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Citation
数理解析研究所講究録 (1997), 982: 52-67

Issue Date
1997-03

URL
http://hdl.handle.net/2433/60921

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
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1 INTRODUCTION

The main problem presented in this paper is to consider a scaling limit of a model in quantum electrodynamics which describes an interaction of $N$-nonrelativistic charged particles and a quantized radiation field in the Coulomb gauge with the dipole approximation. The model we consider is called "the Pauli-Fierz model". Authors in [5,6] have studied a scaling limit of the Pauli-Fierz model with one-nonrelativistic charged particle. We may well extend the scaling limit of one-particle system to $N$-particles system.

The Pauli-Fierz Hamiltonians $H_\rho$ with $N$-nonrelativistic charged particles in the Coulomb gauge with the dipole approximation are defined as operators acting in the Hilbert space

$$L^2(\mathbb{R}^d) \otimes \ldots \otimes L^2(\mathbb{R}^d) \otimes \mathcal{F}(\mathcal{W}) \cong L^2(\mathbb{R}^{dN}) \otimes \mathcal{F}(\mathcal{W})$$

by

$$H_\rho = \frac{1}{2m} \sum_{j=1}^{N} \sum_{\mu=1}^{d} \left( -i\hbar D_{j}^\mu \otimes I - eI \otimes A_\mu(\rho_j) \right)^2 + I \otimes H_b,$$

where $D_{j}^\mu$ is the differential operator with respect to the $j$-th variable in the $\mu$-th direction, $A_\mu(\rho_j)$ the quantized radiation field in the $\mu$-th direction with an ultraviolet cut-off function $\rho_j$ in the Coulomb gauge, $H_b$ the free Hamiltonian in $\mathcal{F}(\mathcal{W})$, and $m, e, \hbar$ the mass of the particles, the charge of the particles, the Planck constant divided $2\pi$, respectively.

Note that $A_\mu$ is depend on the speed of light $c$. We introduce the following scaling.

$$c(\kappa) = c\kappa, e(\kappa) = e\kappa^{-\frac{1}{2}}, m(\kappa) = m\kappa^{-2}. \quad (1.1)$$
Then the scaled Hamiltonian $H_{\vec{\rho}}(\kappa)$ amounts to

$$\frac{\hbar^2 \kappa^2}{2m} \Delta \otimes I + \kappa I \otimes H_b + \frac{1}{2m} \sum_{j=1}^{N} \sum_{\mu=1}^{d} \left( \kappa^2 e \hat{h}_{\mu} D_{\mu}^j \otimes A_{\mu}(\rho_j) + e^2 I \otimes A_{\mu}^2(\rho_j) \right).$$

Defining a pseudo differential operator $E^{\text{REN}}(D, \kappa)$ in $L^2(\mathbb{R}^{dN})$ with a symbol $E^{\text{REN}}(p, \kappa)$ such that $E^{\text{REN}}(p, \kappa) \to \infty$ as $\kappa \to \infty$, we define a Hamiltonian $H_{\vec{\rho}}^{\text{REN}}(\kappa)$ by

$$-E^{\text{REN}}(D, \kappa) \otimes I + \kappa I \otimes H_b + \frac{1}{2m} \sum_{j=1}^{N} \sum_{\mu=1}^{d} \left( \kappa^2 e \hat{h}_{\mu} D_{\mu}^j \otimes A_{\mu}(\rho_j) + e^2 I \otimes A_{\mu}^2(\rho_j) \right).$$

Consequently, we shall show the following for some $\vec{\rho} = (\rho_1, \ldots, \rho_N)$ and scalar potentials $V$ with some conditions (Theorem 3.7):

$$s - \lim_{\kappa \to \infty} (H_{\vec{\rho}}^{\text{REN}}(\kappa) + V \otimes I - z)^{-1} = \mathcal{U}(\infty) \left\{ (E^{\infty}(D) + V_{\text{eff}} - z)^{-1} \otimes P_0 \right\} \mathcal{U}^{-1}(\infty),$$

where $E^{\infty}(D)$ is a pseudo differential operator in $L^2(\mathbb{R}^{dN})$, $V_{\text{eff}}$ a multiplication operator, which is called "effective potential", and $P_0$ a projection on $\mathcal{F}(\mathcal{W})$. Despite the fact that in the case of one-particle system the effective potential $V_{\text{eff}}$ is the Gaussian transformation of a given scalar potential $V$, we shall show that in $N$-particles system, it is not necessary to be the Gaussian transformation. Actually it is determined by a matrix $\tilde{\Lambda}^{\infty} = (\tilde{\Lambda}_{ij}^{\infty})_{1 \leq i, j \leq N}$ which is defined by the ultraviolet cut-off functions $\rho_j$;

$$\tilde{\Lambda}_{ij}^{\infty} = \frac{1}{2} \frac{d-1}{d} \left( \frac{\hbar}{mc} \right) \frac{e^2}{\hbar c} \int_{\mathbb{R}^d} dk \hat{\rho}_i(k) \hat{\rho}_j(k) \omega(k)^3.$$

2 THE PAULI-FIERZ MODEL

To begin with, let us introduce some preliminary notations. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. We denote the inner product and the associated norm by $\langle *, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$ respectively. The inner product is linear in $\cdot$ and antilinear in $\ast$. The domain of an operator $A$ in $\mathcal{H}$ is denoted by $D(A)$. A notation $\hat{f}$ (resp. $\check{f}$) denotes the Fourier transformation (resp. the inverse Fourier transformation) of $f$ and $\bar{f}$ the complex conjugate of $f$. Let

$$\mathcal{W} \equiv \left( L^2(\mathbb{R}^d) \oplus \ldots \oplus L^2(\mathbb{R}^d) \right)_{d-1}. $$
We define the Boson Fock space over $\mathcal{W}$ by

$$\mathcal{F}(\mathcal{W}) \equiv \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{W}),$$

where $\otimes_{s}^{0}\mathcal{W} \equiv \mathbb{C}$ and $\otimes_{s}^{n}\mathcal{W}$ ($n \geq 1$) denotes the n-fold symmetric tensor product. Put

$$\mathcal{F}^\infty(\mathcal{W}) \equiv \bigcup_{N=0}^{\infty} \mathcal{F}_N(\mathcal{W}) \bigoplus \{0\}.$$

The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ ($f \in \mathcal{W}$) act on $\mathcal{F}^\infty(\mathcal{W})$ and leave it invariant with the canonical commutation relations (CCR): for $f, g \in \mathcal{W}$

$$[a(f), a^\dagger(g)] = \langle \bar{f}, g \rangle_{\mathcal{W}},$$

$$[a^\dagger(f), a^\dagger(g)] = 0,$$

where $[A, B] = AB - BA$, $a^\dagger$ denotes either $a$ or $a^\dagger$. Furthermore,

$$\langle a^\dagger(f)\Phi, \Psi \rangle_{\mathcal{F}(\mathcal{W})} = \langle \Phi, a(\bar{f})\Psi \rangle_{\mathcal{F}(\mathcal{W})}, \quad \Phi, \Psi \in \mathcal{F}^\infty(\mathcal{W}).$$

We define polarization vectors $\epsilon^r (r = 1, \ldots, d-1)$ as measurable functions $\epsilon^r : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$\epsilon^r(k)\epsilon^s(k) = \delta_{rs}, \quad \epsilon^r(k)k = 0, \quad a.e. k \in \mathbb{R}^d.$$

The $\mu$-th direction time-zero smeared radiation field in the Coulomb gauge with the dipole approximation is defined as operators acting in $\mathcal{F}(\mathcal{W})$ by

$$A_\mu(f) = \frac{1}{\sqrt{2}} \left\{ a^\dagger \left( \oplus_{r=1}^{d-1} \frac{\sqrt{\omega_k e^\mu_{\bar{r}}}}{\sqrt{\omega}} \right) + a \left( \oplus_{r=1}^{d-1} \frac{\sqrt{\omega_k e^\mu_{\bar{r}}}}{\sqrt{\omega}} \right) \right\},$$

where $\omega(k) = |k|$ and $\bar{g}(k) = g(-k)$. Let $\Omega = (1, 0, 0, \ldots) \in \mathcal{F}(\mathcal{W})$. For a nonnegative self-adjoint operator $h : \mathcal{W} \rightarrow \mathcal{W}$, we denote “the second quantization of $h$” by $d\Gamma(h)$. Put $\tilde{\omega} = \omega \oplus \ldots \oplus \omega$. The free Hamiltonian $H_0$ in $\mathcal{F}(\mathcal{W})$ is defined by

$$H_0 \equiv \hbar cd\Gamma(\tilde{\omega}).$$
The Pauli-Fierz Hamiltonians with $N$-nonrelativistic charged particles interacting with the quantized radiation field with the dipole approximation in the Coulomb gauge read as follows:

$$H_{\vec{\rho}} \equiv H_{\rho_{1},\ldots,\rho_{N}} \equiv \frac{1}{2m} \sum_{j=1}^{N} \sum_{\mu=1}^{d} \left(-i\hbar D_{\mu}^{j} \otimes I - eI \otimes A_{\mu}(\rho_{j})\right)^{2} + I \otimes H_{b},$$

acting in

$$\frac{L^{2}(\mathbb{R}^{d}) \otimes \ldots \otimes L^{2}(\mathbb{R}^{d})}{N} \otimes \mathcal{F}(\mathcal{W}) \cong \int_{\mathbb{R}^{dN}}^{\mathbb{R}^{dN}} \mathcal{F}(\mathcal{W}) dx.$$

We introduce the scaling (1.1). For objects $A$ containing the parameters $c, e, m$, we denote the scaled object by $A(\kappa)$ throughout this paper. We define classes $P$ and $\tilde{P}$ of sets of functions as follows:

**Definition 2.1** $\vec{\rho} = (\rho_{1}, \ldots, \rho_{N})$ is in $P$ if and only if

1. $\hat{\rho}_{j}, j=1, \ldots, N$ are rotation invariant, $\hat{\rho}_{j}(k) = \hat{\rho}_{j}(\|k\|)$, and real-valued,
2. $\hat{\rho}_{j}/\omega, \sqrt{\omega} \hat{\rho}_{j}, \sqrt{\omega} \hat{\rho}_{j} \in L^{2}(\mathbb{R}^{d}).$

Moreover $\vec{\rho}$ is in $\tilde{P}$ if and only if in addition to (1) and (2) above

3. $\hat{\rho}_{j}/\omega \sqrt{\omega} \in L^{2}(\mathbb{R}^{d})$ and there exist $0 < \alpha < 1$ and $1 \leq \epsilon$ such that $\hat{\rho}_{i}(\sqrt{\cdot}) \hat{\rho}_{j}(\sqrt{\cdot})(\sqrt{\cdot})^{d-2} \in \text{Lip}(\alpha) \cap L^{\epsilon}([0, \infty))$, where $\text{Lip}(\alpha)$ is the set of the Lipschitz continuous functions on $[0, \infty)$ with the degree $\alpha$,
4. $\sup_{k} |\hat{\rho}_{j}(k)\omega^{d_{2}-\frac{1}{2}}(k)| < \infty, \sup_{k} |\hat{\rho}_{j}(k)\omega^{d-\frac{1}{2}}(k)| < \infty, j = 1, \ldots, N.$

Put

$$H_{0} = -\frac{1}{2m} \hbar^{2} \Delta \otimes I + I \otimes H_{b},$$

where $\Delta$ is the Laplacian in $\mathbb{R}^{dN}$. It is well known that $H_{0}$ is a nonnegative self-adjoint operator on $D(H_{0}) = D\left(-\frac{1}{2m} \hbar^{2} \Delta \otimes I\right) \cap D(I \otimes H_{b}).$
Proposition 2.2 ([3,4]) For $\vec{\rho} \in P$ and $\kappa > 0$, the operator $H_{\rho}(\kappa)$ is self-adjoint on $D(H_{0})$ and essentially self-adjoint on any core of $H_{0}$ and nonnegative.

Let $F = F \otimes I$, where $F$ denotes the Fourier transform in $L^{2}(\mathbb{R}^{dN})$. It is clear that operators $FH_{\rho}F^{-1}$ can be decomposable as follows:

$$FH_{\rho}(\kappa)F^{-1} = \int_{\mathbb{R}^{dN}} H_{0}(p, \kappa) dp,$$

where

$$H_{0}(p, \kappa) = \frac{1}{2m} \sum_{j=1}^{N} \sum_{\mu=1}^{d} \left( \kappa \hbar p_{\mu}^{2} - eA_{\mu}(\rho) \right)^{2} + \kappa H_{b}. $$

Proposition 2.3 ([3,4]) For $\vec{\rho} \in P$ and $\kappa > 0$, the operator $H_{\rho}(p, \kappa)$ is self-adjoint on $D(H_{b})$ and essentially self-adjoint on any core of $H_{b}$ and nonnegative.

Set Hilbert spaces $M_{d} = \{ f \mid \int |f(k)|^{2} \omega(k)^{d} dk < \infty \}$ and put $\mathcal{W}_{\alpha} = M_{\alpha} \oplus \ldots \oplus M_{\alpha}$, $\alpha \in \mathbb{R}$.

The following lemma is the key lemma to investigating the scaling limits.

Lemma 2.4 ([9]) Let $\vec{\rho} \in \tilde{P}$ and $\kappa > 0$ be sufficiently large. Then there exist a Hilbert Schmidt operator $W_{-}$, a bounded operator $W_{+}$, and $L_{j} = (L_{j}^{1}, ..., L_{j}^{d}), L_{j}^{\mu} \in \mathcal{W}, j = 1, ..., N, \mu = 1, ..., d$ such that, if we put for $p^{j} \in \mathbb{R}^{d}, j = 1, ..., N$

$$B(f, p) = a^{1}(W_{-}f) + a(W_{+}f) + \sum_{j=1}^{N} \langle L_{j}p^{j}, f \rangle_{\mathcal{W}},$$

$$B^{1}(f, p) = a^{1}(W_{+}f) + a(W_{-}f) + \sum_{j=1}^{N} \langle L_{j}p^{j}, f \rangle_{\mathcal{W}},$$

then

$$[B(f, p), B^{1}(g, p)] = \langle f, g \rangle_{\mathcal{W}},$$

$$[B^{1}(f, p), B^{1}(g, p)] = 0, \text{ on } \mathcal{F}^{\infty}(\mathcal{W}),$$

and for $\Phi, \Psi \in \mathcal{F}^{\infty}(\mathcal{W})$,

$$\langle B^{1}(f, p) \Phi, \Psi \rangle_{\mathcal{F}(\mathcal{W})} = \langle \Phi, B(f, p) \Psi \rangle_{\mathcal{F}(\mathcal{W})},$$
moreover

\[ [ H_{\vec{\rho}}(p), B^\#(f, p) ] = \pm B^\#(\hbar \omega f, p), \text{ on } \mathcal{F}^\infty(\mathcal{W}) \cap D(H_b^{\frac{3}{2}}), \]

where \( f \in \mathcal{W}_0 \cap \mathcal{W}_2 \) and + (resp.-) corresponds to \( B^\dagger \) (resp. \( B \)).

By virtue of Lemma 2.4, we see the following.

**Corollary 2.5** Let \( \vec{\rho} \in \tilde{P} \) and \( \kappa \) be sufficiently large. Then for \( \Phi \in D(H_b) \),

\[
\exp \left( i \frac{t}{\hbar} H_{\vec{\rho}}(p) \right) B^\#(f, p) \exp \left( -i \frac{t}{\hbar} H_{\vec{\rho}}(p) \right) \Phi = B^\#(e^{i\omega t} f, p) \Phi.
\]

## 3 SCALING LIMITS

In this section, we construct a unitary operator which implements unitary equivalence of the Pauli-Fierz Hamiltonian and a decoupled Hamiltonian. Moreover we investigate a scaling limit of the Pauli-Fierz Hamiltonian. Unless otherwise stated in this section, we suppose that \( \kappa > 0 \) is sufficiently large. From Lemma 2.4 (1) it follows that there exist two unitary operators \( U(\kappa) \) (\( p \) independent) and \( S(p, \kappa) \) such that ([6,Section III])

\[
U^{-1}(\kappa)S(p, \kappa)^{-1}B^\#(f, p, \kappa)S(p, \kappa)U(\kappa) = a^\#(f), \quad f \in \mathcal{W}.
\]

Concretely \( S(p, \kappa) \) is given by

\[
S(p, \kappa) = \exp \left( \sum_{i,j=1}^{N} \frac{e^2}{\kappa^2} \rho_{ij} \left( a \left( \sum_{\tau=1}^{d-1} e^r \frac{M_{ij}(\kappa) \hat{\rho}_j}{\sqrt{2hc^3 \omega^3}} \right) - a^\dagger \left( \sum_{\tau=1}^{d-1} e^r \frac{M_{ij}(\kappa) \hat{\rho}_j}{\sqrt{2hc^3 \omega^3}} \right) \right) \right),
\]

where \( (M_{ij}(\kappa))_{1 \leq i,j \leq N} \) is a matrix such that

\[
\lim_{\kappa \to \infty} \frac{M_{ij}(\kappa)}{\kappa^2} = \delta_{ij} \frac{1}{m}.
\]

**Theorem 3.1** Suppose \( \vec{\rho} \in \tilde{P} \). Then putting \( S(p, \kappa)U(\kappa) = \mathcal{U}(p, \kappa) \), we see that \( \mathcal{U}(p, \kappa) \) maps \( D(H_b) \) onto itself with

\[
\mathcal{U}(p, \kappa)H_{\vec{\rho}}(p, \kappa)\mathcal{U}^{-1}(p, \kappa) = \kappa H_b + E(p, \kappa),
\]

(3.2)
where

\[
E(p, \kappa) = \frac{\hbar^2}{2m} \sum_{i=1}^{N} \sum_{\mu=1}^{d} \left( \kappa p_{\mu} + \kappa \sum_{j=1}^{N} p_{\mu} \Delta_{\mu \mu}^{ij} (\kappa) \right)^2 + \Box(\kappa),
\]

\[
\Delta_{\mu \mu}^{ij} (\kappa) = \frac{1}{\kappa^3} \frac{e^2}{2c^2} \sum_{k=1}^{N} \sum_{r,s=1}^{d} \left( \frac{\epsilon_{\mu}^{r} \hat{p}_{i}}{\sqrt{\omega}}, (I + W_{-}(\kappa) W_{-}^{-1}(\kappa))^{(r,s)} \frac{\epsilon_{\mu}^{s} \hat{p}_{i}}{\sqrt{\omega}} \right)_{L^2(\mathbb{R}^d)},
\]

\[
\Box(\kappa) = \frac{e^2}{4mc} \sum_{i=1}^{N} \sum_{r,s=1}^{d} \left( \frac{\epsilon_{\mu}^{r} \hat{p}_{i}}{\sqrt{\omega}}, (I - W_{-}(\kappa) W_{-}^{-1}(\kappa))^{(r,s)} \frac{\epsilon_{\mu}^{s} \hat{p}_{i}}{\sqrt{\omega}} \right)_{L^2(\mathbb{R}^d)}.
\]

**Proof:** For simplicity, we omit the symbol \( \kappa \). Put \( \mathcal{U}(p) \Omega \equiv \Omega(p) \). From [6, Proposition 2.4, Lemma 5.9] it follows that \( \Omega(p) \in D(H) \). Then \( \Omega(p) \in D(B(f,p)) \). By virtue of Corollary 2.5 and (3.1), we can see that for all \( f \in \mathcal{W} \)

\[
B(f,p) \exp \left( i \frac{t}{\hbar} H(f,p) \right) \Omega(p) = 0.
\]

(3.3)

The equation (3.3) implies that there exists a positive constant \( E(p) \) such that

\[
\exp \left( i \frac{t}{\hbar} H(p) \right) \Omega(p) = \exp \left( i \frac{t}{\hbar} E(p) \right) \Omega(p).
\]

(3.4)

Hence from Corollary 2.5, (3.1), (3.4) and the denseness of

\[
\mathcal{L} \left\{ B^t(f_1) \ldots B^t(f_n) \Omega(p), \Omega(p) \mid f_j \in \mathcal{W}, j = 1, \ldots, n, n \geq 1 \right\},
\]

one can get (3.2). The constant \( E(p) \) is explicitly given by

\[
E(p) = \frac{<H(p)\Omega(p), \Omega>_\mathcal{W}}{<\Omega(p), \Omega>_\mathcal{W}}.
\]

It completes the proof. \( \square \)

The positive constant \( E(p, \kappa) \) can be rewritten by:

\[
E(p, \kappa) = \frac{\kappa^2 \hbar^2}{2m} p^2 + E^{RE}(p, \kappa) + \tilde{E}(p, \kappa),
\]

where

\[
\tilde{E}(p, \kappa) = \frac{\kappa^2 \hbar^2}{2m} \sum_{i,j=1}^{N} \sum_{\mu, \nu=1}^{d} p_{\mu} b_{\mu \nu}^{ij} (\kappa) p_{\nu}.
\]

(3.5)
\[ b_{\mu\nu}^{ij}(\kappa) = \sum_{k=1}^{N} \sum_{\alpha=1}^{d} \left( \frac{\Delta_{\nu\alpha}^{jk}(\mathcal{K}) + \overline{\Delta_{\nu}^{j}\alpha}(k\kappa)}{2} \right) \left( \frac{\Delta_{\mu\alpha}^{ik}(\mathcal{K}) + \overline{\Delta ik}(\mu\alpha\kappa)}{2} \right), \]

\[ E^{\text{REN}}(p, \kappa) = E(p, \kappa) - \frac{\kappa^{2}k^{2}}{2m}p^{2} - \tilde{E}(p, \kappa). \]

Note that since \((b_{\mu\nu}^{ij}(\kappa))_{1 \leq i,j \leq N, 1 \leq \mu, \nu \leq d}\) is nonnegative and symmetric \(dN \times dN\) matrix, we have \(\tilde{E}(p, \kappa) \geq 0\) for any \(p \in \mathbb{R}^{dN}\). We define

\[ H_{\rho}\overset{\text{REN}}{\rightarrow}(\kappa) = -E^{\text{REN}}(D, \kappa) \otimes I + \kappa I \otimes H_{b} \]

\[ + \frac{1}{2m} \sum_{j=1}^{N} \sum_{\mu=1}^{d} \left( -2\kappa e_{\mu}D_{\mu}^{j} \otimes A_{\mu}(p_{j}) + e^{2}I \otimes A_{\mu}(p_{j})^{2} \right), \]

\[ \overline{H_{\rho}}(\kappa) = \tilde{E}(D, \kappa) \otimes I + \kappa I \otimes H_{b}, \]

where \(E^{\text{REN}}(D, \kappa)\) and \(\tilde{E}(D, \kappa)\) are pseudo differential operators on \(L^{2}(\mathbb{R}^{dN})\) with symbols \(E^{\text{REN}}(p, \kappa)\) and \(\tilde{E}(p, \kappa)\) respectively.

**Theorem 3.2** Suppose \(\rho \in \bar{P}\). Then \(H_{\rho}\overset{\text{REN}}{\rightarrow}(\kappa)\) and \(\overline{H_{\rho}}(\kappa)\) are essentially self-adjoint on any core of \(H_{0}\) and bounded from below.

**Remark 3.3** Write

\[ E(p, \kappa) = \frac{\hbar^{2}\kappa^{2}}{2m}p^{2} + \sum_{\mu=1}^{d} \sum_{i=1}^{N} \frac{\hbar^{2}\kappa^{2}}{m}p_{\mu}^{2}p_{\mu}(\kappa) + \sum_{\mu=1}^{d} \sum_{i=1}^{N} \frac{\hbar^{2}\kappa^{2}}{2m}p_{\mu}^{2}(\kappa)^{2} + \Box(\kappa). \]  

(3.6)

Then the first and second terms on the right hand side of (3.6) diverge as \(\kappa \rightarrow \infty\) for \(p \neq 0\), but the rest terms not. Actually we see that

\[ \lim_{\kappa \rightarrow \infty} \frac{\hbar^{2}\kappa^{2}}{2m} \sum_{\mu=1}^{d} \sum_{i=1}^{N} \overline{p}_{\mu}(\kappa)^{2} \]

\[ = \frac{1}{2m} \left( \frac{e^{2}}{2mc^{2}} \right) \left( \frac{d-1}{d} \right)^{2} \sum_{\alpha=1}^{d} \sum_{k=1}^{N} \left( \sum_{j=1}^{N} \frac{\hbar p_{\alpha}^{2} \left( \frac{\hat{\rho}_{j}}{\sqrt{\omega^{3}}}, \frac{\hat{\rho}_{k}}{\sqrt{\omega}} \right)}{L^{2}(\mathbb{R}^{d})} \right)^{2}, \]

\[ \equiv E^{\infty}(p). \]

Then, by (3.2), concerning an asymptotic behavior of \(H_{\rho}(\kappa)\) as \(\kappa \rightarrow \infty\), we should subtract the first and second terms in the right hand side of (3.6) from the original Hamiltonian \(H_{\rho}(\kappa)\). However one can not say that \(\overline{p}_{\mu}(\kappa)^{2}\) is real and nonnegative for any \(p \in \mathbb{R}^{dN}\). To
guarantee the nonnegative self-adjointness of the Hamiltonian $H_{\vec{\rho}}^{\text{REN}}(\kappa)$ with the divergence terms subtracted, we should define $\tilde{E}(p, \kappa)$ such as (3.5). In this sense, we may say that the operator $H_{\vec{\rho}}^{\text{REN}}(\kappa)$ has an interpretation of the Hamiltonian $H_{\rho}(\kappa)$ with the infinite self-energy of the nonrelativistic particles subtracted.

We define

$$U(\kappa) = F^{-1} \left( \int_{\mathbb{R}^{dN}} U(\kappa, p) dp \right) F.$$

Then we have the following theorem.

**Theorem 3.4** ([6]) Suppose that $\vec{\rho} \in \tilde{P}$. Then

$$s - \lim_{\kappa \to \infty} U(\kappa) = \exp \left( \sum_{j=1}^{N} \frac{e}{m} D^j \otimes \left\{ a \left( \frac{e_{\mu} \hat{\rho}_j}{\sqrt{2\hbar c^3 \omega^3}} \right) - a^\dagger \left( \frac{e_{\mu} \hat{\rho}_j}{\sqrt{2\hbar c^3 \omega^3}} \right) \right\} \right),$$

$$\equiv U(\infty).$$

We take scalar potentials $V$ to be real-valued measurable functions on $\mathbb{R}^{dN}$ and put

$$C_\kappa(V) = U^{-1}(\kappa)(V \otimes I)U(\kappa), \quad C(V) = U^{-1}(\infty)(V \otimes I)U(\infty).$$

We introduce conditions (V–1) and (V–2) as follows.

(V–1) For sufficiently large $\kappa > 0$, $D(\tilde{E}(D, \kappa)) \subset D(V)$ and for $\lambda > 0$, $V(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is bounded with

$$\lim_{\lambda \to \infty} ||V(\tilde{E}(D, \kappa) + \lambda)^{-1}|| = 0,$$

where the convergence is uniform in sufficiently large $\kappa > 0$.

(V–2) For $\lambda > 0$, $V(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is strongly continuous in $\kappa$ and

$$s - \lim_{\kappa \to \infty} V(\tilde{E}(D, \kappa) + \lambda)^{-1} = V(E^\infty(D) + \lambda)^{-1}.$$
The condition (3.7) yields that, by the Kato-Rellich theorem and commutativity of $\mathcal{U}(\kappa)$ and $(\tilde{E}(D, \kappa) + \lambda)^{-1}$, operators $\tilde{E}(D, \kappa) \otimes I + C_{\kappa}(V)$ are essentially self-adjoint on any core of $D(\tilde{E}(D, \kappa) \otimes I)$ and uniformly bounded from below in sufficiently large $\kappa > 0$. Moreover since $I \otimes H_b$ is nonnegative and commute with $\tilde{E}(D, \kappa) \otimes I$, one can see that

$$\overline{H}_{\vec{\rho}}(V, \kappa) \equiv \tilde{E}(D, \kappa) \otimes I + C_{\kappa}(V) + \kappa I \otimes H_b$$

is essentially self-adjoint on any core of $D(\tilde{E}(D, \kappa) \otimes I + \kappa I \otimes H_b)$ and uniformly bounded from below in sufficiently large $\kappa > 0$. In particular, $D(H_0)$ is a core of $\overline{H}_{\vec{\rho}}(V, \kappa)$. Put

$$H_{\vec{\rho}}^{\text{REN}}(V, \kappa) \equiv H_{\vec{\rho}}^{\text{REN}}(\kappa) + V \otimes I.$$

**Theorem 3.5** Let $\vec{\rho} \in \tilde{P}$. Suppose that $V$ satisfies (V–1) and (V–2). Then, for sufficiently large $\kappa > 0$, the operator $H_{\vec{\rho}}^{\text{REN}}(V, \kappa)$ is essentially self-adjoint on $D(H_0)$ and bounded from below uniformly in sufficiently large $\kappa > 0$. Moreover the unitary operator $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself and for $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large,

$$(3.8) \quad (H_{\vec{\rho}}^{\text{REN}}(V, \kappa) - z)^{-1} = \mathcal{U}(\kappa) (\overline{H}_{\vec{\rho}}(V, \kappa) - z)^{-1} \mathcal{U}^{-1}(\kappa).$$

**Proof:** Since $\mathcal{U}(\kappa)$ maps $D(I \otimes H_b)$ onto itself (see Theorem 3.1) and $-\Delta \otimes I$ commutes with $\mathcal{U}(\kappa)$ on $D(-\Delta \otimes I)$, $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself. Put

$$S_0^\infty(\mathbb{R}^d) = \{ f \in L^2(\mathbb{R}^d); \hat{f} \in C_0^\infty(\mathbb{R}^d) \}.$$

At first, by Theorem 3.1, we see that for $\Phi \in S_0^\infty(\mathbb{R}^d) \hat{\otimes} D(H_b),$

$$(3.9) \quad H_{\vec{\rho}}^{\text{REN}}(V, \kappa)\Phi = \mathcal{U}(\kappa)\overline{H}_{\vec{\rho}}(V, \kappa)\mathcal{U}^{-1}(\kappa)\Phi.$$ 

By a limiting argument we can extend (3.9) to $\Phi \in D(H_0)$. Since $D(H_0)$ is a core of $\overline{H}_{\vec{\rho}}(V, \kappa)$ and $\mathcal{U}(\kappa)$ maps $D(H_0)$ onto itself, the right hand side of (3.9) is essentially self-adjoint on $D(H_0)$. So is the left hand side of (3.9). (3.8) can be easily shown. $\square$
We want to consider a scaling limit of $H_{\tilde{\rho}}^{\text{REN}}(V, \kappa)$ as $\kappa \to \infty$. Let $V$ satisfy (V - 1). Then since $D(C(V)) \supset D(-\Delta) \otimes D(H_b)$, one can define, for $\Phi \in \mathcal{F}(\mathcal{W})$ and $\Psi \in D(H_b)$, a symmetric operator $E_{\Phi, \Psi}(C(V))$ with $D(E_{\Phi, \Psi}(C(V))) = D(-\Delta)$ by

$$\langle f, E_{\Phi, \Psi}(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} = \langle f \otimes \Phi, C(V)(g \otimes \Psi) \rangle_{\mathcal{F}}, \quad f \in L^2(\mathbb{R}^{dN}), g \in D(-\Delta).$$

In particular, we call $E_{\Omega, \Omega}(C(V)) \equiv E_{\Omega}(C(V))$ “the partial expectation of $C(V)$ with respect to $\Omega$”.

**Theorem 3.6** Let $\tilde{\rho} \in \tilde{P}$. Suppose that $V$ satisfies the conditions (V - 1) and (V - 2). Then for $z \in \mathbb{C} \setminus \mathbb{R}$ or $z < 0$ with $|z|$ sufficiently large,

$$s - \lim_{\kappa \to \infty} (H_{\tilde{\rho}}^{\text{REN}}(V, \kappa) - z)^{-1} = \mathcal{U}(\infty) \left( \{E^\infty(D) + E_{\Omega}(C(V)) - z\}^{-1} \otimes P_0 \right) \mathcal{U}^{-1}(\infty),$$

where $P_0$ is the projection from $\mathcal{F}(\mathcal{W})$ to the one dimensional subspace $\{\alpha \Omega | \alpha \in \mathbb{C}\}$.

**Proof:** By (V - 1) and (V - 2), we see that

(V-1)' For sufficiently large $\kappa > 0$, $D(\tilde{E}(D, \kappa)) \subset D(C_\kappa(V))$ and for $\lambda > 0$,

$$C_\kappa(V)(\tilde{E}(D, \kappa) + \lambda)^{-1}$$

is bounded with

$$\lim_{\lambda \to \infty} ||C_\kappa(V)(\tilde{E}(D, \kappa) + \lambda)^{-1}|| = 0,$$

where the convergence is uniform in sufficiently large $\kappa > 0$.

(V-2)' For $\lambda > 0$, $C_\kappa(V)(\tilde{E}(D, \kappa) + \lambda)^{-1}$ is strongly continuous in $\kappa$ and

$$s - \lim_{\kappa \to \infty} C_\kappa(V)(\tilde{E}(D, \kappa) + \lambda)^{-1} = C(V)(E^\infty(D) + \lambda)^{-1}.$$

From (V - 1)', (V - 2)' and iterating the second resolvent formula with respect to the pair $(\overline{H_\rho}(\kappa), \overline{H_\rho}(V, \kappa))$, it follows that

$$s - \lim_{\kappa \to \infty} \left( \overline{H_\rho}(V, \kappa) - z \right)^{-1} = (E^\infty(D) \otimes I + (I \otimes P_0)C(V)(I \otimes P_0) - z)^{-1} I \otimes P_0.$$
Since
\[(I \otimes P_0)C(V)(I \otimes P_0) = E_\Omega(C(V)),\]
we see that
\[
s-\lim_{\kappa \to \infty} \left( \overline{H\rho(V, \kappa)} - z \right)^{-1} = (E^\infty(D) + E_\Omega(C(V)) - z)^{-1} \otimes P_0.
\]
Thus by Theorems 3.4 and 3.5, we get (3.10).

We want to see $E_\Omega(C(V))$ more explicitly. For $\bar{\rho} \in \tilde{P}$, let $\tilde{\Delta}^\infty = (\tilde{\Delta}_{ij}^\infty)_{1 \leq i, j \leq d}$, where
\[
\tilde{\Delta}_{ij}^\infty = \frac{1}{2} \frac{d-1}{d} \left( \frac{\hbar}{mc} \right)^2 \int_{\mathbb{R}^d} dk \frac{\hat{p}_i(k) \hat{p}_j(k)}{\omega(k)^3}.
\]
Let $I_{d \times d}$ denote $d \times d$-identity matrix. Since $\Delta^\infty \equiv \Delta^\infty \otimes I_{d \times d}$ is a nonnegative symmetric matrix, there exist unitary matrices $T$ so that
\[
T\Delta^\infty T^{-1} = \begin{pmatrix} \lambda_1 I_{d \times d} & & \\ & \lambda_2 I_{d \times d} & \\ & & \ddots \\ & & & \lambda_N I_{d \times d} \end{pmatrix}, \tag{3.11}
\]
where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$.

**Theorem 3.7** Suppose $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_K > 0$, $\lambda_{K+1} = \ldots = \lambda_N = 0$ and fix a unitary operator $T$ in (3.11). Let $x = (x_1, \ldots, x_N)$, $x_j \in \mathbb{R}^d$, $j = 1, \ldots, N$ and $V$ satisfy
\[
\int_{\mathbb{R}^{dK}} dy_1 \cdots dy_K |V| \circ T^{-1} (y_1, \ldots, y_K, (Tx)_{K+1}, \ldots, (Tx)_N) \exp \left( -\frac{\sum_{j=1}^{K} |(Tx)_j - y_j|^2}{2\lambda_1 \ldots \lambda_K} \right) < \infty. \tag{3.12}
\]
Moreover we suppose that the left hand side of (3.12) is locally bounded. Then the partial expectation $E_\Omega(C(V))$ is given by a multiplication operator $V_{\text{eff}}$;
\[
V_{\text{eff}}(x) = (2\pi \lambda_1 \ldots \lambda_K)^{-\frac{d}{2}} \int_{\mathbb{R}^{dK}} dy_1 \cdots dy_K V \circ T^{-1} (y_1, \ldots, y_K, (Tx)_{K+1}, \ldots, (Tx)_N) \times \exp \left( -\frac{\sum_{j=1}^{K} |(Tx)_j - y_j|^2}{2\lambda_1 \ldots \lambda_K} \right).
\]
In particular, in the case where $\tilde{\Delta}^\infty$ is non-degenerate, $V_{\text{eff}}$ is given by

$$V_{\text{eff}}(x) = (2\pi \det \Delta^\infty)^{-\frac{d}{2}} \int_{\mathbb{R}^{dN}} V(y) \exp \left(-\frac{|x-y|^2}{2 \det \Delta^\infty} \right) dy.$$ 

**Proof:** Suppose $V \in \mathcal{S}(\mathbb{R}^{dN})$, which is the set of the rapidly decreasing infinitely continuously differentiable functions on $\mathbb{R}^{dN}$. Then the direct calculation shows that for $f, g \in L^2(\mathbb{R}^{dN})$

$$\langle f, E_\Omega(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dk \tilde{f}(x)g(x)e^{ikx} \tilde{V}(k)e^{-\frac{1}{2} \sum_{\mu=1}^{d} \sum_{i,j=1}^{N} \Delta^\infty_{ij} k_i^\mu k_j^\mu}.$$ 

Hence we have

$$\langle f, E_\Omega(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} = \langle f, V_{\text{eff}}g \rangle_{L^2(\mathbb{R}^{dN})}. \quad (3.13)$$

We next consider the case where $V$ is bounded. In this case we can approximate $V$ by a sequence $\{V_n\}_{n=1}^{\infty}$, $V_n \in \mathcal{S}(\mathbb{R}^{dN})$, such that

$$\|V - V_n\|_\infty \to 0 \ (n \to \infty),$$

where $\| \cdot \|_\infty$ denotes the sup norm. Then we have

$$E_\Omega(C(V_n)) \to E_\Omega(C(V)) \ (n \to \infty),$$

strongly. Moreover $(V_n)_{\text{eff}}(x) \to V_{\text{eff}}(x)$ for all $x \in \mathbb{R}^{dN}$. Thus for $f, g \in L^2(\mathbb{R}^{dN})$, (3.13) follows for such $V$. Finally, let $V$ satisfy (3.12). Define

$$V_n = \begin{cases} V(x) & |V(x)| \leq n, \\ n & |V(x)| > n. \end{cases}$$

Hence for $f \in L^2(\mathbb{R}^{dN})$ and $g \in D(-\Delta)$, we have

$$\langle f, E_\Omega(C(V_n))g \rangle_{L^2(\mathbb{R}^{dN})} \to \langle f, E_\Omega(C(V))g \rangle_{L^2(\mathbb{R}^{dN})} \ (n \to \infty).$$

On the other hand, since the left hand side of (3.12) is locally bounded, we can see that for $f \in C_0^\infty(\mathbb{R}^{dN})$ and $g \in D(-\Delta)$,

$$\langle f, (V_n)_{\text{eff}}g \rangle_{L^2(\mathbb{R}^{dN})} \to \langle f, V_{\text{eff}}g \rangle_{L^2(\mathbb{R}^{dN})} \ (n \to \infty),$$

which completes the proof. \qed
Remark 3.8 In Theorem 3.7, in the case where $\tilde{\Delta}^\infty$ is non-degenerate, since the left hand side of (3.12) is continuous in $x \in \mathbb{R}^{dN}$, it is necessarily locally bounded.

We call $V_{eff}$ “the effective potential with respect to V”. We give a typical example of scalar potentials $V$ and ultraviolet cut-off functions $\vec{\rho}$.

Example 3.9 Let

$$\tilde{\Delta}_{ij}^\infty = \delta_{ij} \frac{1}{2} \frac{d-1}{d} \left( \frac{\hbar}{mc} \right)^2 \frac{e^2}{\hat{p}_i(k)^2} \int_{\mathbb{R}^d} dk \omega(k)^3.$$

Then there exist positive constants $\delta_1$ and $\delta_2$ such that for sufficiently large $\kappa > 0$

$$\delta_1 |p|^2 \leq \tilde{E}(p, \kappa) \leq \delta_2 |p|^2. \quad (3.14)$$

Let $d = 3$ and $V$ be the Coulomb potential;

$$V(x_1, \ldots, x_N) = -\sum_{j=1}^{N} \frac{\alpha_j}{|x_j|} + \sum_{i \neq j} \frac{\beta_{ij}}{|x_i - x_j|}, \quad \alpha_j \geq 0, \beta_{ij} \geq 0.$$

Then $V$ is the Kato class potential ([10], Theorem X.16). Namely for any $\epsilon > 0$, there exists $b \geq 0$ such that $D(V) \supset D(\Delta)$ and

$$||V\Phi||_{L^2(\mathbb{R}^{3N})} \leq \epsilon ||-\Delta\Phi||_{L^2(\mathbb{R}^{3N})} + b||\Phi||_{L^2(\mathbb{R}^{3N})}. \quad (3.15)$$

Together with (3.14) and (3.15), one can see that $V$ satisfies $(V - 1)$, $(V - 2)$ and for any $t > 0$

$$\int_{\mathbb{R}^{3d}} |V'(y)e^{-i|x-y|^2} dy < \infty.$$

Then the scaling limit of the Pauli-Fierz Hamiltonian with the Coulomb potential exists and has the effective potential given by

$$V_{eff}(x) = (2\pi \gamma)^{-\frac{3}{2}} \int_{\mathbb{R}^{3N}} V(y)e^{-i|x-y|^2} dy,$$

$$\gamma = \left\{ \frac{1}{3} \left( \frac{\hbar}{mc} \right)^2 \frac{e^2}{\hat{p}_j(k)^2} \right\}^N \prod_{j=1}^{N} \left( \int_{\mathbb{R}^3} dk \frac{\hat{p}_j^2(k)}{\omega(k)^3} \right).$$
4 CONCLUDING REMARK

As is seen in Theorem 3.7, the effective potential $V_{ef}$ is characterized by the matrix-valued functional $\tilde{\Delta}^\infty = \tilde{\Delta}^\infty(\tilde{\rho})$, which has the following mathematical meaning; putting

$$U(\infty)(x_i \otimes I)U^{-1}(\infty) - x_i \otimes I \equiv \Delta x_i, \quad i = 1, ..., N,$$

we see that the partial expectation of $\Delta x_i \Delta x_j$ with respect to $\Omega$ is as follows;

$$E_\Omega[(\Delta x_i \Delta x_j)] = \tilde{\Delta}_{ij}^\infty(\tilde{\rho})I.$$

In one-nonrelativistic particle case, the author in [5] show that the partial expectation $E_\Omega[(\Delta x)^2]$ with respect to $\Omega$ may be interpreted as the mean square fluctuation in position of one-nonrelativistic particle ([2]). In this sense, $\tilde{\Delta}_{ij}^\infty(\tilde{\rho})$ may also be interpreted as correlation of fluctuations in position of the $i$-th and the $j$-th nonrelativistic particles under the action of quantized radiation fields.

5 REFERENCES


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