<table>
<thead>
<tr>
<th>Title</th>
<th>On the Time-Independence of Entropy dimensions associated with a $W^*$-Dynamical System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>AKASHI, Shigeo</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1997, 980: 32-34</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60875">http://hdl.handle.net/2433/60875</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
On the Time-Independence of Entropy dimensions associated with a $W^*$-Dynamical System

Shigeo AKASHI
Department of Mathematics, Faculty of Science, Niigata University
8050, 2-nomachi, Igarashi, Niigata-shi, 950-21 JAPAN

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all positive integers, the set of all real numbers and the set of all complex numbers, respectively, $\mathcal{H}$ and $B(\mathcal{H})$ denote a separable Hilbert space and the algebra of all bounded operators on $\mathcal{H}$, respectively. Let $\mathcal{N}_{*,+,1}(B(\mathcal{H}))$ be the set of all normal states on $B(\mathcal{H})$. If $S$ is a weak* compact and convex subset of $\mathcal{N}_{*,+,1}(B(\mathcal{H}))$, then the set of all extremal points belonging to $S$, which is denoted by $\text{ex}S$, is non-empty. For any normal state $\phi \in S$, if there exist both a non-negative sequence $\{\lambda_k; k \in \mathbb{N}\}$ satisfying $\sum_k \lambda_k = 1$ and a sequence of normal states $\{\phi_k; k \in \mathbb{N}\} \subset \text{ex}S$, which enable $\phi$ to be represented by the following countable convex combination:

$$\phi = \sum_{k=1}^{\infty} \lambda_k \phi_k,$$

then, we define $D(\phi, S)$ by the set of all non-negative sequences that enable $\phi$ to be represented by the above way. Now, for any positive number $\alpha \neq 1$, Ohya's $(S, \alpha)$-entropy of $\phi$ is defined by

$$S(\phi, S, \alpha) = \inf \left\{ \frac{\log \sum_{k=1}^{\infty} \lambda_k^\alpha}{1 - \alpha}; \{\lambda_k; k \in \mathbb{N}\} \in D(\phi, S) \right\}.$$

Here, Ohya's $S$-entropy dimension of $\phi$ is defined by

$$d(\phi, S) = \inf \{\alpha > 0; S(\phi, S, \alpha) < \infty\}.$$

Throughout this paper, we will treat the case that $S = \mathcal{N}_{*,+,1}(B(\mathcal{H}))$ holds and we will abbreviate $d(\phi, \mathcal{N}_{*,+,1}(B(\mathcal{H})))$ to $d(\phi)$ for simplicity.

Let $(B(\mathcal{H}), \mathbb{R}, \alpha)$ be a $W^*$-dynamical system, and $\alpha$ be a surjective continuous action defined on $\mathbb{R}$ with values in the set of all surjective *-automorphism group on $B(\mathcal{H})$, that is, for any $s, t \in \mathbb{R}$, $\alpha_s \circ \alpha_t = \alpha_{s+t}$ holds and $\alpha_t$ is a surjective *-homomorphism defined on $B(\mathcal{H})$ with values in $B(\mathcal{H})$ which is continuous in the $\sigma$-weak operator topology and satisfies the following condition:

$$\lim_{t \to \alpha} \langle x, \alpha_t(A)y \rangle = \langle x, \alpha_s(A)y \rangle, \quad x, y \in \mathcal{H}, \quad A \in B(\mathcal{H}).$$

Then, it follows from the following theorem that the entropy dimensions of the normal states constructed by the combination with the initial states and the continuous action
associated with the given $W^*$-dynamical system are time-independent.

**Theorem.** Let $\alpha$ be a surjective continuous action. Then, for any normal state $\phi$, $d(\phi) = d(\phi \circ \alpha_t)$ holds for any $t \in \mathbb{R}$.

**Proof.** For any $x \in \mathcal{H}$, the vector state constructed by $x$, which is denoted by $\omega_x$ is defined by

$$\omega_x(A) = \langle x|A|x \rangle, \quad A \in B(\mathcal{H}).$$

Here, we can assume that $\phi$ is represented by

$$\phi = \sum_{k=1}^{\infty} \lambda_k |f_k><f_k|,$$

where $\{\lambda_k\}$ is a non-negative sequence satisfying $\Sigma_k \lambda_k = 1$, and $\{f_k\}$ is an orthonormal system of $\mathcal{H}$. Then, $\phi$ can be represented by

$$\phi = \sum_{k=1}^{\infty} \lambda_k \omega_{e_k}.$$

Since $\phi \circ \alpha_t = 0$ implies that $\phi = 0$ holds, $j \neq k$ implies that $\omega_j \circ \alpha_t \neq \omega_k \circ \alpha_t$ holds. Therefore, it is sufficient to prove that, for any positive integer $k$, $\omega_k \circ \alpha_t$ belongs to $ex\mathcal{N}_{s,+1}(B(\mathcal{H}))$ holds. Let $\omega$ be an element of $ex\mathcal{N}_{s,+1}(B(\mathcal{H}))$ and $\psi$ be $\omega \circ \alpha_t$ and $\{\mathcal{H}_\omega, \pi_\omega, x_\omega\}$ (resp. $\{\mathcal{H}_\psi, \pi_\psi, x_\psi\}$) be the cyclic representation of $B(\mathcal{H})$ (resp. $B(\mathcal{H})$) constructed by $\omega$ (resp. $\psi$). Let $(\alpha_t)_{\omega,\psi}$ be an operator on $\{\pi_\psi(B)x_\psi; B \in B(\mathcal{H})\}$ with values in $\{\pi_{\omega}(A)x_\omega; A \in B(\mathcal{H})\}$ defined by

$$(\alpha_t)_{\omega,\psi}^* \pi_\psi(B)x_\psi = \pi_\psi(\alpha_t(B))x_\psi, \quad B \in B(\mathcal{H}).$$

Then, for any $B, C \in B(\mathcal{H})$, we have

$$< (\alpha_t)_{\omega,\psi}^* \pi_\psi(B)x_\psi | (\alpha_t)_{\omega,\psi}^* \pi_\psi(C)x_\psi > = < \pi_\psi(\alpha_t(B))x_\psi | \pi_\psi(\alpha_t(C))x_\psi >$$

$$= < x_\psi | \pi_\psi(\alpha_t(B)^*\alpha_t(C))x_\psi >$$

$$= < x_\psi | \pi_\psi(B^*C)x_\psi >$$

$$= < \pi_\psi(B)x_\psi | \pi_\psi(C)x_\psi > .$$

These equalities imply that $(\alpha_t)_{\omega,\psi}^* (\alpha_t)_{\omega,\psi}$ is the identity mapping. It is clear that the uniform closure of $\{\pi_\omega((\alpha_t)(B))x_\omega; B \in B(\mathcal{H})\}$ is exactly equal to $\mathcal{H}_\omega$, because $\alpha_t$ is surjective. Therefore, $(\alpha_t)_{\omega,\psi}$ can be uniquely extended to an isometry defined on $\mathcal{H}_\phi$. Since, for any $B, C \in B(\mathcal{H})$, we have

$$(\alpha_t)_{\omega,\psi}^* \pi_\psi(B)(\alpha_t)_{\omega,\psi}^* \pi_\psi((\alpha_t)(C))x_\omega = (\alpha_t)_{\omega,\psi}^* \pi_\psi(B)(\alpha_t)_{\omega,\psi}^* \pi_\psi((\alpha_t)(C))x_\psi$$

$$= (\alpha_t)_{\omega,\psi}^* \pi_\psi(BC)x_\psi$$

$$= \pi_\omega((\alpha_t)(BC))x_\omega$$

$$= \pi_\omega((\alpha_t)(B))\pi_\omega((\alpha_t)(C))x_\omega,$$
these equalities imply that $(\alpha_t)_{\omega,\psi}\pi_{\psi}(B)(\alpha_t)^*_{\omega,\psi} = \pi_\omega((\alpha_t)(B))$ holds for any $B \in B(\mathcal{H})$, and

$$\{\pi_\psi((\alpha_t)(B))x_{\psi;B}; B \in B(\mathcal{H})\}' = (\alpha_t)^*_{\omega,\psi}\{\pi_\omega((\alpha_t)(B))x_\omega; B \in B(\mathcal{H})\}'(\alpha_t)_{\omega,\psi} = CI,$$

where $I$ means the identity mapping on $\mathcal{H}_\psi$, and $A'$ means the commutant of an algebra $A$. These equalities imply that the cyclic representation $\{\mathcal{H}_\psi, \pi_\psi, x_\psi\}$ is irreducible, therefore, we obtain the conclusion.

References


