

Comments on the Entropy Differential in Extended Irreversible Thermodynamics

Masakazu Ichiyanagi (Gifu Univ. of Econ., Ogaki, Gifu)

1. Introduction

Recently there have appeared a number of theories which purport to extend the usual theory of irreversible thermodynamics [1- 8]. The classical theory, due to Onsager [9], uses the extensive variables as the basic thermodynamic quantities which characterize the condition of a macroscopic aged system. The choice of the thermodynamic state variables, however, is determined not only by the physical nature of the system under study but also by the scheme adopted and hoped-for precision in the description; so the number of thermodynamic state variables may vary from one system and theory to other ones. In the classical theory, time dependence is introduced through the time derivatives of the extensive variables, which are referred to as the thermodynamic (dissipative) fluxes. One introduces the thermodynamic forces which are seen as causing the corresponding fluxes. Near equilibrium the forces are written as linear functions of the deviations of the extensive variables from their equilibrium values. To complete the theory we have to write the constitutive equations relating fluxes and forces in a particular system. These equations are introduced not as a time evolution equation but as a constitutive equation rendering the necessary conditions to yield a complete set of equations for the variables.

In its simplest form extended irreversible thermodynamics (EIT) includes dissipative fluxes in the set of independent thermodynamic variables to characterize the condition of a nonequilibrium open system. EIT uses a generalized entropy which, in addition to the usual extensive variables, includes the dissipation fluxes as independent variables, and is interested in obtaining evolution equations for the dissipative fluxes, compatible with the second law of thermodynamic formulated in terms of the generalized entropy.

The various contributions to EIT describe the work of the groups. The theory has not undoubtedly achieved its final form yet. Indeed, it has been argued strenuously by Eu [3] that some derivations, based on a generalized entropy, are actually incorrect [10]. He shows that the entropy differential for systems away from equilibrium cannot be an exact form.

Our definition of thermodynamic variables is within the spirit of Onsager's, since we are interested in a *discontinuous* system. The method to be used here is based on the principle of maximum entropy [11], which is known to provide a systematic recipe for the calculation of any macroscopic observable character of a system away from equilibrium. It will be shown that the procedure of maximum entropy is 'reasonable' in that it defines a nonequilibrium entropy which enjoys the Gibbs relation of a known form. In order to assign a Gibbs space of thermodynamic variables, we use the notion of observation level by Fick and Sauermann [12]. By this we can find a sufficient condition for the choice of the Gibbs space.

In this paper, we want to find a possible relation between the statistical and the thermodynamic entropies. To do this, we utilize the notion of relative entropy [13,14] which measures an *entropic* distance between two states characterized by density matrixes. This description involves, besides the usual statistical entropy defined in terms of the nonequilibrium density matrix, also another entropy written in terms of the generalized canonical density matrix, the latter of which enjoys the (extended) Gibbs relation.

2. The principle of maximum entropy

In order to characterize the thermodynamic state of an open system we require the expectation values of a set of observables $\{ \mathbf{H}, \mathbf{A}_i ; i=1,2, \dots, f \}$ which are supposed to be known from a measurement. Let the operators \mathbf{A}_i be the operators other than the Hamiltonian of the system, \mathbf{H} , and be linearly independent. Using such a set, we define an observation level [12]. Note that the choice of the thermodynamic state variables is determined not only by the physical nature of the system under study but also by the scheme adopted and hoped-for precision in the description; so the number of thermodynamic state variables may vary from one system and theory to other ones.

These pieces of information represent the following constraints on the nonequilibrium density matrix assignment;

$$\text{Tr}\rho(t) = 1, \quad (2.1)$$

$$\text{Tr}\rho(t)\mathbf{H} = E(t), \quad (2.2)$$

$$\text{Tr}\rho(t)\mathbf{A}_i = \alpha_i(t), \quad (i=1, \dots, f). \quad (2.3)$$

Here, $E(t)$ and $\alpha_i(t)$ are the macroscopic variables to be used in nonequilibrium statistical thermodynamics. $\rho(t)$ denote the density matrix which is a solution of the von Neumann equation characterizing dynamics of an open system interacting with its surroundings. Hence, we will write down the von Neumann equation

$$\partial\rho(t)/\partial t + [i\mathbf{H}, \rho(t)] = \mathbf{L}[\rho(t)]. \quad (2.4)$$

Here, \mathbf{H} represents the entire Hamiltonian of the system. It suffices to think of \mathbf{H} as containing all the terms one can handle dynamically, such as kinetic energy, and external fields which vary slowly in *space* and *time*. Hence, $\mathbf{H} \approx \mathbf{H}_\tau = \mathbf{H} - \sum \mathbf{A}_i \mathbf{E}_i(\tau)$ ($\mathbf{E}_i(\tau)$; external fields and $\tau = \lambda^2 t$; $\lambda \rightarrow 0$). $\mathbf{L}[\rho(t)]$ describe the effects which are attributed to *collisions* and interactions between the system and its surroundings. The precise form of the latter is not irrelevant to the present discussion.

In principle, there must be an extremely large class of density matrixes that fulfill the von Neumann equation and yield the expectation values $E(t)$ and $\alpha_i(t)$. The question of which of these is correct one is answered by maximizing an entropy. A generalized canonical density matrix $\rho_c(t)$ is the density matrix which maximizes the statistical entropy

$$S[\rho(t)] = -\text{Tr}[\rho(t)\ln\rho(t)] \leq S[\rho_c(t)], \quad (k_B = 1) \quad (2.5)$$

subjected to the prescribed manifold of expectation values (2.2) and (2.3). As is well-known, the method of Lagrange multipliers yields

$$\rho_c(t) = \exp[F(t) - \beta(t)\mathbf{H} - \sum \mathbf{X}'_i(t)\mathbf{A}_i], \quad (2.6)$$

where $F(t)$ is the normalization factor defined by

$$\exp[- F(t)] = \text{Tr} \exp[-\beta(t)\mathbf{H} - \sum \mathbf{X}'_i(t)\mathbf{A}_i]. \quad (2.7)$$

and $\beta(t)$ and $\mathbf{X}'_i(t)$ are the Lagrange multipliers. The constraints

$$\text{Tr} \rho_c(t)\mathbf{H} = E(t), \quad (2.8)$$

$$\text{Tr} \rho_c(t)\mathbf{A}_i = \alpha_i(t) \quad (i = 1, 2, \dots, f) \quad (2.9)$$

are employed to express these multipliers

$$\beta(t) = \beta[E(t), \alpha_1(t), \dots, \alpha_f(t)], \quad (2.10)$$

$$\mathbf{X}'_i(t) = \mathbf{X}'_i[E(t), \alpha_1(t), \dots, \alpha_f(t)]. \quad (2.11)$$

By making use of (2.6) in (2.5), we obtain the expression

$$S[\rho_c(t)] = - F(t) + \beta(t)E(t) + \sum \mathbf{X}'_i(t)\alpha_i(t). \quad (2.12)$$

It is noted that, if \mathbf{H} and all \mathbf{A}_i are not explicitl time-dependent, the change in the normalization factor on changing the multipliers is obtained from (2.7); it is

$$\delta F(t) = E(t)\delta\beta(t) + \sum \alpha_i(t)\delta\mathbf{X}'_i(t). \quad (2.13)$$

Hence, from (2.12) and (2.13) we obtain the so-called Gibbs relation:

$$\delta S[\rho_c(t)] = \beta(t)\delta E(t) + \sum \mathbf{X}'_i(t)\delta\alpha_i(t). \quad (2.14)$$

Thus, (2.13) is seen to be a form of the *integrability* condition for the entropy differential (2.14). that is, the maximum entropy differential (2.14) is an exact form with respect to the observation level chosen and (2.13) is a Gibbs-Duhem equation.

Let us note that, by taking the derivative with respect to time of both sides of (2.9), we obtain

$$d\alpha_i(t)/dt = \sum A_{ij}[t] dX'_j(t)/dt, \quad (i, j = 0, 1, \dots, f). \quad (2.15)$$

where $X'_0(t) = \beta(t)$, $A_0 = \mathbf{H}$ and

$$A_{ij}[t] = \text{Tr} \int dx \{ \rho_c(t) \}^{1-x} (A_j - \alpha_j(t)) \{ \rho_c(t) \}^x (A_i - \alpha_i(t)). \quad (2.16)$$

Here we have used (2.13), in which we replaced the symbol δ by d/dt . The coefficients $A_{ij}[t]$ are the (equal time) correlation of fluctuations; $A_i - \alpha_i(t)$. Equations (2.15), in principle, are used to obtain the Lagrange multipliers, $X'_i(t)$, as functions of $E(t)$ and $\alpha_i(t)$.

From eq.(2.14) we obtain the expression for the entropy production

$$S[\rho_c(t)] = \beta(t)E(t) + \sum X'_i(t)\alpha_i(t). \quad (2.17)$$

The overdot signifies differentiation in time. This result is used to define the dissipative fluxes $\alpha_i(t)$ and the corresponding forces $X'_i(t)$. That is, the Lagrange multipliers have the meaning of the thermodynamic forces with respect to the observation level considered. In this paper we consider the case in which we have

$$X'_i(t) = X_i^s + X_i(t). \quad (2.18)$$

Here, X_i^s characterize a stationary state of the system in question.

There is an important question whether it is possible to apply the principle of maximum entropy [11] even if a system is away from an equilibrium. Next, let us consider this. The generalized canonical density matrix $\rho_c(t)$ is used to calculate the average values of operators other than $\{\mathbf{H}, A_i : i=1, \dots, f\}$. It is sufficient for illustration to calculate the fluxes

$$J_i(t) = d\alpha_i(t)/dt \approx \text{Tr} \rho_c(t) [i\mathbf{H}, A_i]. \quad (i=1, \dots, f). \quad (2.19)$$

This is our definition of the so-called *dissipative* fluxes as far as they are not equal to zero; that is, they are the averages of the current operators, $i[\mathbf{H}, A_i]$, with respect to the nonequilibrium density matrix $\rho_c(t)$ in the sense of the

response theory. By definition, they should be equal to zero if $\rho_c(t)$ approaches to an equilibrium density matrix. It is noted here that the definition (2.19) means an approximation in the sense of Fick and Sauermann[13]. The constraints (2.9) and (2.19) are consistent, if and only if we have chosen the variables A_i to be conserved so that $\text{Tr}\rho(t)LA_i$ equal zero.

By making use of the identity

$$\text{Tr}\rho_c(t)[i\mathbf{H}, \mathbf{A}_i] = \text{Tr}[\rho_c(t), i\mathbf{H}]\mathbf{A}_i \quad (2.20)$$

and the approximation which assumes that $X_i(t)$ are small;

$$\rho_c(t) \approx \rho_0(t) \left[1 - \int dx \{\rho_0(t)\}^{-x} [\sum X_j(t)\mathbf{A}_j, \beta(t)\mathbf{H}] \{\rho_0(t)\}^x \right], \quad (2.21)$$

where we have put

$$\rho_0(t) = \exp\{ [F(t) - \beta(t)\mathbf{H}] - \sum X_i^s \mathbf{A}_i \}, \quad (2.22)$$

it is easy to get, within the approximation employed, the linear phenomenological laws:

$$\mathbf{J}_i(t) = \sum B_{ij}[t]\mathbf{X}_j(t), \quad (i,j=1,2,\dots,f), \quad (2.23)$$

where

$$B_{ij}[t] = \beta(t)\text{Tr}\int dx \{\rho_0(t)\}^{1-x} [i\mathbf{H}, \mathbf{A}_j] \{\rho_0(t)\}^x [\mathbf{A}_i, \mathbf{H}]. \quad (2.24)$$

are the transport coefficients in our case. Here, to derive the formulae (2.24) we have used that fact that

$$\text{Tr}\rho_0(t)[\mathbf{A}_i, \mathbf{A}_j] = 0. \quad (2.25)$$

That is, the all operators \mathbf{A}_i are macroscopically commutable.

These coefficients, which are not of the form of a time-correlation function, satisfy the following reciprocity relation

$$B_{ji}[t] = B_{ij}[t], \quad (i,j=1,2,\dots,f). \quad (2.26)$$

Equations (2.23), together with (2.24), are the phenomenological laws in our case. Accordingly, we conclude that the chosen observation level is sufficient in view of the flux operators of the system. This clearly shows that the

procedure of the principle of maximum entropy is applicable to nonequilibrium systems if the constraints (2.8) and (2.9) are properly specified. This is the outline of the principle of maximum entropy. We will now return to our main subject.

3. The generalized entropy

In an approach to irreversible thermodynamics, it is thought that a general theory can be constructed, if the notion that the entropy is a maximum at thermal equilibrium is relaxed so that nonconserved variables are included among the constraints for the principle of maximum entropy. In the previous section we have denoted the set of conserved observables by $\{ A_i ; i = 1, 2, \dots, f \}$. Then the set of the dissipative current operators, denoted by

$$B_i = [iH, A_i], \quad (3.1)$$

is a subset of the set of nonconserved observables. Let us denote the set of those other than the observables corresponding to the dissipative fluxes associated with the conserved observables by $\{ b_k ; k \geq 1 \}$. It should be noted here that by definition we have $\text{Tr}L[\rho(t)]A_i = 0$ whereas $\text{Tr}L[\rho(t)]b_k \neq 0$.

The method employed in the previous section can be extended to the case in which the constraints on the nonequilibrium density matrix assignment are given by (2.1 - 3), and

$$\text{Tr}\rho(t)B_i = \alpha_i(t), \quad (3.2)$$

$$\text{Tr}\rho(t)b_k = G_k(t), \quad (k \geq 1). \quad (3.3)$$

Here, the precise density matrix $\rho(t)$ enjoys the von Neumann equation of the form (2.4). Equations (3.2) and (2.3) are consistent because it is true that, by definition, $\text{Tr}L[\rho(t)]A_i = 0$. As before, maximization of the statistical entropy $S[\rho(t)]$ subject to those constraints, together with (2.1), (2.2) and (2.3), yields the generalized canonical form

$$\rho_c(t) = \exp[F(t) - \beta(t)H - \sum X'_i(t)A_i - \sum Y'_i(t)B_i - \sum y'_k(t)b_k], \quad (3.4)$$

where $F'(t)$ denotes the normalization factor so that $\text{Tr} \rho_c(t) = 1$. $\beta(t)$, $\mathbf{X}'_i(t)$, $\mathbf{Y}'_i(t)$ and $\mathbf{y}'_k(t)$, respectively, are the Lagrange multipliers which are functions of the expectation values \mathbf{H} , $\alpha_i(t)$, $\alpha_i(t)$ and $\mathbf{G}_k(t)$. Then, by employing the density matrix we obtain the expression for the generalized entropy

$$\begin{aligned} S_{\text{eit}}(t) &\equiv S[\rho_c(t)] \\ &= -F'(t) + \beta(t)E(t) + \sum \mathbf{X}'_i(t)\alpha_i(t) + \sum \mathbf{Y}'_i(t)\alpha_i(t) + \sum \mathbf{y}'_k(t)\mathbf{G}_k(t), \end{aligned} \quad (3.5)$$

which is a function of all the expectation values of the observables chosen.

Now it is easy to verify that

$$\delta F'(t) = E(t)\delta\beta(t) + \sum \alpha_i(t)\delta\mathbf{X}'_i(t) + \sum \alpha_i(t)\delta\mathbf{Y}'_i(t) + \sum \mathbf{G}_k(t)\delta\mathbf{y}'_k(t), \quad (3.6)$$

and

$$\delta S_{\text{eit}}(t) = \beta(t)\delta E(t) + \sum \mathbf{X}'_i(t)\delta\alpha_i(t) + \sum \mathbf{Y}'_i(t)\delta\alpha_i(t) + \sum \mathbf{y}'_k(t)\delta\mathbf{G}_k(t). \quad (3.7)$$

Here, *mutatis mutandis* we have used the conventions (2.18).

It is clear that, at present, (3.7) plays a role similar to the Gibbs relation in equilibrium in nonequilibrium thermodynamics. Therefore, (3.7) gives the entropy production if we replace the symbol δ by d/dt ; it is

$$S_{\text{eit}}(t) = \beta(t)E(t) + \sum \mathbf{X}'_i(t)\alpha_i(t) + \sum \mathbf{Y}'_i(t)\alpha_i(t) + \sum \mathbf{y}'_k(t)\mathbf{G}_k(t), \quad (3.8)$$

which is essentially identical to the well-known formula given by Machlup and Onsager [15] for the generalized entropy. The differential form (3.7) is of the form of an extended Gibbs relation in the literature [1-4] in EIT. Equation (3.7) is presumed to be an exact differential in EIT and it has been the starting point of many theories.

The generalized canonical density matrix (3.3) yields, by definition, the correct expectation values of \mathbf{H} , $\{\alpha_i(t)\}$, $\{\alpha_i(t)\}$ and $\{\mathbf{G}_k(t)\}$. The expectation values of other observables are easily evaluated. For instance, we have

$$\begin{aligned} d^2\alpha_i(t)/dt^2 &= \text{Tr} d\rho_c(t)/dt \mathbf{B}_i \\ &= \sum C_{ij}[t]\mathbf{X}_j(t) + \sum D_{ij}[t]\mathbf{Y}_j(t) + \sum d_{ik}[t]\mathbf{y}_k(t), \end{aligned} \quad (3.9)$$

where

$$C_{ij}[t] = \text{Tr} \int dx \{ \rho_c(t) \}^{1-x} \mathbf{A}_j \{ \rho_c(t) \}^x \mathbf{B}_i, \quad (3.10)$$

$$D_{ij}[t] = \text{Tr} \int dx \{ \rho_c(t) \}^{1-x} \mathbf{B}_j \{ \rho_c(t) \}^x \mathbf{B}_i, \quad (3.11)$$

$$d_{ik}[t] = \text{Tr} \int dx \{ \rho_c(t) \}^{1-x} \mathbf{b}_k \{ \rho_c(t) \}^x \mathbf{B}_i. \quad (3.12)$$

Here, we have used the convention $\mathbf{A}_0 = \mathbf{H}$ and $\mathbf{X}_0(t) = \beta(t)$. It is worth to note here that the coefficients $D_{ij}[t]$ enjoy the reciprocity relation;

$$D_{ij}[t] = D_{ji}[t], \quad (i, j = 1, 2, \dots, f). \quad (3.13)$$

However, the coefficients $C_{ij}[t]$ have such a reciprocity only in an approximate sense in which we use the Gibbsian density matrix in place of the generalized canonical density matrix in (3.10). The equations of motion of $\mathbf{G}_k(t)$ are calculated as

$$d\mathbf{G}_k(t)/dt = \sum T_{ki}[t] \mathbf{X}_i(t) + \sum U_{ki}[t] \mathbf{Y}_i(t) + \sum V_{kj}[t] \mathbf{y}_j(t). \quad (3.14)$$

Here, the coefficients, $T_{ki}(t)$, $U_{ki}(t)$, and $V_{kj}(t)$ are given by

$$T_{ki}[t] = \text{Tr} \int dx \{ \rho_c(t) \}^{1-x} \mathbf{A}_i \{ \rho_c(t) \}^x \mathbf{b}_k, \quad (3.15)$$

$$U_{ki}[t] = \text{Tr} \int dx \{ \rho_c(t) \}^{1-x} \mathbf{B}_i \{ \rho_c(t) \}^x \mathbf{b}_k, \quad (3.16)$$

$$V_{kj}[t] = \text{Tr} \int dx \{ \rho_c(t) \}^{1-x} \mathbf{b}_j \{ \rho_c(t) \}^x \mathbf{b}_k (= V_{jk}[t]). \quad (3.17)$$

On the other hand, we obtain from (3.6)

$$F'(t) = E(t)\beta(t) + \sum \alpha_i(t) \mathbf{X}_i(t) + \sum \alpha_i(t) \mathbf{Y}_i(t) + \sum \mathbf{G}_k(t) \mathbf{y}_k(t). \quad (3.18)$$

Equations (3.9) and (3.14), together with (3.18), constitute the set of differential equations, first order in time, for all the Lagrange multipliers involved.

The extended observation level chosen is sufficient in view of the observables, $[i\mathbf{H}, \mathbf{B}_j]$, when

$$\text{Tr} \rho(t) [i\mathbf{H}, \mathbf{B}_j] \approx \text{Tr} \rho_c(t) [i\mathbf{H}, \mathbf{B}_j], \quad (3.19)$$

hold to a good approximation. In consequence, all we can say here is that, under the mild conditions (3.19), the application of the principle of maximum entropy recovers some useful results. By making use of the approximation similar to (2.21) for $\rho_c(t)$ into (3.19), one obtains at once

$$\alpha_i(t) = \sum E_{ij}[t] \mathbf{X}_j(t) + \sum F_{ij}[t] \mathbf{Y}_j(t) + \sum e_{ij}[t] y_j(t), \quad (3.20)$$

$$(i = 0, 1, \dots, f).$$

Here, the transport coefficients are given by

$$E_{ij}[t] = \beta(t) \int dx \{ \rho_0(t) \}^{1-x} [\mathbf{H}, \mathbf{A}_j] \{ \rho_0(t) \}^x [i\mathbf{H}, \mathbf{B}_i], \quad (3.21)$$

$$F_{ij}[t] = \beta(t) \int dx \{ \rho_0(t) \}^{1-x} [\mathbf{H}, \mathbf{B}_j] \{ \rho_0(t) \}^x [i\mathbf{H}, \mathbf{B}_i], \quad (3.22)$$

$$e_{ij}[t] = \beta(t) \int dx \{ \rho_0(t) \}^{1-x} [\mathbf{H}, \mathbf{b}_j] \{ \rho_0(t) \}^x [i\mathbf{H}, \mathbf{B}_i], \quad (3.23)$$

with

$$\rho_0(t) = \exp\{ F(t) - \beta(t)\mathbf{H} - \sum \mathbf{X}_i^s \mathbf{A}_i - \sum \mathbf{Y}_i^s \mathbf{B}_i - \sum y_j^s \mathbf{b}_j \}. \quad (3.24)$$

$F_{ij}[t]$ s enjoy the following reciprocity relations;

$$F_{ij}[t] = F_{ji}[t]. \quad (i, j = 0, 1, \dots, f). \quad (3.25)$$

On the other hand, $E_{ij}[t]$ have the following properties

$$E_{ij}[t] = -E_{ji}[t], \quad (i, j = 0, 1, \dots, f), \quad (3.25')$$

only when all the terms in the exponential are discarded except the first two terms, $F(t)$ and $-\beta(t)\mathbf{H}$. $E_{ij}(t)$ are antisymmetric only in the approximate sense. This is a desired property, because close to an equilibrium the relaxation times of the fluxes would be macroscopically long. Since $\mathbf{X}'_i(t)$, $\mathbf{Y}'_i(t)$ and $y'_i(t)$ are the functions of all the expectation values $\alpha_i(t)$, $\alpha_i(t)$ and $\mathbf{G}_i(t)$, (3.20) are differential equations, second order in time, which are sought-for equations for dissipative fluxes in EIT.

In the many theories the differential equations for fluxes are derived, not from a time evolution but from the form of the extended entropy production and the requirement of the second law of thermodynamics. The

resulting equations so obtained involve the two kinds of conception, transport coefficient and relaxation time. In the present theory, we have conclusively demonstrated how these fundamental quantities are calculated in statistical mechanics. We believe that the present formalism in its generality can be justified. The only important point is that the phenomenological laws, *second order in time*, does not contradict the microscopic equations of motion. The choice of the observables set in the principle of maximum entropy has played a prominent role in our discussion.

In the present development, (3.19) is necessary to establish an equivalence between the two entities, $\rho(t)$ and $\rho_c(t)$, with respect to the chosen observation level. From the manner of definition $\rho(t)$ possesses more information about the dynamics of the system than is actually contained in the generalized canonical density matrix $\rho_c(t)$. This corresponds to a coarse graining of the density matrix which could contribute to an additional entropy production other than is presented by (3.8).

In the classical theory the contraction of information of this kind is not of importance for the formulating irreversible thermodynamics because it might be microscopic in nature. Hence, we often identify the statistical entropy $S[\rho(t)]$ with the maximized entropy $S[\rho_c(t)]$. This type of information loss would be irrelevant only if the conserved variables are chosen as the basic thermodynamic observables, since it is assumed that there is a clear separation of dynamical behavior of the system into microscopically time and macroscopically long time behavior. Such a hierarchical structure regarding time in dynamics, albeit assumed as fulfilled very generally by macroscopic systems, may be proved only very few cases. In the realm of EIT in which we focus our attention not only to the conserved observables but also to the nonconserved observables, thus, we cannot neglect its contribution to the over all entropy production. We must notice the fact that the relaxation times of any nonconserved variables cannot be shorter than the collision time, if it

existed at all. Thus, the relaxation times of the higher fluxes, such as $[i\mathbf{H}, \mathbf{B}_i]$, $[i\mathbf{H}, [i\mathbf{H}, \mathbf{B}_i]]$, and etc. might be of the same order. Accordingly, in retrospect the conditions are not independent of our setting of the observation level. This information loss due to the coarse graining can be measured in terms of a relative entropy which is defined by [14]

$$S[\rho(t) | \rho_c(t)] \equiv \text{Tr} \rho(t) [\ln \rho(t) - \ln \rho_c(t)] \geq 0. \quad (3.26)$$

This is non-negative by Klein's inequality. Since $\rho(t)$ and $\rho_c(t)$ are equivalent in the sense mentioned above, it is readily verified that

$$S[\rho(t) | \rho_c(t)] = S[\rho_c(t)] - S[\rho(t)], \quad (3.27)$$

or equivalently,

$$S[\rho(t)] = S[\rho_c(t)] - S[\rho(t) | \rho_c(t)]. \quad (3.27')$$

Hence, taking the derivatives with respect to time of both sides of (3.27') yields at once

$$\begin{aligned} \sigma_{\text{ent}}(t) &\equiv dS[\rho(t)]/dt \\ &= \sum \mathbf{X}_i(t) \alpha_i(t) + \sum \mathbf{Y}_i(t) \alpha_i(t) + \sum \mathbf{y}_j(t) \mathbf{G}_j(t) - \sigma_L(t), \end{aligned} \quad (3.28)$$

where

$$\sigma_L(t) = dS[\rho(t) | \rho_c(t)]/dt \geq 0. \quad (3.29)$$

Here it should be realized that

These results are in agreement with Eu's proposals [16]. The two quantities, $S[\rho_c(t)]$ and $-S[\rho(t) | \rho_c(t)]$, respectively, are corresponding to the compensation and the β functions in Eu's theory [16].

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