Introducing a metric on the space of fuzzy continuous mappings and the completeness of the space

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Introducing a metric on the space of fuzzy continuous mappings

and the completeness of the space

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We consider the space of the mappings which take their values in the set of fuzzy numbers, and introduce a metric on the space. We prove that the space constitutes a complete space under the metric.

A fuzzy number we treat in this paper is as follows.

Definition 1. A fuzzy number is a fuzzy set with a membership function \( \mu : \mathbb{R} \rightarrow [0,1] \) satisfying the following conditions:

1. there are real numbers \( a \) and \( b \) such that \( \text{cl} \{ t \in \mathbb{R} | \mu(t) > 0 \} = [a, b] \),
2. there exists a unique real number \( m(a \leq m \leq b) \) such that \( \mu(m) = 1 \),
3. \( \mu(t) \) is upper semi-continuous on \([a, b]\),
4. \( \mu(t) \) is nondecreasing on \([a, m]\) and nonincreasing on \([m, b]\).

The set of all fuzzy numbers is denoted by \( \mathbb{F}(\mathbb{R}) \). Let \( \rho \) denote the Hausdorff distance among bounded closed intervals in \( \mathbb{R} \). We introduce a distance on \( \mathbb{F}(\mathbb{R}) \) by the following:

Definition 2. For two fuzzy numbers \( \tilde{a} \) and \( \tilde{b} \) in \( \mathbb{F}(\mathbb{R}) \), the distance \( d(\tilde{a}, \tilde{b}) \) between \( \tilde{a} \) and \( \tilde{b} \) is defined by

\[
d(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \rho(\tilde{a}_\alpha, \tilde{b}_\alpha),
\]

where \( \tilde{a}_\alpha \) and \( \tilde{b}_\alpha \) denote the \( \alpha \)-cuts of \( \tilde{a} \) and \( \tilde{b} \), respectively.

Definition 3. For \( \epsilon > 0 \) and \( \tilde{a} \in \mathbb{F}(\mathbb{R}) \), two kinds of \( \epsilon \)-neighborhoods of \( \tilde{a} \) are defined by

\[
\mathbb{B}(\tilde{a}; \epsilon) = \{ \tilde{b} \in \mathbb{F}(\mathbb{R}) | d(\tilde{a}, \tilde{b}) < \epsilon \},
\]

\[
\overline{\mathbb{B}}(\tilde{a}; \epsilon) = \{ \tilde{b} \in \mathbb{F}(\mathbb{R}) | d(\tilde{a}, \tilde{b}) \leq \epsilon \}.
\]

Definition 4. Let \( \tilde{a} \) and \( \tilde{b} \) be two fuzzy numbers. Then

\( \tilde{a} \preceq \tilde{b} \) iff \( (\sup \tilde{a}_\alpha \leq \sup \tilde{b}_\alpha) \) and \( (\inf \tilde{a}_\alpha \leq \inf \tilde{b}_\alpha) \) for \( \forall \alpha \in [0,1] \),

and

\( \tilde{a} \prec \tilde{b} \) iff \( (\sup \tilde{a}_\alpha < \sup \tilde{b}_\alpha) \) and \( (\inf \tilde{a}_\alpha < \inf \tilde{b}_\alpha) \) for \( \forall \alpha \in [0,1] \).
Proposition 1. For $\epsilon > 0$ and $\bar{a} \in \mathbb{F}(\mathbb{R})$, it holds that

(i) $\bar{b} \in \overline{B}(\bar{a}; \epsilon) \Leftrightarrow \bar{a} - \epsilon \preceq \bar{b} \preceq \bar{a} + \epsilon$,
(ii) $\bar{b} \in B(\bar{a}; \epsilon) \Rightarrow \bar{a} - \epsilon \prec \bar{b} \prec \bar{a} + \epsilon$.

The condition (iv) in Definition 1 is sometimes exchanged by the following:

(iv)$'$ $\mu(t)$ is strictly increasing on $[a, m]$ and strictly decreasing on $[m, b]$.

Denote the set of all fuzzy sets satisfying (i), (ii), (iii) in Definition 1 and (iv)$'$ by $\mathbb{F}'(\mathbb{R})$. For $\bar{a} \in \mathbb{F}'(\mathbb{R})$, let

$B'(\bar{a}; \epsilon) = \{\bar{b} \in \mathbb{F}'(\mathbb{R}) \mid d(\bar{a}, \bar{b}) < \epsilon\}$.

Proposition 2. For $\epsilon > 0$ and $\bar{a} \in \mathbb{F}'(\mathbb{R})$, it holds that

$\bar{b} \in B'(\bar{a}; \epsilon) \Leftrightarrow \bar{a} - \epsilon \prec \bar{b} \prec \bar{a} + \epsilon$.

Proposition 3. For $\bar{a} \in \mathbb{F}(\mathbb{R})$, let

$i(\alpha) = \inf \bar{a}_\alpha, \quad s(\alpha) = \sup \bar{a}_\alpha, \quad \alpha \in [0, 1]$.

Then $i(\alpha)$ and $s(\alpha)$ are lower semi-continuous and upper semi-continuous on $[0, 1]$, respectively.

Proposition 4. Let $X$ be a metric space. Let $f_n (n = 1, 2, \cdots)$ be a real-valued function defined on $X$. Suppose that the sequence $\{f_n\}$ converges uniformly to a function $f$ defined on $X$. If, for each $n$, $f_n$ is lower (resp. upper) semi-continuous on $X$, then $f$ is lower (resp. upper) semi-continuous on $X$.

Theorem 1. $$(\mathbb{F}(\mathbb{R}), d)$$ is a complete metric space.

Definition 5. Let $X$ be a metric space, and let $\tilde{f}$ a mapping from $X$ to $\mathbb{F}(\mathbb{R})$. Let $x$ be a point of $X$. Then, $\tilde{f}$ is said to be continuous at $x$, iff for every $\epsilon > 0$, there exists a positive number $\delta = \delta(x)$ satisfying that

$y \in S(x; \delta) \Rightarrow \tilde{f}(y) \in B(\tilde{f}(x); \epsilon)$.

If $\tilde{f}$ is continuous at every $x$ in $X$, then $\tilde{f}$ is said to be continuous on $X$.

Proposition 5. Every continuous mapping from a compact metric space $X$ to $\mathbb{F}(\mathbb{R})$ is uniformly continuous on $X$. 
Definition 6. Let $X$ be a metric space. Denote the class of all continuous mappings from $X$ to $\mathbb{F}(\mathbb{R})$ by $\text{CF}[X]$. For two members $\tilde{f}$ and $\tilde{g}$ in $\text{CF}[X]$, define the distance between $\tilde{f}$ and $\tilde{g}$ by

$$\delta(\tilde{f}, \tilde{g}) = \sup_{x \in X} d(\tilde{f}(x), \tilde{g}(x)).$$

Proposition 3. Let $X$ be a compact metric space. Then, for every pair $(\tilde{f}, \tilde{g})$ of fuzzy mappings in $\text{CF}[X]$, $\delta(\tilde{f}, \tilde{g})$ assumes a finite value and is represented by

$$\delta(\tilde{f}, \tilde{g}) = \max_{x \in X} d(\tilde{f}(x), \tilde{g}(x)).$$

Theorem 2. Let $X$ be a compact metric space. Then $(\text{CF}[X], \delta)$ is a complete metric space.

References

