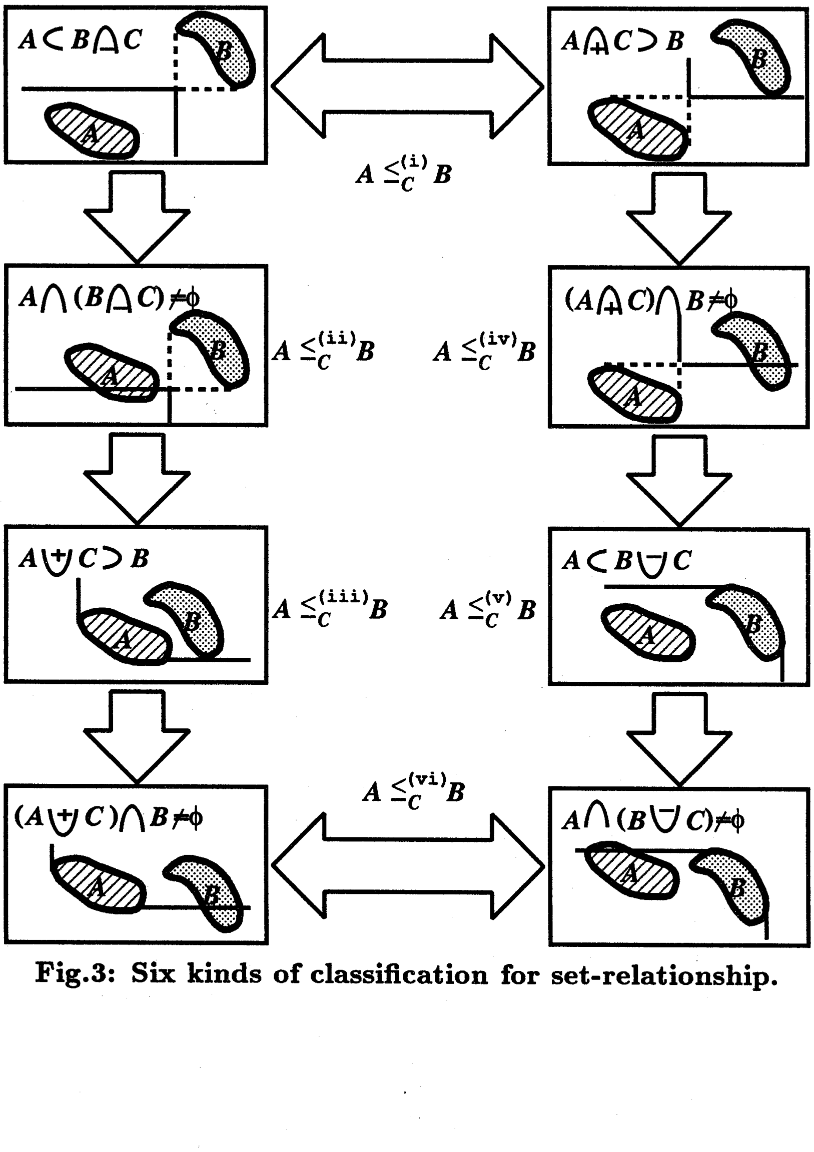

Fig.3: Six kinds of classification for set-relationship.