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Optimization Games on Graphs *

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Abstract

We introduce a general integer programming formulation for a class of combinatorial optimization games, which include many interesting problems on graphs. The formulation immediately allows us to improve the algorithmic result for finding imputations in the core (an important solution concept in cooperative game theory) of the network flow game on unit networks. An important result is a general theorem that the core for this class of games is nonempty if and only if a related linear program has an integer optimal solution. We study the properties for this mathematical condition to hold for several problems on graphs, and apply them to resolve algorithmic and complexity issues for their cores: decide whether the core is empty; if the core is empty, find an imputation in the core; given an imputation x, test whether x is in the core.

1 Introduction

Game theory has a profound influence on methodologies of many different branches of sciences, especially those of economics, operations research and management sciences. The thesis of bounded rationality is introduced as a crucial concept for game theoretical strategies to have practically meanful implementations in real life situations [15, 17]. In particular, computational complexity (existence of polynomial time algorithms) has been singled out as one important factor for the bounded rationality of the participating agents in a game, and several authors have taken algorithmic and complexity issues as the main focus in solutions for game theory problems [1, 4, 8, 9, 12, 13, 15].

Recently, Kalai has extensively discussed the interplays of operations research, game theories, and theoretical computer science [8]. An interesting problem discussed by Kalai is a network flow game. Consider a digraph $D = (V, E)$ with a source vertex s and a sink vertex t. Kalai and Zemel [8, 10, 11] consider a cooperative game associated with the maximum flow from s to t by identifying each arc as a player, and defining the value $v(S)$ of a subset (i.e., a coalition) $S \subseteq E$ to be the value of a maximum flow in the subgraph $D[S] = (V, S)$. Call a mapping $z : E \rightarrow R_+$ (R_+ is the set of nonnegative reals) as an *imputation* if $z(V) = v(V)$ holds, where $z(S)$ for $S \subseteq V$ represents $\sum_{u \in S} z(u)$. The *core* is defined to be the set of imputations z such that $z(S) \geq v(S)$ holds for all $S \subseteq E$ [16, 18]. The maximum flow game distributes the profit

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from the maximum flow to players who control the arcs in the network. In practice, this can be regarded, for example, as a measure of reliability for maintaining a communication network between s and t. An imputation is a way to distribute the credit for a subset of arcs toward the level of connectivity maintained by them. The concept of core provides us a fundamental principle for such imputation to be rational, as it says that any subgroup of players would acquire at least as much payment as they can collectively obtain as a subgroup.

For a special case of the maximum flow game on unit networks, i.e., those with capacity ones, Kalai and Zemel [11] showed a convex characterization theorem of the core: An imputation is in the core if and only if it is a convex combination of the characteristic vectors of minimum cuts. This immediately implies that the core is always nonempty and leads to polynomial time algorithms for those problems related to the core; that is, one can find an imputation in the core in polynomial time, as well as one can test in polynomial time whether a given imputation is in the core or not. The maximum flow game enjoys a further nice property that it is *totally balanced*, i.e., the cores of all the subgames are nonempty.

It is not accidental that the above special case of the maximum flow game always has a nonempty core and allows polynomial time solution algorithms. In Section 2, we introduce a general (and different from that of Kalai and Zemel for the network game on unit networks) integer programming formulation of combinatorial optimization games, and show that the game has a nonempty core if and only if the corresponding linear programming relaxation has an integer optimal solution. In the network flow game on unit networks, it turns out that the linear program relaxation always has an integer solution, as guaranteed by Menger's Theorem [6].

In Section 3, we introduce several interesting examples. For the single source single sink network game, the algorithm of Kalai and Zemel only works for edge players in a directed unit network. As an advantage of our integer programming formulation, this can be immediately extended to an undirected graph, to vertex players and s-t vertex-connectivity as the game value, as well as the matching game for bipartite graphs. This approach can also be applied to solve several other games. Of course, the integrality condition does not always hold. A specially interesting case is the matching/vertex-cover game for general graphs. Though one integer program is polynomially solvable and the other is NP-hard for this pair of graph optimization problem, the linear programming relaxations of the pair are dual to each other. However, we will see that the condition for the nonemptiness of the core is polynomially checkable for both games in each pair. This is not necessarily true of all the NP-hard combinatorial optimization problems. There are cases in which none of the integrality and polynomial time solvability holds. For the graph coloring game, it is an NP-hard problem to decide whether the core is empty or decide whether an imputation is in the core. Exactly in such situations, the complexity concept grown out of theoretical computer science provides a better understanding on why a good mathematical characterization is not obtainable [5]. For the graph coloring game, we could reveal the equivalence between perfect graphs and totally balanced games.

2 Maximization and Minimization Games

2.1 Definitions

Let A be an $m \times n$ $\{0,1\}$-matrix. Let 1_k and 0_k denote the column vectors with all ones and all zeros, respectively, of dimension k. We may denote these vectors by 1 and 0 for simplicity. Let $M = \{1, 2, \cdots, m\}$ and $N = \{1, 2, \cdots, n\}$ be the corresponding index sets, and let t denote the transposition. Consider the following linear program,

$$\begin{aligned} LP(c, A, \max): \quad & \max y^t c \\ s.t. \quad & y^t A \leq 1_n^t, \quad y \geq 0_m, \end{aligned}$$

and its dual,

$$\begin{aligned} DLP(c, A, \max): \quad & \min 1_n^t x \\ s.t. \quad & Ax \geq c, \quad x \geq 0_n, \end{aligned}$$

where c is an m-dimensional column vector $\in R^m$, y is an m-dimensional column vector of variables and x is an n-dimensional column vector of variables.

We denote the corresponding integer programming version of $LP(c, A, \max)$ by $ILP(c, A, \max)$. Since A is a $\{0,1\}$-matrix, the integrality constraints are equivalent to require y to have $\{0,1\}$ values. We define the *packing game* $Game(c, A, \max)$ as follows, where $\overline{S} = N - S$:

1. The player set is N.
2. For each subset $S \subseteq N$, $v(S)$ is defined as the value of the following integer program:

$$\begin{aligned} ILP(c, A_{M,S}, \max): \quad & \max y^t c \\ s.t. \quad & y^t A_{M,S} \leq 1_{|S|}^t, \quad y^t A_{M,\overline{S}} \leq 0_{n-|S|}^t, \\ & y \in \{0,1\}^m, \end{aligned}$$

where $A_{T,S}$ is the submatrix of A with row set T and column set S, and $v(\emptyset)$ is defined to be 0.

Since this is a maximization problem, we may as well assume that $c_j > 0$ for j with $A_{j\cdot} \neq 0$. Otherwise, we can always choose $y_j = 0$. For a vector $z : N \to R_+$ and a subset $S \subseteq N$, let $z(S)$ denote $\sum_{j \in S} z(j)$. Let us recall that a vector $z : N \to R_+$ is an *imputation* if $z(N) = v(N)$, and an imputation is in the *core* if $z(S) \geq v(S)$ holds for all $S \subseteq N$.

We then introduce a *covering game* $Game(c, A, \min)$ for the minimization problem in the similar manner:

1. The player set is M.
2. For each subset $T \subseteq M$, $v(T)$ is defined as the value of the following integer program:

$$\begin{aligned} ILP(d, A_{T,N}, \min): \quad & \min d^t x \\ s.t. \quad & A_{T,N} x \geq 1_{|T|}, \quad x \in \{0,1\}^n, \end{aligned}$$

   where $v(\emptyset)$ is defined to be 0.

Again we can assume $d_j > 0$ for all j. Otherwise we may always choose $x_j = 1$ to simplify the problem. Since the value of the game is defined by a solution to the minimization problem, this is in fact a problem of sharing the cost of the game. Thus, we would revise the definitions of imputation and core. A vector $w : M \rightarrow R_+$ is an *imputation* if $w(M) = v(M)$, and an imputation is in the *core* if $w(T) \le v(T)$ holds for all $T \subseteq M$.

From definition, we easily see that both packing game and covering game are monotone (i.e., $v(S') \le v(S)$, $S' \subseteq S \subseteq N$ holds for $Game(c, A, \max)$, and $v(T') \le v(T)$, $T' \subseteq T \subseteq M$ holds for $Game(d, A, \min)$).

As a justification of introducing these general formulations, note that the maximum flow game on a digraph $D = (V, E)$ with source s and sink t, studied in [11], can be formulated as $Game(1_m, A, \max)$ if we interpret that A is the path-arc incidence matrix: $A_{ij} = 1$ if arc j is in the i-th directed s-t path, and 0 otherwise. Constraint $y^t A \le 1_n^t$ requires that, for each arc j, there is at most one path chosen by y, which goes through the arc (i.e., arc capacity is 1).

In this paper, we shall introduce a number of optimization games on graphs, which are formulated as the above maximization and/or minimization games, and study the following properties and questions concerning their cores.

1. *Nonemptyness*: Is the core of the game always nonempty?

2. *Convex characterization*: Can any imputation in the core be represented as a convex combination of some well-defined dual objects (such as minimum s-t cuts)?

3. *Testing nonemptiness*: Can it be tested in polynomial time whether a given instance of the game has nonempty core?

4. *Checking membership*: Can it be checked in polynomial time whether a given imputation belongs to the core?

5. *Finding a core member*: Is it possible to find an imputation in the core in polynomial time?

As our discussion will focus on games on graphs, the polynomiality is determined in terms of the input size of the graph (i.e., the number of vertices $|V|$ and the number of arcs or edges $|E|$). Even though the sizes of the constraint matrices A in the above formulations are sometimes exponential in $|V|$ and $|E|$ (e.g., the A for the maximum flow game has exponentially many rows), we would like to obtain algorithms of polynomial time in the input size of the graphs.

We note at this point that two games $Game(c, A, \max)$ and $Game(c, A, \min)$ are *not* dual in the sense of the underlying linear programs since the roles of objective function and the right hand side of the constraint are not interchanged. In the case of $c = 1$, however, the corresponding linear relaxations become dual to each other.

2.2 Main theorems

We give important lemmas and theorems that characterize the cores of the maximization and minimization games defined in the previous sebsection.

Lemma 1 *A vector $z : N \rightarrow R_+$ is in the core of $Game(c, A, \max)$ if and only if*

1. $z(N) = v(N)$ *(i.e., z is an imputation),*

2. $z(S_i) \geq c_i$ *for all* $i \in M$*, where* $S_i = \{j \in N \mid A_{ij} = 1\}$ *(i.e., z is feasible to* $DLP(c, A, \max)$ *of* $LP(c, A, \max)$*.* □

Theorem 1 *The core for* $Game(c, A, \max)$ *is nonempty if and only if* $LP(c, A, \max)$ *has an integer optimal solution. In such case, a vector* $z : N \to R_+$ *is in the core if and only if it is an optimal solution to* $DLP(c, A, \max)$*.* □

Lemma 2 *A vector* $w : M \to R_+$ *is in the core of* $Game(d, A, \min)$ *if and only if*

1. $w(M) = v(M)$ *(i.e., w is an imputation),*

2. $w(T_j) \leq d_j$ *for all* $j \in N$*, where* $T_j = \{i \in M \mid A_{ij} = 1\}$ *(i.e., z is feasible to the dual* $DLP(d, A, \min)$ *of* $LP(d, A, \min)$*).* □

Theorem 2 *The core for* $Game(d, A, \min)$ *is nonempty if and only if* $LP(d, A, \min)$ *has an integer optimal solution. In such case, a vector* $w : M \to R_+$ *is in the core if and only if it is an optimal solution to* $DLP(d, A, \min)$*.* □

3 A Selection of Examples

There are many interesting optimization games on graphs, which can be formulated as in Section 2. We will focus on the following games.

1. Maximum flow game in unit networks, s-t edge connectivity game in undirected graphs, s-t vertex connectivity game in undirected graphs, and maximum matching game in bipartite graphs.

2. Maximum r-arborescence game.

3. Maximum matching game and minimum vertex cover game.

4. Maximum independent set game and minimum edge cover game.

5. Minimum coloring game.

3.1 Network Flow Game and Its Variants

Let us consider the maximum flow game in a unit directed network $D = (V, E)$ with source $s \in V$ and sink $t \in V$, which is also denoted simply by $D = (V, E, s, t)$. We call a unit network (i.e., with arc capacity one) also as a digraph. The value of a maximum flow in the case of a digraph D is the number of arc disjoint s-t paths in D. For this reason, it is also called the s-t *arc-connectivity* of D. A partition of V, $C = (U, V - U)$, is called a *cut* in G, and represents the set of arcs $\{(i, j) \in E \mid i \in U, j \in V - U\}$. A cut is an s-t *cut* if $s \in U$ and $t \in V - U$. A minimum s-t cut is an s-t cut with the minimum cardinality as an arc set. The following property is well known as Menger's theorem (which is a special case of a more general result called the max-flow min-cut theorem for capacitated networks) [3].

Lemma 3 *Given a digraph $D = (V, E, s, t)$, the s-t arc-connectivity of D (i.e., the value of a maximum flow) is equal to the size of a minimum s-t cut in D.* □

Based on this lemma, we see that there is a polynomial time algorithm to generate a imputation in the core. Take a minimum s-t cut C in D, and let I_C be its characteristic vector; $I_C(j) := 1$ if $j \in C$ and 0 otherwise. Let $z := I_C$. Then z is an imputation since $z(E) = |C| = v(E)$ by Lemma 3. Furthermore, for any $S \subseteq E$, we have $z(S) = |C \cap S| \leq v(S)$ by Lemma 3 and the fact that $C \cap S$ is an s-t cut in $D[S]$. Therefore, this I_C is indeed in the core, and the core is nonempty. Now a vector $z \in R^{|E|}$ is a *convex combination* of a family of C if $z = \sum_C \lambda_C I_C$ holds for some λ_C such that $\sum_C \lambda_C = 1$ and $\lambda_C \geq 0$ for all C. If the family of C is finite, the set of such z forms a convex polyhedron whose extremal points are precisely those characteristic vectors I_C. Kalai and Zemel [10] went one step further to show that an imputation z is in the core if and only if it is a convex combination of I_C of minimum s-t cuts C.

In some cases, there may be *dummy players* in the game in the sense that those players j always get $z(j) = 0$ in an imputation $z : N \to R_+$. We say that an imputation z is in the core of a game with a set $\hat{N} \subseteq N$ of dummy players if it is in the core of the game and $z(j) = 0$, $j \in \hat{N}$ holds. Of course, a game may not have a core for some set $\hat{N}$ of players, even if it has a nonempty core (if there is no dummy player).

To make $Game(1_{|M|}, A, \max)$ more general for the purpose of utilizing it in discussing the s-t vertex-connectivity game and some other games later, we introduce a set $\hat{E} \subseteq E$ of dummy players. A set $\hat{E}$ of arcs (for dummy players) is called *valid* if $F = E - \hat{E}$ contains at least one minimum s-t cut $C \subseteq E$ of D. A game is *trivial* if $v(N) = 0$ for the set of entire players.

Theorem 3 *For a digraph $D = (V, E, s, t)$ and a set $\hat{E}$ of dummy players, the nontrivial s-t arc connectivity game has nonempty core if and only if $\hat{E}$ is valid.* □

Theorem 4 *Let $z : E \to R_+$ be an imputation of the nontrivial s-t arc-connectivity game on a digraph $D = (V, E, s, t)$ with a valid set $\hat{E}$ of dummy players. Then z is in the core with respect to $\hat{E}$ (i.e., $z(e) = 0$, $e \in \hat{E}$) if and only if it is a convex combination of the set of the characteristic vectors for minimum s-t cuts C contained in $F = E - \hat{E}$.* □

Corollary 1 *For a valid set $\hat{E}$ of dummy players, testing nonemptiness, checking membership and finding a core member of the s-t arc-connectivity game, can all be answered in polynomial time.* □

We emphasize at this point that the results in Theorems 3 and 4 can be extended to other optimization games on graphs, which can be reducible to the maximum flow game in a directed network. Those problems include:

P1: s-t edge-connectivity game in an undirected graph $G = (V, E, s, t)$, where players are on edges and $v(S)$, $S \subseteq E$ is defined to be the size of maximum flow from s to t in the induced network $G[S]$,

P2: s-t vertex-connectivity game in a digraph $D = (V, E, s, t)$ (resp. undirected graph $G = (V, E, s, t)$), where players are on vertices except s and t, and $v(S)$, $S \subseteq V - \{s, t\}$ is defined to be the maximum number of arc (resp., edge) disjoint paths from s to t in the induced digraph $D[S]$ (resp., graph $G[S]$), P3: maximum matching game with edge players on a bipartite graph $G = (V_1, V_1, E)$, where $v(S)$, $S \subseteq E$ is defined to be the size of maximum matching in the induced graph $G[S]$.

For a digraph $D = (V, E, s, t)$ (or undirected graph $G = (V, E, s, t)$), a subset $W \subseteq V - \{s, t\}$ is called an *s-t vertex-cut* if the graph induced by $V - W$ has no path from s to t.

Corollary 2 *For a game in the above P1 (resp., P2 and P3), the core is always nonempty, and if the game is not trivial, the core is a convex combination of a set of characteristic vectors of minimum s-t cuts (resp., minimum s-t vertex-cuts for P2 and minimum vertex-covers for P3). Furthermore, testing nonemptiness, checking membership and finding a core member of all these games, can be answered in polynomial time.* □

One may define a minimum cut game in a digraph $G = (V, E, s, t)$ as the covering game $Game(1_{|E|}, A, \min)$, which is the dual game of the s-t arc connectivity game $Game(1_{|\mathcal{P}|}, A, \max)$, where $A = A_{\mathcal{P},E}$ is the incidence matrix between the arc set E and the s-t path set $\mathcal{P}$. This game has players on s-t paths in $\mathcal{P}$, which may be exponentially many in the size of $|V|$ and $|E|$. Recall that, for any subset $S \subseteq E$, $LP(1_{|\mathcal{P}|}, A_{\mathcal{P},S}, \min)$ naturally represents the maximum flow problem in the induced digraph $G[S] = (V, S, s, t)$, and hence it has an integer optimal solution due to Lemma 3, which is now applied to $G[S]$. However, this is not the case in the minimum cut game. Although $LP(1_{|E|}, A, \min)$ has an integer optimal solution due to Lemma 3, the corresponding linear programs $LP(1_{|E|}, A_{T,E}, \min)$ for subsets $T \subset \mathcal{P}$ may not enjoy such an itegral property. For example, G possibly contains three s-t paths P_i, $i = 1, 2, 3$ such that $P_i \cap P_j = \{e_{ij}\}$, $1 \le i < j \le 3$ hold for some arcs $\{e_{12}, e_{23}, e_{13}\} \subseteq E$. For $T = \{P_1, P_2, P_3\} \subseteq \mathcal{P}$, $LP(1_{|E|}, A_{T,E}, \min)$ has an optimal solution $x : E \to R_+$ such that $x(e_{12}) = x(e_{23}) = x(e_{13}) = 0.5$ and $x(e) = 0$ for other arcs, implying that it has no integer optimal soltion (with the optimal value 1.5).

3.2 Arborescence Game

The maximum r-arborescence game and minimum r-cut game is played on a digraph $D = (V, E)$ with a root $r \in V$. Recall that an r-arborescence in D is a spanning directed tree such that every vertex in D is reachable from r. For each subset $S \subseteq E$ of arcs (i.e., players), the game value $v(S)$ is defined to be the size of the maximum number of r-arborescences on the subgraph $G[S] = (V, S)$. This game can be formulated as a packing game $Game(1_{|M|}, A, \max)$ by matrix A such that the rows correspond to all r-arborescences and the columns correspond to all arcs; $A_{ij} = 1$ if and only if arc j is in the i-th r-arborescence.

The model of the r-arborescence game can arise when we want to maintain paths from a distinguished source r to all the vertices in the network. For each arc in D, there is one player in control of this arc. Note that such k is equal to the maximum number of pairwise disjoint r-arborescences. For this reason, we call the maximum r-arborescence game also as the *single source connectivity game* (on digraphs). By Lemma 1, an imputation is in the core of this game if and only if there is no r-arborescence such that the sum of the imputation on the r-arborescence is less than one.

The questions about of the core of this game can be answered in polynomial time, similar to that of s-t arc-connectivity, because the integrality of the corresponding linear programs for both the r-arborescences and the r-cuts follows from the next well known result of Edmonds [2]. A cut $C = (U, V - U)$ in a digraph represents the set of arcs from U to $V - U$. It is called an *r-cut* if $r \in U$ holds.

Lemma 4 [2] *In a digraph $D = (V, E)$ with root $r \in V$, the maximum number of pairwise disjoint r-arborescences is equal to the minimum cardinality of an r-cut.* □

Let the maximum number of disjoint r-arborescences be k. Since these k pairwise disjoint r-arborescences form a solution to the primal linear program $LP(1_{|M|}, A, \max)$ of this game, and the minimum cardinality r-cut of size k is a solution to its dual linear program, it follows that they are optimal solutions to the primal and the dual, respectively, by the duality theory of linear programming. Therefore the primal linear program has an integer solution.

A set $\hat{E}$ of arcs (for dummy players) is called *valid* if $F = E - \hat{E}$ contains at least one minimum r-cut $C \subseteq E$ of D. Analogously with Theorems 3 and 4, we have the following results.

Theorem 5 *For a digraph $D = (V, E)$ with root $r \in V$ and a set $\hat{E}$ of dummy players, the maximum r-arborescence game with $v(E) > 0$ has nonempty core if and only if $\hat{E}$ is valid.* □

Theorem 6 *Let $z : E \to R_+$ be an imputation of the maximum r-arborescence game on a digraph $D = (V, E)$ with root $r \in V$ and a valid set $\hat{E}$ of dummy players and let $v(E) > 0$. Then z is in the core with respect to $\hat{E}$ (i.e., $z(e) = 0$, $e \in \hat{E}$) if and only if it is a convex combination of the set of the characteristic vectors for minimum r-cuts C contained in $F = E - \hat{E}$.* □

Corollary 3 *For a set $\hat{E}$ of dummy players, testing nonemptiness, checking membership and finding a core member of the maximum r-arborescence game, can all be answered in polynomial time.* □

3.3 Matching and Vertex Cover

Given a graph $G = (V, E)$, we define the maximum matching game by a game such that the players are on vertices and the game value $v(S)$ for a subset $S \subseteq V$ is the maximum matching size in the subgraph $G[S]$ induced by S. Similarly, the minimum vertex cover game is defined by a game such that players are on edges and $v(S)$ for $S \subseteq E$ is the minimum vertex cover size in the subgraph $G[S] = (V, S)$. These games are formulated by packing game $Game(1_{|E|}, A, \max)$ and covering game $Game(1_{|V|}, A, \min)$, respectively, where the constraint matrix A is the incidence matrix of G in which the rows correspond to edges E and the columns correspond to vertices V; $A_{ij} = 1$ if and only if edge i and vertex j are incident.

For bipartite graphs as discussed at the end of Section 3.1, the maximum matching game can be formulated as a special case of the s-t arc-connectivity problem with a subset of edges as players. Thus, we have the convex characterization of the core, and all problems about the core can be answered in polynomial time. Similar results also hold for the minimum vertex cover game (as will be discussed in Section 3.3.2). However, these nice properties break down for general graphs, and the core is nonempty only for some special classes of graphs.

By Lemma 1, an imputation z is in the core of the matching game if and only if $z(u)+z(u') \geq 1$ holds for all edges $(u, u') \in E$. Based on this observation, we can easily find two classes of graphs for which the cores are always nonempty. The class of graphs for which the size of a minimum vertex cover is the same as the size of a maximum matching, and the class of graphs with perfect matching. For a graph $G = (V, E)$ in the first class, we assign $z(v) = 1$ if v is in the minimum vertex cover and $z(v) = 0$ otherwise. It is easy to see that this z is indeed in the core. For a graph $G = (V, E)$ in the second class, we assign every vertex $v \in V$ with $z(v) = 0.5$. Then $z(V) = |V|/2 = v(E)$, since G has a perfect matching. Furthermore, since the size of a maximum matching in any subgraph $G[S]$ induced by $S \subseteq V$ is no more than $|S|/2$, this z is indeed in the core.

On the other hand, one can easily construct graphs with non-empty cores which are not in the above two classes. For example, take two graphs one from each of the above classes, and connect them with edges between the vertices in the minimum cover and the vertices in the perfect matching. However, the next theorem says that these are essentially all graphs which have nonempty cores for the maximum matching game.

Theorem 7 *An undirected graph $G = (V, E)$ has a nonempty core for the maximum matching game if and only if there exists a subset $V_1 \subseteq V$ such that*

1. *the subgraph $G_1 = G[V_1]$ induced by V_1 has a minimum vertex cover W with the same size as its maximum matching,*
2. *the subgraph $G_2 = G[V - V_1]$ induced by $V - V_1$ has a perfect matching,*
3. *all the remaining edges $(u, u') \in E$ between G_1 and G_2 satisfy $u \in W$ for the vertex cover W in 1.* □

Corollary 4 *For the core of the maximum matching game, testing nonemptiness, checking membership and finding a core member, can be done in polynomial time.* □

Recall that an integer solution of its dual LP problem $DLP(1_{|E|}, A, \max) = LP(1_{|V|}, A, \min)$ of the maximum matching game implies a minimum vertex cover. In some class of graphs which have a nonempty core, the size of a maximum matching is not equal to the size of a minimum vertex cover (e.g., K_4 has a perfect matching, but its minimum size of a vertex cover is 3). In such a case, the core of the matching game cannot be represented by a convex combination of integer solutions of its dual problem $LP(1_{|V|}, A, \min)$, i.e., minimum vertex covers (because their optimal values are different), implying that the convex characterization of the core of the maximum matching game is not possible.

Theorem 8 *The core for the minimum vertex cover game on graph $G = (V, E)$ is nonempty if and only if the size of a maximum matching is equal to the size of a minimum vertex cover.* □

Theorem 9 *For the core of the minimum vertex cover game, testing nonemptiness, checking membership and finding a core member, can be done in polynomial time.* □

3.4 Edge Cover and Independent Set

For an undirected graph $G = (V, E)$, we can define a mutually dual pair of the minimum edge cover game and the independent set game by $Game(1_{|E|}, A', \min)$ and $Game(1_{|V|}, A', \max)$, respectively, where the constraint matrix A' is the incidence matrix of G in which the rows correspond to vertices and the columns correspond to edges (i.e., the transposition of the matrix A used for the pair of the maximum matching game and the minimum vertex cover game). Thus, for the minimum edge cover game, the players are on vertices, and the game value $v(S)$ for $S \subseteq V$ is the minimum number of edges that cover all vertices in S, i.e.,

$$\min\{|F| \mid F \cap E(u) \neq \emptyset, \ \forall u \in S\},$$

where $E(u)$ denotes the set of edges in E which are incident to u. Note that $v(S)$ is not necessarily the size of a minimum edge cover in the subgraph $G[S]$ induced by vertex set S. For the minimum edge cover game, we assume that G has no isolated vertex to prevent it from becoming infeasible.

Similarly, the players for the maximum independent set game are on edges and the game value $v'(T)$ for $T \subseteq E$ is the size of a maximum independent set in the subgraph $G[V\langle T\rangle]$ induced by $V\langle T\rangle$, where $V\langle T\rangle$ is defined by

$$V\langle T\rangle = \{i \in V \mid i \text{ is adjacent only to edges in } T\}.$$

(Note that $v'(T)$ is not the size of a maximum independent set in the subgraph $G[T]$.)

Theorem 10 *Let $G = (V, E)$ an undirected graph with no isolated vertex. Then $w : V \to R_+$ is in the core of the minimum edge cover game on G if and only if $\overline{w} = 1_{|V|} - w$ is in the core of the maximum matching game on G.* □

Corollary 5 *An undirected graph $G = (V, E)$ has a nonempty core for the minimum edge cover game if and only if there exists a subset $V_1 \subseteq V$ such that*

1′. *the subgraph $G_1 = G[V_1]$ induced by V_1 has a maximum independent set W' with the same size as its minimum edge cover,*

2′. *the subgraph $G_2 = G[V - V_1]$ induced by $V - V_1$ has a perfect matching (i.e., an edge cover with $|V|/2$ edges),*

3′. *all the remaining edges $(u, u') \in E$ between G_1 and G_2 satisfy $u \in V - W'$ for the maximum independent set W' in 1′.* □

Theorem 11 *Let $G = (V, E)$ be an undirected graph with no isolated vertex. Then the core for the maximum independent set game on graph G is nonempty if and only if the size of a maximum independent set is equal to the size of a minimum edge cover in G.* □

Theorem 12 *For the core of the maximum independent set game, testing nonemptiness, checking membership and finding a core member, can be done in polynomial time.* □

Theorem 13 *Given an undirected graph $G = (V, E)$ with $V \neq \emptyset$ but no isolated vertex, an imputation is in the core of the maximum independent set game if and only if it is a convex combination of the characteristic vectors of minimum edge covers.* □

One may define the maximum clique problem in an undirected graph $G = (V, E)$ as the maximum independent set problem on its complement graph $\overline{G} = (V, \overline{E})$. Obviously, such clique game is given by a packing game $Game(1_{|\overline{E}|}, A'', \max)$ which has players on the edges in $\overline{G}$, where A'' is the vertex-edge incidence matrix A'' of the complement graph $\overline{G}$. Therefore, all the results in this subsection can be generalized to the maximum clique problem.

3.5 Chromatic Number

Let $\chi(G')$ denote the chromatic number of an undirected graph G' (i.e., the minimum number of maximal independent set which together covers all vertices of G'). For the minimum coloring game on a graph $G = (V, E)$, we define the game value $v(S)$, $S \subseteq V$ as $\chi(G[S])$, i.e., the size of a minimum coloring of the subgraph $G[S]$ induced from G by S. This game can be represented by a covering game $Game(1_{|\mathcal{I}|}, A, \min)$, the rows of the matrix A correspond to the vertices in a graph G, and the columns correspond to maximal independent sets, where $\mathcal{I}$ denotes the set of all maximal independent sets in G. The coloring game arises frequently in applications if the smallest number of conflict-free groups are sought in a system where vertices represent members and edges represent conflicts between members. This type of conflict graphs, for example, can be found in many resource sharing problems.

By Lemma 2, a vector $w : V \rightarrow R_+$ is in the core of the minimum coloring game if and only if

1. $w(V) = \chi(G)$,
2. $w(S) \leq 1$ for any independent set $S \subseteq V$.

Let $\omega(G)$ denote the size of a maximum clique in G, which satisfies $\omega(G) \leq \chi(G)$, as widely known in the coloring problem. We can easily observe that the characteristic vector I_C of a maximum clique $C \subseteq V$ is a core of the coloring game if $\omega(G) = \chi(G)$ holds. Therefore the minimum coloring game on such a graph has nonempty core. However, the converse is not true. That is, there is a graph $G = (V, E)$ such that $\omega(G) < \chi(G)$ but the core of the coloring game is nonempty. For example, for a graph G with $\omega(G) < \chi(G)$ and $\alpha(G)\chi(G) = |V|$, the z defined by $z(u) := \chi(G)/|V|$ for all $u \in V$ is in the core, where $\alpha(G)$ is the stable number of G,i.e., the size of a maximum independent set. Therefore, in general, the coloring game has no convex characterization by a set of maximum cliques. Also, from such a graph G, construct the graph $G' = G + K_{\chi(G)}$ by adding complete graph $K_{\chi(G)}$ via a single common vertex. Then G' satisfies $\omega(G') = \chi(G')$, but has nonempty core because its core is the same as that of G, which is not a convex combination of cliques. That is, in general, the coincidence of the optimum values of $ILP(1_m, A, \max)$ and $IPL(1_n, A, \min)$ does not imply the convex characterization of the core of a covering game $Game(1_n, A, \min)$.

Theorem 14 *If a graph $G = (V, E)$ is bipartite and $E \neq \emptyset$, an imputation $w : V \rightarrow R_+$ is in the core of the minimum coloring game if and only if it is a convex combination of the characteristic vectors of edges in E; i.e., $w = \sum_{e \in E} \lambda_e I_e$, with $\sum_{e \in E} \lambda_e = 1$ and $\lambda_e \geq 0$ for all $e \in E$. This can be tested in polynomial time.* □

Theorem 15 *For the minimum coloring game, it is NP-complete to decide whether the core is empty or not. It is also NP-complete to decide whether a given imputation is in the core or not.* □

Theorem 16 *Let $G = (V, E)$ be a perfect graph. Then the core of the minimum coloring game is always nonempty. Furthermore it can be tested in polynomial time whether an imputation w is in the core or not.* □

Theorem 17 *A graph $G = (V, E)$ is perfect if and only if the minimum coloring game on G is totally balanced.* □

4 Conclusion

The computational issues in game theory have received much attention recently, and have been a motivation of our investigation into the classes of optimization games on graphs. We conclude the paper by giving a table of the results considered for the five properties/questions raised in Section 2.

Table 1. Summary of the results for optimization games on graphs.

Games	Core nonemptiness	Convex characterization of the core	Testing nonemptiness of the core	Checking if an imputation is in the core	Finding an imputation in the core
Max flow (G, D)	yes	yes	—	P	P
s-t connectivity (G, D)	yes	yes	—	P	P
r-arborescence (D)	yes	yes	—	P	P
Max matching (G)	no	no	P	P	P
Min vertex cover (G)	no	yes	P	P	P
Min edge cover (G)	no	no	P	P	P
Max indep. set (G)	no	yes	P	P	P
Max clique (G)	no	yes	P	P	P
Min coloring (G)	no	no	NPC	NPC	NPH

D: digraphs, G: undirected graphs, P: polynomial time, NPC: NP-complete, NPH: NP-hard, —: trivial

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