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One-sided phase constraint $x(t) \geq 0$ forms an envelope

相条件 $x(t) \geq 0$ は包絡線を生成する

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Introduction

In this paper, we are concerned with the following max-type function:

$$S(x) := \max_{t \in T} G(x(t), t), \quad x \in X,$$

where $T$ is a compact metric space, $X$ is a subspace of the set of all $n$-dimensional vector-valued continuous functions $C(T)^n$ equipped with the uniform norm. We denote by $G_x$ and $G_{xx}$ the gradient (row) vector and the Hesse matrix of $f$ w.r.t. $x$, respectively, and assume them to be continuous on $R^n \times T$. This max-type function is induced from a phase constraint

$$G(x(t), t) \leq 0 \quad \forall t \in T,$$

which appears in variational problems and optimal control problems. For instance, a variational problem to find the shortest path in $R^2$ joining two given points $P$ and $Q$ that does not transverse the unit ball is formulated as follows:

Minimize

$$\int_0^1 \sqrt{x_1^2 + x_2^2} dt$$

subject to

$$(x_1(0), x_2(0)) = P, \quad (x_1(1), x_2(1)) = Q,$$

$$1 - x_1^2(t) - x_2(t)^2 \leq 0 \quad \forall t \in [0, 1].$$

There are two aims in this paper. First one is to give formulae for first- and second-order directional derivatives of $S(x)$. Second one is to show that one-sided phase constraint $x(t) \geq a(t)$, where $a(t)$ is a given continuous function, always forms an envelope except two trivial cases:

$$x(t) \equiv a(t),$$

$$x(t) > a(t) \quad \text{for every } t.$$
By the way, there are a lot of papers that dealt with another max-type function:

\[ S_0(x) := \max_{t \in T} G(x, t) \quad x \in \mathbb{R}^n, \quad (2) \]

Clarke[1], Correa and Seeger[2], Danskin [3], Dem'yanov and Malozemov[4] Dem'yanov and Zabrodin[5], Hettich and Jongen[6], Ioffe[7], Kawasaki[8][9] [10][11][13], Shiraishi[17], Seeger[16], Wetterling[18]. We encounter this max-type function, for example, in Tchebycheff approximation. The latter max-type function \( S_0(x) \) is a special case of \( S(x) \). Indeed, if we take as \( X \) \( \{x(t) \equiv a | a \in \mathbb{R}^n \} \), then \( S(x) \) reduces to \( S_0(x) \). So \( S(x) \) inherits a lot of properties from \( S_0(x) \).

論文の概要

次の max-型関数の 1 次と 2 次の方向微分について考察する。

\[ S(x) := \max_{t \in T} G(x(t), t) \quad x \in X, \quad (3) \]

ただし \( T \) はコンパクト距離空間, \( X \) は \( n \) 次元ベクトル値連続関数全体 \( C(T)^n \) の部分空間とする。この max-型関数は変分問題や最適制御問題の相対条件

\[ G(x(t), t) \leq 0 \quad \forall t \in T \]

を考察するとき出会う。本論文では、この相条件から包絡線が生成されるかどうかを調べるために、max-型関数 \( S(x) \) の 2 次の方向微分を表す公式を与える。

ところで、従来よく研究されてきた max-型関数は次の関数である。

\[ S_0(x) := \max_{t \in T} G(x, t) \quad x \in \mathbb{R}^n, \quad (4) \]

この関数はチェビシェフ近似問題と密接に関係する。さらに、集合 \( T \) が \( x \) に依存してよいとすれば、\( S_0(x) \) の最小化問題はパラメトリック最適化問題になる。\( S(x) \) が \( S_0(x) \) と本質的に異なる点は、後者は \( x \) と \( t \) が独立に動けるのに対し、前者は \( x \) が \( t \) に依存することである。しかしながら、\( S_0(x) \) は \( S(x) \) のスペシャルケースを見なすこともできる。つまり、\( X \) として \( n \) 次元ベクトル値定数関数全体 \( \{x(t) \equiv a | a \in \mathbb{R}^n \} \) をとればよい。従って、\( S(x) \) は \( S_0(x) \) の多くの性質を受け継ぐことになる。その結果、相条件も包絡線を生成する。より正確に言えば、片側相制約 \( x(t) \geq a(t) \) について、二つの自明なケース:

\[ \overline{x}(t) \equiv a(t), \]

\[ \overline{x}(t) > a(t) \quad \text{for every } t. \]

を除いて、点 \( x \) において包絡線を生成する方向 \( y \) を選ぶ事が出来る。
In the following, we denote by $T(x)$ the set of all extreme points $G(x(\cdot), \cdot)$, that is,

$$T(x) := \{t \in T ; G(x(t), t) = S(x)\}, \quad x \in C(T)^n.$$  

**Theorem 1** The function $S(x)$ is continuous.

**Theorem 2** The function $S(x)$ is directionally differentiable in any direction $y \in X$, and its directional derivative is given by

$$S'(x; y) = \max\{G_x(x(t), t)y(t) ; t \in T(x)\}. \quad (5)$$

Taking constant functions as $x(t)$ and $y(t)$ in Theorem 2, we get Danskin's formula.

**Corollary 1** (Danskin[3]) The function $S_0(x)$ is directionally differentiable in any direction $y \in \mathbb{R}^n$ and its directional derivative is given by

$$S'_0(x; y) = \max\{G_x(x(t), t)y(t) ; t \in T(x)\}. \quad (6)$$

Next, we consider a second-order directional derivative of $S(x)$.

**Definition 1** The upper second-order directional derivative of $S(x)$ at $x$ in the direction $y$ is defined by

$$S''^u(x; y) = \lim_{\epsilon \to 0^+} \sup\left\{ \frac{S(x + \epsilon y) - S(x) - \epsilon S'(x, y)}{\epsilon^2} \right\} \quad (7)$$

**Definition 2** ([9]) For any functions $u, v \in C(T)$ satisfying

$$\{ u(t) \geq 0 \forall t \in T, \quad v(t) \geq 0 \text{ if } u(t) = 0 \}, \quad (8)$$

we define a function $E : T \to [-\infty, +\infty]$ by

$$E(t) := \begin{cases} 
\sup\left\{ \limsup_{n \to \infty} u(t_n)^2 \right\} & \text{if } u(t) = v(t) = 0 \text{ and } t \notin T_0, \\
0 & \text{if } u(t) = v(t) = 0 \text{ and } t \in T_0, \\
-\infty & \text{otherwise},
\end{cases} \quad (9)$$

$$T_0 := \left\{ t \in T ; \exists t_n \to t \text{ s.t. } u(t_n) > 0, \quad \frac{v(t_n)}{u(t_n)} \to +\infty \right\}. \quad (10)$$

**Theorem 3** Let $x$ and $y$ be arbitrary functions in $C(T)^n$. Then it holds that

$$S''^u(x; y) = \max\left\{ \frac{y(t)^T G_{xx}(x(t), t)y(t)}{2} + E(t) ; t \in T(x; y) \right\}, \quad (11)$$

where $T(x; y) := \{t \in T(x) ; S'(x; y) = G_x(x(t), t)y(t)\}$ and $E(t)$ is defined via Definition 2 by taking

$$u(t) = S(x) - G(x(t), t), \quad v(t) = S'(x; y) - G_x(x(t), t)y(t). \quad (12)$$
Taking constant functions as $x(t)$ and $y(t)$ in Theorem 3, we get the following formula due to [9].

**COROLLARY 2** Let $x$ and $y$ be arbitrary points in $\mathbb{R}^n$. Then it holds that

$$\overline{S}'(x; y) = \max \left\{ \frac{y^T G_{xx}(x, t)y}{2} + E(t) \ ; \ t \in T(x; y) \right\},$$

(13)

where $E(t)$ is defined via Definition 2 by taking

$$u(t) = S(x) - G(x, t), \quad v(t) = S'(x; y) - G_x(x, t)y.$$  

(14)

We proved in [9] [10] that an envelope is formed from $G(x, t)$ when $E(t) > 0$ at some point $t \in T(x; y)$.

**EXAMPLE 1** Let us consider a family of straight lines $f(x, t) = 2tx - t^2$, $t \in [0, 1]$, $x \in \mathbb{R}$. It is evident that it forms an envelope $S_0(x) = x^2$, $(0 \leq x \leq 1)$.

Hence $S_0''(0; y) = y^2$ for any $y \geq 0$ and $f_{xx}(0, t) \equiv 0$. Thus there is a gap between the second-order directional derivatives of the max-type function $S_0(x)$ and its constituent functions $f(x, t)$. On the other hand, it is directly computed from the definition that $E(0) = y^2$ for every $y > 0$, which fills the gap.

$$S_0'(0; y) = \max \left\{ \frac{1}{2} y^T f_{xx}(0, t)y + E(t) \ ; \ t \in T(0; y) \right\}$$

$$= E(0) = y^2$$

It is reasonable to guess that an envelope is formed from the phase constraint $x(t) \geq a(t)$ when the function $E(t)$ is positive at some $t \in T(x; y)$ as well as the max-type function $S_0(x)$. However this is not a proof but a guess. So we next give a proof that the one-sided phase constraint $x(t) \geq a(t)$ certainly forms an envelope for a certain direction $y(t)$ except two trivial cases.
**Theorem 4** Let $T$ be a connected compact metric space. Assume that $\overline{x}(t)$ does not satisfy neither

$$x(t) \equiv a(t),$$  
(15)

nor

$$x(t) > a(t) \text{ for every } t.$$  
(16)

Then there exists a function $y \in C(T)$ such that the one-sided phase constraint $x(t) \geq a(t)$ forms an envelope in the direction $y$.

**Proof.** Let $y(t) := -2\sqrt{x(t) - a(t)}$ and put for $\xi \in \mathbb{R}$

$$s(\xi) := S(\overline{x} + \xi y) = \max_{t} \{a(t) - \overline{x}(t) - \xi y(t)\}$$

$$= \max_{t} \{a(t) - \overline{x}(t) + 2\xi \sqrt{\overline{x}(t) - a(t)}\}$$

Then $s(\xi)$ becomes a standard max-function.

$$s(\xi) = \max_{\tau \in T} \{2\xi \tau - \tau^2\}$$

Furthermore, from the assumption, the image of $T$ by the continuous function $\sqrt{x(t) - a(t)}$ is a compact interval $T' := [0, t_1]$ with $t_1 > 0$. Hence $s(\xi)$ is same with Example 1, so that an envelope is formed.

**References**


