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Duality Theorems on an Infinite Network

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1 Introduction and Preliminaries

Let $N = \{X, Y, K\}$ be an infinite network which is locally finite and has no self-loops. Here X is the countable set of nodes, Y is the countable set of arcs, $K : X \times Y \longmapsto \{-1, 0, +1\}$ is the node-arc incidence matrix. Local finiteness means that $K(x, \cdot)$ has finite support in Y for every $x \in X$.

We denote by $\mathcal{X}$ the set of all real-valued functions on X, and by $\mathcal{X}^*$ the set of all real-valued functions on X with finite support. Likewise we denote by $\mathcal{Y}$ the set of all real-valued functions on Y, and by $\mathcal{Y}^*$ the set of all real-valued functions on Y with finite support. For each $w \in \mathcal{Y}$, the divergence $\partial w \in \mathcal{X}$ is defined as

$$\partial w(x) := \sum_{y\in Y} K(x, y)w(y).$$

For each $u \in \mathcal{X}$, the discrete derivative $du \in \mathcal{Y}$ is defined as

$$du(y) := \sum_{x\in X} K(x, y)u(x) = u(b(y)) - u(a(y)),$$

where $a(y)$ is the initial node and $b(y)$ is the terminal node of arc y. Clearly, if $w \in \mathcal{Y}^*$, then $\partial w \in \mathcal{X}^*$, and if $u \in \mathcal{X}^*$, then $du \in \mathcal{Y}^*$, since N is locally finite.

For $w_1, w_2 \in \mathcal{Y}$ with either w_1 or w_2 in $\mathcal{Y}^*$, we define the inner product

$$< w_1, w_2 >:= \sum_{y\in Y} w_1(y)w_2(y).$$

For $u, v \in \mathcal{X}$ with either u or v in $\mathcal{X}^*$, we define the inner product

$$((u, v)) := \sum_{x\in X} u(x)v(x).$$

Note that the fundamental formula

$$((u, \partial w)) =< du, w >$$

holds if $u \in \mathcal{X}^*$ or $w \in \mathcal{Y}^*$.

The space $\mathcal{X}$ can be identified with the product space $\mathbf{R}^X$, and therefore can be given the product topology of $\mathbf{R}^X$. As usual, we call this the weak topology on $\mathcal{X}$. It is the topology of pointwise convergence, i.e., a sequence $\{\xi_\nu\}$ in $\mathcal{X}$ converges weakly to some $\xi \in \mathcal{X}$ if and only if $\xi_\nu(x) \to \xi(x)$ for all $x \in X$. If $\mathcal{X}$ is given the weak topology, then $\mathcal{X}^*$ becomes the topological dual of $\mathcal{X}$, which means that the continuous linear functionals on $\mathcal{X}$ are precisely those of the form $< u, \cdot >$ with $u \in \mathcal{X}^*$. Henceforth, without exception, $\mathcal{X}$ will bear the weak topology. Likewise $\mathcal{Y}$ will always bear the weak topology, so that $\mathcal{Y}^*$ becomes the topological dual of $\mathcal{Y}$. We observe that the mappings $w \longmapsto \partial w$ and $u \longmapsto du$ are continuous, if $\mathcal{X}$ and $\mathcal{Y}$ carry the weak topology. This follows from the fact that $K(x, \cdot)$ and $K(\cdot, y)$ have finite support.

2 Weak Duality

Let $F, G : \mathcal{Y} \longmapsto \mathbf{R} \cup \{+\infty\}$ be two convex, weakly lower semicontinuous functions which are mutually conjugate in the following sense:

For every $w_1 \in \mathcal{Y}^*$,

$$G(w_1) = \sup\{< w_1, w > -F(w); w \in \mathcal{Y}\}, \tag{2.1}$$

and for every $w_2 \in \mathcal{Y}^*$,

$$F(w_2) = \sup\{< w, w_2 > -G(w); w \in \mathcal{Y}\}. \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$< w_1, w_2 > \leq G(w_1) + F(w_2) \tag{2.3}$$

for all w_1, w_2 in $\mathcal{Y}$ with either w_1 or w_2 in $\mathcal{Y}^*$.

Now let X_1 and X_2 be two disjoint subsets X such that $X = X_1 \cup X_2$. Let $f_1, f_2 \in \mathcal{X}$ be given such that the support of f_1 is contained in X_1 and the support of f_2 is contained in X_2. In order to introduce dual pairs of optimization problems on the network N we define a primal objective function $E : \mathcal{Y} \longmapsto \mathbf{R} \cup \{+\infty\}$ as

$$E(w) := F(w) - ((f_1, \partial w)) \text{ for all } w \in \mathcal{Y},$$

and we define a dual objective function $E^* : \mathcal{X} \longmapsto \mathbf{R} \cup \{+\infty\}$ as

$$E^*(u) := -G(du) + ((u, f_2)) \text{ for all } u \in \mathcal{X}.$$

In order to make E well-defined we shall employ the following hypothesis:

$(E.1)$ $$f_1 \in \mathcal{X}^*.$$

In order to make E^* well-defined we shall employ the following hypothesis:

$$(E.2) \qquad f_2 \in \mathcal{X}^*.$$

However, if E is restricted to $\mathcal{Y}^*$, then (E.1) is not needed, and if E^* is restricted to $\mathcal{X}^*$, then (E.2) is not needed. The functions E and $-E^*$ are convex and weakly lower semicontinuous, with values in $\mathbf{R} \cup \{+\infty\}$.

If $w \in \mathcal{Y}$ is a flow on the arcs $y \in Y$, then $F(w)$ may be considered as a generalized energy of w. And if $u \in \mathcal{X}$ is a potential on the nodes $x \in X$, then $G(du)$ may be considered as a generalized Dirichlet sum of u.

We consider two pairs of optimization problems as follows:
To the primal problem

$$(P) \qquad \inf\{E(w); w \in \mathcal{Y}, \partial w(x) = f_2(x) \text{ on } X_2\}$$

we associate the dual problem

$$(D_0) \qquad \sup\{E^*(u); u \in \mathcal{X}^*, u(x) = f_1(x) \text{ on } X_1\}.$$

And to the primal problem

$$(P_0) \qquad \inf\{E(w); w \in \mathcal{Y}^*, \partial w(x) = f_2(x) \text{ on } X_2\}$$

we associate the dual problem

$$(D) \qquad \sup\{E^*(u); u \in \mathcal{X}, u(x) = f_1(x) \text{ on } X_1\}.$$

We adopt the convention that the infimum over the empty set equals $+\infty$, and the supremum over the empty set equals $-\infty$. Obviously the only difference between (P) and (P_0) and between (D) and (D_0) consists in the underlying spaces. In case N is a finite network, a similar problem was treated in [1], p. 162.

Henceforth we denote by $V(P), V(D_0), V(P_0), V(D)$ the optimal values of the problems $(P), (D_0), (P_0), (D)$ respectively. We shall study duality relations between (P) and (D_0) and between (P_0) and (D), and describe an application of our results to the potential theory on locally finite networks.

We have the following weak duality result:

Theorem 2. 1 *(1) Assume that (E.1) holds. Then* $V(P) \geq V(D_0)$.
(2) Assume that (E.2) holds. Then $V(P_0) \geq V(D)$.

Proof. (1) The claim is obviously true, if (P) or (D_0) have no feasible solutions. So let w and u be feasible solutions for (P) and (D_0) respectively. Then

$$\begin{aligned} E(w) - E^*(u) &= F(w) + G(du) - ((f_1, \partial w)) - ((u, f_2)) \\ &= F(w) + G(du) - ((u, \partial w)) \\ &= F(w) + G(du) - < du, w > \\ &\geq 0, \end{aligned}$$

from (2.3), since $u \in \mathcal{X}^*$. Thus $E(w) \geq E^*(u)$ for all feasible w and u, which implies $V(P) \geq V(D_0)$. The proof of (2) is similar. □

From (E.1) it follows that problem (D_0) has a feasible solution, i.e., there exists $u \in \mathcal{X}^*$ such that $u = f_1$ on X_1. Likewise we have

Proposition 2. 1 *Assume that (E.2) holds and that $X_1 \neq \emptyset$. Then problem (P_0) has a feasible solution, i.e., there exists $w \in \mathcal{Y}^*$ such that $\partial w(x) = f_2(x)$ on X_2.*

Proof. Fix $x_0 \in X_1$. For every $a \in X_2$ select a finite path $p_a \in \mathcal{Y}^*$ from x_0 to a, i.e., p_a is the path index of a path from x_0 to a (cf. [6]). Then p_a is a unit flow from x_0 to a, i.e., $\partial p_a(a) = +1$, $\partial p_a(x_0) = -1$ and $\partial p_a(x) = 0$ for all other x. Let us consider

$$w(y) := \sum\nolimits_{a \in X_2} f_2(a) p_a(y).$$

Then $w(y)$ is well-defined, since f_2 has finite support in X_2, and it is easily seen that w has the requested properties. □

For later use we denote by ε_A the characteristic function of a subset $A \subset X$, i.e., $\varepsilon_A(x) = 1$ for $x \in A$ and $\varepsilon_A(x) = 0$ for $x \in X \setminus A$.

3 A General Duality Theorem

Our main tool will be a general duality result studied in [5](cf. [4]). We prepare it below for the sake of completeness.

Let $\mathcal{U}$ be a real vector space, let $\mathcal{Z}$ be a locally convex topological vector space, and let $\mathcal{W}$ be the topological dual of $\mathcal{Z}$. Let $\varphi : \mathcal{U} \to \mathbf{R} \cup \{+\infty\}$ and $\psi : \mathcal{Z} \to \mathbf{R} \cup \{-\infty\}$ be given. Let C be a nonempty subset of $\mathcal{U}$ and Q be a nonempty subset of $\mathcal{Z}$. Let T be a transformation from $\mathcal{U}$ into $\mathcal{Z}$.

Let us consider the following general extremum problem (V) and its dual problem (V*):

$$(V) \qquad V := \inf\{\varphi(\xi) - \psi(T\xi); \xi \in C,\ T\xi \in Q\},$$

$$(V^*) \qquad V^* := \sup\{\psi^*(\zeta) - \varphi_T^*(\zeta); \zeta \in \mathcal{W}\},$$

where

$$\begin{aligned} \psi^*(\zeta) &:= \inf\{\zeta(\eta) - \psi(\eta); \eta \in Q\}, \\ \varphi_T^*(\zeta) &:= \sup\{\zeta(T\xi) - \varphi(\xi); \xi \in C\}. \end{aligned}$$

It is always true that $V \geq V^*$. We have by [5]

Theorem 3. 1 *Assume that the set*

$$\mathcal{E} := \{(z,s) \in \mathcal{Z} \times \mathbf{R}; z = \eta - T\xi, s \geq \varphi(\xi) - \psi(\eta), \xi \in C, \eta \in Q\}$$

is convex and closed in $\mathcal{Z} \times \mathbf{R}$. *If* V *is finite, then* $V = V^*$ *holds and there exists* $\xi \in C$ *such that* $T\xi \in Q$ *and* $V = \varphi(\xi) - \psi(T\xi)$.

Proof. Clearly, $V = \inf\{s; (0,s) \in \mathcal{E}\}$. Let V be finite. Then $(0,V) \in \mathcal{E}$, since $\mathcal{E}$ is closed, and this gives the existence of $\xi \in C$ with the claimed property. In order to prove $V \leq V^*$, let $t < V$. Then $(0,t) \notin \mathcal{E}$. Hence from the strong separation theorem there exists $(\zeta, \tau) \in \mathcal{W} \times \mathbf{R}$ such that

$$\zeta(0) + \tau t < \zeta(z) + \tau s \quad \forall (z,s) \in \mathcal{E}. \tag{3.1}$$

Since $(0, V + r) \in \mathcal{E}$ for all $r \geq 0$, we obtain from (3.1) that $\tau > 0$. Dividing (3.1) by τ and rewriting ζ/τ as ζ, we obtain

$$t \leq \zeta(z) + s \quad \forall (z,s) \in \mathcal{E},$$

hence in particular

$$t \leq \zeta(\eta - T\xi) + \varphi(\xi) - \psi(\eta)$$

for all $\xi \in C$, $\eta \in Q$, and therefore $t \leq \psi^*(\zeta) - \varphi_T^*(\eta) \leq V^*$. Since $t < V$ was arbitrary, we obtain $V \leq V^*$. □

4 Duality between (P) and (D_0)

We are going to derive the strong duality relation $V(P) = V(D_0)$ from Theorem 3.1. We assume (E.1) and specify the data of Theorem 3.1 as follows:
$\mathcal{U} := \mathcal{Y}$, $\mathcal{Z} := \mathcal{X}$, $\mathcal{W} := \mathcal{X}^*$; $C := \mathcal{Y}$, $Q := \{\eta \in \mathcal{X}; \eta = f_2 \text{ on } X_2\}$;

$$T\xi := \partial\xi, \quad \varphi(\xi) := F(\xi), \quad \psi(\eta) := ((f_1, \eta)), \quad \zeta(\eta) := ((\eta, \zeta))$$

for all $\xi \in \mathcal{Y}, \eta \in \mathcal{X}, \zeta \in \mathcal{X}^*$. Then we have for all $\xi \in \mathcal{Y}$

$$\varphi(\xi) - \psi(T\xi) = F(\xi) - ((f_1, \partial\xi)) = E(\xi).$$

Therefore $V = V(P)$. For all $\zeta \in \mathcal{X}^*$ we have

$$\begin{aligned}
\varphi_T^*(\zeta) &= \sup\{((\zeta, \partial\xi)) - F(\xi); \xi \in C\} \\
&= \sup\{< d\zeta, \xi > -F(\xi); \xi \in \mathcal{Y}\} = G(d\zeta), \\
\psi^*(\zeta) &= \inf\{((\zeta - f_1, \eta)); \eta \in Q\} \\
&= \inf\{((\zeta - f_1, \eta\varepsilon_{X_2} + \eta\varepsilon_{X_1})); \eta \in Q\} \\
&= ((\zeta, f_2)) + \inf\{((\zeta - f_1, \eta\varepsilon_{X_1})); \eta \in Q\}.
\end{aligned}$$

Therefore $\psi^*(\zeta) = ((\zeta, f_2))$ if $\zeta - f_1 = 0$ on X_1, and $\psi^*(\zeta) = -\infty$ otherwise. Thus $V^* = V(D_0)$.

In order to apply Theorem 3.1 we need another hypothesis:

$$(H.1) \qquad \text{The level sets} \quad \{\xi \in \mathcal{Y}; F(\xi) - < w, \xi > \leq \alpha\} \quad (\alpha \in \mathbf{R})$$

are weakly compact in $\mathcal{Y}$ for all $w \in \mathcal{Y}^*$.

Theorem 4. 1 *Assume that (E.1) holds, that $V(P)$ is finite and that (H.1) is satisfied. Then $V(P) = V(D_0)$ and problem (P) has an optimal solution.*

Proof. The result follows from Theorem 3.1. We only have to show that the convex set

$$\mathcal{E} = \{(z, s) \in \mathcal{X} \times \mathbf{R}; z = \eta - \partial\xi, s \geq \varphi(\xi) - \psi(\eta), \xi \in C, \eta \in Q\}$$

is closed in $\mathcal{X} \times \mathbf{R}$, where $\mathcal{X}$ bears the weak topology. Since the set X of nodes is countable, $\mathcal{X}$ is a metrizable space under the weak topology (cf. [2], p. 32). Therefore the weak closedness in $\mathcal{X}$ means the sequential weak closedness (cf. [2], p. 20). Thus we have to show that $\mathcal{E}$ is sequentially closed. Let $\{(z_n, s_n)\}$ be a sequence in $\mathcal{E}$ such that $z_n \to \bar{z}$ pointwise and $s_n \to \bar{s}$ in $\mathbf{R}$. There exist $\xi_n \in C$ and $\eta_n \in Q$ such that $z_n = \eta_n - \partial\xi_n$, $s_n \geq F(\xi_n) - ((f_1, \eta_n))$. Then

$$\begin{aligned} s_n &\geq F(\xi_n) - ((f_1, \partial\xi_n + z_n)) \\ &= F(\xi_n) - < df_1, \xi_n > -((f_1, z_n)). \end{aligned}$$

Because of (E.1), $\{((f_1, z_n))\}$ converges to $((f_1, \bar{z}))$. Thus the sequence $\{((f_1, z_n))\}$ remains bounded. Since $\{s_n\}$ is also bounded, we see that the sequence $\{F(\xi_n) - < df_1, \xi_n >\}$ is bounded from above. Thus, because of (H.1), all ξ_n are contained in a weakly compact subset of $\mathcal{Y}$. Since the set Y of arcs is countable, $\mathcal{Y}$ is metrizable under the weak topology. Hence the weak compactness of a closed set in $\mathcal{Y}$ means the sequential weak compactness (cf. [2], p. 21). So, by choosing a subsequence if necessary, we may assume that $\{\xi_n\}$ converges pointwise to some $\bar{\xi} \in C$. Then $\partial\xi_n \to \partial\bar{\xi}$ pointwise, and $\eta_n = \partial\xi_n + z_n \to \bar{\eta} = \partial\bar{\xi} + \bar{z} \in Q$ pointwise. Thus $\bar{z} = \bar{\eta} - \partial\bar{\xi}$ and $\bar{s} \geq F(\bar{\xi}) - ((f_1, \bar{\eta}))$, since F is weakly lower semicontinuous. Thus $(\bar{z}, \bar{s}) \in \mathcal{E}$, and $\mathcal{E}$ is closed. □

5 Duality between (P_0) and (D)

Now we are going to derive the duality relation $V(P_0) = V(D)$. We assume (E.2) and specify the data of Theorem 3.1 as follows:

$\mathcal{U} := \mathcal{X}$, $\mathcal{Z} := \mathcal{Y}$, $\mathcal{W} := \mathcal{Y}^*$; $C := \{\xi \in \mathcal{X}; \xi = f_1 \text{ on } X_1\}$, $Q := \mathcal{Y}$;

$$T\xi := d\xi, \; \psi(\eta) := -G(\eta), \; \varphi(\xi) := -((\xi, f_2)), \; \zeta(\eta) := - < \eta, \zeta >$$

for all $\xi \in \mathcal{X}, \eta \in \mathcal{Y}, \zeta \in \mathcal{Y}^*$.

Then for all $\xi \in \mathcal{X}$ there holds

$$\varphi(\xi) - \psi(T\xi) = -((\xi, f_2)) + G(d\xi) = -E^*(\xi).$$

Therefore $V = -V(D)$. For all $\zeta \in \mathcal{Y}^*$ there holds

$$\begin{aligned} \psi^*(\zeta) &= \inf\{- < \eta, \zeta > + G(\eta); \eta \in Q\} = -F(\zeta), \\ \varphi_T^*(\zeta) &= \sup\{- < d\xi, \zeta > + ((\xi, f_2)); \xi \in C\} \\ &= \sup\{((\xi, -\partial\zeta + f_2)); \xi \in C\} \\ &= \sup\{((\xi\varepsilon_{X_2}, -\partial\zeta + f_2)); \xi \in C\} - ((f_1, \partial\zeta)). \end{aligned}$$

Therefore $\varphi_T^*(\zeta) = -((f_1, \partial\zeta))$ if $-\partial\zeta + f_2 = 0$ on X_2, and $\varphi_T^*(\zeta) = +\infty$ otherwise. Hence $\psi^*(\zeta) - \varphi_T^*(\zeta) = -E(\zeta)$ for all $\zeta \in \mathcal{Y}^*$ which are feasible for (P_0), and $\psi^*(\zeta) - \varphi_T^*(\zeta) = -\infty$ otherwise. Thus $V^* = -V(P_0)$.

We prepare

Proposition 5. 1 *Let $\{\xi_n\} \subset \mathcal{X}$, and let $a \in X$. If $\{d\xi_n\}$ converges pointwise and if $\{\xi_n(a)\}$ converges, then $\{\xi_n\}$ converges pointwise to some $\xi \in \mathcal{X}$.*

Proof. For every $x \in X$ select a finite path $p_x \in \mathcal{Y}^*$ from a to x. Then

$$< d\xi_n, p_x > = ((\xi_n, \partial p_x)) = \xi_n(x) - \xi_n(a).$$

Since $\{< d\xi_n, p_x >\}$ converges and $\{\xi_n(a)\}$ converges, $\{\xi_n(x)\}$ converges, too. Since this holds for every $x \in X$, $\{\xi_n\}$ converges pointwise to some $\xi \in \mathcal{X}$. □

We further introduce the following hypothesis:

$(H.2)$ The level sets $\{\eta \in \mathcal{Y}; G(\eta) - < \eta, w > \leq \alpha\}$ $(\alpha \in \mathbf{R})$

are weakly compact in $\mathcal{Y}$ for all $w \in \mathcal{Y}^*$.

Theorem 5. 1 *Assume that (E.2) holds, that $V(D)$ is finite, that $X_1 \neq \emptyset$, and that (H.2) is satisfied. Then $V(P_0) = V(D)$ and problem (D) has an optimal solution.*

Proof. This follows from Theorem 3.1. As in the proof of Theorem 4.1, we shall show that the convex set

$$\mathcal{E} = \{(z, s) \in \mathcal{Y} \times \mathbf{R}; z = \eta - d\xi, s \geq \varphi(\xi) - \psi(\eta), \xi \in C, \eta \in Q\}$$

is sequencially weakly closed in $\mathcal{Y} \times \mathbf{R}$. Let $\{(z_n, s_n)\}$ be a sequence in $\mathcal{E}$ such that $z_n \to \bar{z}$ pointwise, and $s_n \to \bar{s}$. There exist $\xi_n \in C$ and $\eta_n \in Q$ such that

$$z_n = \eta_n - d\xi_n, \; s_n \geq -((\xi_n, f_2)) + G(\eta_n).$$

By Proposition 2.1, there exists $w \in \mathcal{Y}^*$ such that $\partial w = f_2$ on X_2. From $\xi_n \in C$ we obtain then

$$\begin{aligned} ((\xi_n, f_2)) &= ((\xi_n, \partial w)) - ((f_1, \partial w)) \\ &= < d\xi_n, w > -((f_1, \partial w)) \\ &= < \eta_n - z_n, w > -((f_1, \partial w)). \end{aligned}$$

Thus

$$s_n \geq < z_n, w > +((f_1, \partial w)) - < \eta_n, w > +G(\eta_n).$$

Since $\{< z_n, w >\}$ converges to $< \bar{z}, w >$, we see that the sequence $\{- < \eta_n, w > +G(\eta_n)\}$ is bounded from above. Using hypothesis (H.2), by the same reasoing as in the proof of Theorem 4.1, we may assume that $\{\eta_n\}$ converges pointwise to some $\bar{\eta} \in \mathcal{Y}$. Then $\{d\xi_n\}$ converges also pointwise to $\bar{\eta} - \bar{z}$. Since $X_1 \neq \emptyset$ and $\xi_n \in C$, we see that $\xi_n(a) = f_1(a)$ for some $a \in X_1$. From Proposition 5.1 it follows that $\{\xi_n\}$ converges poitwise to some $\bar{\xi} \in C$. Then $\{d\xi_n\}$ converges pointwise to $d\bar{\xi}$, so that $d\bar{\xi} = \bar{\eta} - \bar{z}$. Altogether we obtain that

$$\bar{z} = \bar{\eta} - d\bar{\xi}, \ \bar{s} \geq -((\bar{\xi}, f_2)) + G(\bar{\eta}),$$

since G is weakly lower semicontinuous. Thus $(\bar{z}, \bar{s}) \in \mathcal{E}$, and $\mathcal{E}$ is closed. □

6 Applications

As applications of our duality results, we obtain generalizations of some fundamental inverse relations from [3] and [6] which play important roles in the discrete potential theory (cf. [7]).

We let F and G be as before. In addition we assume that F and G are nonnegative and symmetric, and that G is homogeneous of degree $q > 1$ and G is homogeneous of degree $p > 1$, with $1/p + 1/q = 1$.

In connection with problems (P_0) and (D) we choose $f_2 = 0$ (so that (E.2) holds), and we assume that $f_1 \neq 0$ (so that $X_1 \neq \emptyset$). For all $\eta \in \mathcal{Y}^*$ we let $I(\eta) := ((f_1, \partial\eta))$. We define

$$\begin{aligned} \beta &:= \inf\{pG(du); u \in \mathcal{X}, \ u = f_1 \text{ on } X_1\} \\ \alpha_0 &:= \inf\{qE(\eta); \eta \in \mathcal{Y}^*, \ \partial\eta = 0 \text{ on } X_2, \ I(\eta) = 1\}. \end{aligned}$$

It is obvious that $\beta \geq 0, \ \alpha_0 \geq 0$, and

$$V(D) = \frac{-1}{p}\beta.$$

Moreover we have

$$\begin{aligned} V(P_0) &= \inf\{F(w) - I(w); w \in \mathcal{Y}^*,\ \partial w = 0 \text{ on } X_2\} \\ &= \inf\{\inf\{|\,t\,|^q F(\eta) - t; \eta \in \mathcal{Y}^*, \partial\eta = 0 \text{ on } X_2,\ I(\eta) = 1\}; t \in \mathbf{R}\} \\ &= \inf\{\frac{|\,t\,|^q \alpha_0}{q} - t; t \in \mathbf{R}\} \\ &= -\frac{1}{p}\alpha_0^{-p/q}. \end{aligned}$$

So, if $V(D)$ is finite and $\neq 0$, the duality relation $V(P_0) = V(D)$ takes the form

$$\beta^{1/p}\alpha_0^{1/q} = 1.$$

From Theorem 5.1 we obtain therefore

Corollary 6. 1 *Assume that β is finite and $\neq$ 0, and that (H.2) is satisfied. Then* $\beta^{1/p}\alpha_0^{1/q} = 1$.

On the other hand, if we define

$$\begin{aligned} \beta_0 &:= \inf\{pG(du); u \in \mathcal{X}^*,\ u = f_1 \text{ on } X_1\} \\ \alpha &:= \inf\{qE(\eta); \eta \in \mathcal{Y},\ \partial\eta = 0 \text{ on } X_2,\ I(\eta) = 1\}. \end{aligned}$$

then $V(D_0) = -\beta_0/p$ and $V(P) = -\alpha^{-p/q}/p$. We obtain from Theorem 4.1

Corollary 6. 2 *Assume that (E.1) and (H.1) are satisfied, and that α is finite and $\neq$ 0. Then* $\beta_0^{1/p}\alpha^{1/q} = 1$.

From Corollary 6.1 we can obtain Theorem 5.1 in [3]. To be more specific, assume that A is an arbitrary subset of X, and B is a nonempty subset of X which is disjoint with A. Let us take

$$X_1 := A \cup B,\ X_2 := X \setminus (A \cup B),\ f_1 := \varepsilon_B,\ f_2 = 0.$$

Then

$$I(\eta) = \textstyle\sum_{x\in B} \partial\eta(x).$$

In case $\partial\eta = 0$ on X_2, $I(\eta)$ is called the strength of η on B. Let, as in [3],

$$\begin{aligned} d_p(A,B) &:= \inf\{pG(du); u \in \mathcal{X},\ u = 0 \text{ on } A,\ u = 1 \text{ on } B\} = \beta \\ d^*_{q,0}(A,B) &:= \inf\{qF(\eta); \eta \in \mathcal{Y}^*, \partial\eta = 0 \text{ on } X \setminus (A \cup B),\ I(\eta) = 1\} = \alpha_0. \end{aligned}$$

Notice that Corollary 6.1 gives a sufficient condition for the validity of the inverse relation

$$(d_p(A,B))^{1/p} \cdot (d^*_{q,0}(A,B))^{1/q} = 1.$$

Observe that from $\eta \in \mathcal{Y}^*$ and $\partial\eta = 0$ on $X \setminus (A \cup B)$ it follows that

$$\sum_{x \in B} \partial\eta(x) = -\sum_{x \in A} \partial\eta(x).$$

Remark 6.1 Let $r \in \mathcal{Y}$ be strictly positive and take F as

$$F(w) := \frac{1}{q} \sum_{y \in Y} r(y) |w(y)|^q.$$

Then we have

$$G(w) = \frac{1}{p} \sum_{y \in Y} r(y)^{1-p} |w(y)|^p.$$

Notice that $pG(du) = D_p(u)$(Dirichlet sum of u of order p) and $qF(w) = H_q(w)$(the energy of w of order q) (cf. [3]). We see that F satisfies (H.1) and that G satisfies (H.2).

References

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