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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1997), 983: 175-187</td>
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<td>Issue Date</td>
<td>1997-03</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60930">http://hdl.handle.net/2433/60930</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
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GOURSAT PROBLEM FOR A MICRONDIFFERENTIAL OPERATOR OF FUCHSIAN TYPE AND ITS APPLICATION

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§0. INTRODUCTION.

The Goursat problem in the holomorphic (or the real analytic) category is treated by several authors and studied in depth. Moreover, C. Wagschal [W] extended the problem to the case of a system of integro-differential operators and obtained the Cauchy-Kovalevskaja type (the unique solvability) theorem. However, it seems that the study of the Goursat problem is not so satisfactory from the microlocal point of view. Therefore in this article, we treat a microdifferential operator of Fuchsian type with respect to several variables and consider the Goursat problem in the framework of holomorphic (or micro-) functions.

The notion of Fuchsian type (with respect to one variable) was introduced by M. S. Baouendi and C. Goulaouic [Ba-G] for a partial differential operator. This includes non characteristic type as a special case, and the Cauchy-Kovalevskaja type theorem was proved in [Ba-G]. Seeing this result, N. S. Madi [M] generalized Fuchsian type to several variables case
by the name of "a Goursat operator of several Fuchsian variables" and obtained the Cauchy-Kovalevskaja type theorem for the Goursat problem in the framework of holomorphic functions. Note that Y. Laurent-T. Monterio Fernandes [La-MF] and Z. Szmydt and B. Ziemian [Sz-Zi] gave different definitions of Fuchsian type with respect to several variables respectively. On the other hand, succeeding to Baouendi-Goulaouic [Ba-G], many mathematicians have obtained almost sufficient results in Fuchsian type with respect to one variable. For example, H. Tahara [Ta] treated a Fuchsian system in the sense of Volević and proved the Cauchy-Kovalevskaja type theorem in the complex domain. Further, as an application he obtained the existence and uniqueness theorem on an initial value problem for a Fuchsian hyperbolic system in the framework of hyperfunctions. Moreover, he proved the existence theorem on a homogeneous initial value problem for a Fuchsian microhyperbolic system of microdifferential operators in the framework of microfunctions. On the other hand, T. Ōaku proved the existence theorem on an inhomogeneous initial value problem for a Fuchsian hyperbolic microdifferential operator in [O1] and the uniqueness theorem under the F-mildness condition (but without the hyperbolicity assumption) in [O3] in the framework of microfunctions (cf. [O2]).

In this article, we define a matrix of microdifferential operators of Fuchsian type with respect to several variables as a natural generalization of one variable case due to Tahara [Ta] or non-microlocal case due to Madi
Moreover, we prove the Cauchy-Kovalevskaja type theorem for the Goursat problem in the space of holomorphic functions under the action of microdifferential operators due to J. M. Bony and P. Schapira [Bo-Sc]. As an application we solve the Goursat problem in the framework of micro-(or hyper-)functions; we prove the existence theorem for sufficiently "regular" initial data under suitable assumptions.

§1. STATEMENT OF MAIN THEOREM.

In this article, we use the following notation: $\mathbb{N}$ denotes the set of natural numbers (not containing 0) and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a subset $D$ of some topological space, $[D]$ denotes the closure. For natural numbers $M, N \in \mathbb{N}$, and a linear space $L$ we denote by $\text{Mat}(M \times N; L)$ the space of matrices of size $N \times N$ whose components are in $L$. Further set

\[
\begin{align*}
\text{Mat}(N; L) & := \text{Mat}(N \times N; L), \\
L^N & := \text{Mat}(1 \times N; L), \\
L^{\oplus N} & := \text{Mat}(N \times 1; L).
\end{align*}
\]

In addition, if $A$ has a norm $\| \|$, for $P = (P^{(\mu, \nu)})^{N}_{\mu, \nu=1} \in \text{Mat}(M \times N; A)$ we set $\|P\| := \max \{\|P^{(\mu, \nu)}\|; 1 \leq \mu \leq M, 0 \leq \nu \leq N\}$. For natural numbers $d, n \in \mathbb{N}$, we use coordinates $\tau = (\tau_1, \ldots, \tau_d) \in \mathbb{C}^d$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Moreover for multi-indices $\gamma = (\gamma_1, \ldots, \gamma_n)$ and
$\alpha = (\alpha_1, \ldots, \alpha_d)$, we set
\[
\begin{aligned}
\partial_z^\gamma &:= \partial_{z_1}^{\gamma_1} \cdots \partial_{z_n}^{\gamma_n}, & \partial_r^\alpha &:= \partial_{r_1}^{\alpha_1} \cdots \partial_{r_d}^{\alpha_d}, \\
z^\gamma &:= z_1^{\gamma_1} \cdots z_n^{\gamma_n}, & \tau^\alpha &:= \tau_1^{\alpha_1} \cdots \tau_d^{\alpha_d}, \\
\gamma! &:= \gamma_1! \cdots \gamma_n!, & \alpha! &:= \alpha_1! \cdots \alpha_d!, \\
|\gamma| &:= \sum_{j=1}^n \gamma_j, & |\alpha| &:= \sum_{j=1}^d \alpha_j,
\end{aligned}
\]
as usual. For vectors $R = (R_1, \ldots, R_d)$ and $R' = (R'_1, \ldots, R'_d) \in \mathbb{R}^d$, we define an order relation as follows:
\[
R' \leq R \overset{\text{def.}}{\iff} R_j' \leq R_j \quad \text{for all } j,
\]
\[
R' < R \overset{\text{def.}}{\iff} R' \leq R \text{ and } R' \neq R,
\]
\[
R' \prec R \overset{\text{def.}}{\iff} R_j' < R_j \quad \text{for all } j.
\]
For a vector $r = (r_1, \ldots, r_d) \in \mathbb{R}^d$, we set $[r]_+ := ([r_1]_+, \ldots, [r_d]_+)$, where $[r_j]_+ = \max\{r_j, 0\}$. We fix $m^{(\nu)} = (m_1^{(\nu)}, \ldots, m_d^{(N)})$ and $k^{(\nu)} = (k^{(1)}, \ldots, k^{(N)}) \in \mathbb{N}_0^d$ with $m^{(\nu)} \geq k^{(\nu)}$ ($1 \leq \nu \leq N$) and set $m = (m^{(1)}, \ldots, m^{(N)})$ and $k = (k^{(1)}, \ldots, k^{(N)}) \in (\mathbb{N}_0^d)^N$. For any $N$-tuple of (generalized) functions $f(z, \tau) = (f_1(z, \tau), \ldots, f_N(z, \tau))$, we mean $f = O(\tau^{m-k})$ by
\[
\partial^i_{\tau_j} f_{\nu}|_{\tau_j=0} = 0 \quad (1 \leq j \leq d, 1 \leq \nu \leq N, 0 \leq i \leq m^{(\nu)}_{j} - k^{(\nu)}_{j} - 1).
\]
Set $1_d := (1, \ldots, 1) \in \mathbb{N}^d$. For a vector $R = (R_1, \ldots, R_d) \in \mathbb{R}^d$ with $0 \prec R$, we set $B(R) := \{\tau \in \mathbb{C}^d; |	au_j| < R_j (1 \leq j \leq d)\}$. Let $V \subset \mathbb{C}^n$ be
a relatively compact open neighborhood of the origin and $h_0$ a positive number. We set

$$U = \{(z; \zeta) \in T^*\mathbb{C}^n; z \in V, \zeta_1 = 1, |\zeta_j| < h_0 (2 \leq j \leq n)\}.$$  

We denote the sheaf of rings of microdifferential operators of finite order (resp. of order at most $\nu$) by $\mathcal{E}$ (resp. $\mathcal{E}(\nu)$) as usual.

1.1 Definition. Let $P(z, \tau; \partial_z, \partial_\tau) = (P^{(\mu,\nu)}(z, \tau; \partial_z, \partial_\tau))_{\mu,\nu=1}^N$ be a matrix in $\text{Mat}(N; \Gamma([U \times B(R)]; \mathcal{E}_{\mathbb{C}^{n+d}}))$; that is, each $P^{(\mu,\nu)}$ is a microdifferential operator of finite order defined in some neighborhood of $[U \times B(R)]$. Then, $P$ is said to be of Fuchsian type with weight $(k, m)$ (with respect to $\tau$-variables) if it has the following form:

$$P^{(\mu,\nu)}(z, \tau; \partial_z, \partial_\tau) = \sum_{0 \leq \alpha \leq m^{(\nu)}} P^{(\mu,\nu)}_{\alpha}(z, \tau; \partial_z) \partial_\tau^\alpha,$$

where each $P^{(\mu,\nu)}_{\alpha}$ is a microdifferential operator with holomorphic parameters $\tau$ and satisfies the following:

1. The order $\text{ord} P^{(\mu,\nu)}_{\alpha}$ of $P^{(\mu,\nu)}_{\alpha}$ is at most $|m^{(\nu)}| - |\alpha|$;

2. There exist $P^{1,(\mu,\nu)}_{\alpha}(z, \tau; \partial_z)$ and $P^{2,(\mu,\nu)}_{\alpha}(z, \tau; \partial_z) (0 \leq \alpha \leq m^{(\nu)})$ such that $\text{ord} P^{1,(\mu,\nu)}_{\alpha} \leq 0$ and

$$P^{(\mu,\nu)}_{\alpha}(z, \tau; \partial_z) = \tau^{[\alpha - m^{(\nu)} + k^{(\nu)}]} P^{1,(\mu,\nu)}_{\alpha}(z, \tau; \partial_z)$$

$$+ \tau^{[\alpha - m^{(\nu)} + k^{(\nu)} + 1_d]} P^{2,(\mu,\nu)}_{\alpha}(z, \tau; \partial_z).$$
1.2 Remark. (1) The Fuchsian property above is invariant under any coordinate change of $z$-variables, or more generally an arbitrary quantized contact transformation for $(z; \zeta)$-variables.

(2) The Fuchsian type defined in Definition 1.1 is a natural generalization of differential operators of Fuchsian type introduced by Madi [M]; that is, if $P$ is a differential operator of Fuchsian type in the sense of Definition 1.1, then $P$ is of Fuchsian type in the sense of Madi. Further if $d = N = 1$, a microdifferential operator of Fuchsian type is nothing but of Fuchsian type defined by Tahara (see [Ta]).

Let $T^{(\nu)} = (T^{(\nu)}_1, \ldots, T^{(\nu)}_d)$ $(1 \leq \nu \leq N)$ be indeterminates and set

$$\overline{T} := (T^{(1)}, \ldots, T^{(N)}).$$

If $P$ is of Fuchsian type with weight $(k, m)$, we define the indicial polynomial of $P$ by

$$\mathcal{I}_P(z; \zeta; \overline{T}) := \det \left( \sum_{m^{(\nu)} - k^{(\nu)} \leq \alpha \leq m^{(\nu)}} \sigma_0(P^{1, (\mu, \nu)}_\alpha)(z, 0; \zeta) \mathcal{I}_\alpha(T^{(\nu)}) \right),$$

where $\mathcal{I}_\alpha(T^{(\nu)}) = \prod_{j=1}^{d} \mathcal{I}_{\alpha_j}(T^{(\nu)}_j)$ with

$$\mathcal{I}_{\alpha_j}(T^{(\nu)}_j) := \begin{cases} T^{(\nu)}_j(T^{(\nu)}_j - 1) \cdots (T^{(\nu)}_j - \alpha_j + 1) & (\alpha_j \geq 1), \\ 1 & (\alpha_j = 0). \end{cases}$$

Let $A(z, \tau; \partial_z)$ be a microdifferential operator of finite order with holomorphic parameters $\tau$ defined in a neighborhood of $[U \times B(R)]$. Let
$c \in \mathbb{C}$ and set $\Sigma := \{z \in \mathbb{C}^n; z_1 = c\}$. Let $\Omega \subset V$ be an open convex set and assume that $\Omega$ is $h_0$-$\Sigma$-flat in the sense of Bony-Schapira; that is, if $z \in \Omega$, $w \in \Sigma$ and $h_0|z_j - w_j| \leq |z_1 - w_1|$ ($2 \leq j \leq n$), then it follows that $w \in \Omega \cap \Sigma$. Let $f(z, \tau)$ be a holomorphic function defined on $\Omega \times B(R)$. If $p \in \mathbb{N}$, there exists a unique holomorphic function $g(z, \tau)$ on

$$\begin{align*}
\begin{cases}
\partial_{z_1}^p g(z, \tau) = f(z, \tau), \\
\partial_{z_1}^j g|_{z_1 = c} = 0 \quad (0 \leq j \leq p - 1).
\end{cases}
\end{align*}$$

Then, we define $(\partial_{z_1}^{-p})\Sigma f(z, \tau) := g(z, \tau)$; that is,

$$(\partial_{z_1}^{-p})\Sigma f(z, \tau) := \int_c^{z_1} \frac{(z_1 - w_1)^{p-1}}{(p-1)!} f(w_1, z') dw_1,$$

where $z' := (z_2, \ldots, z_n)$. We write formally

$$A(z, \tau; \partial_z) = \sum_{\gamma_1 \in \mathbb{Z}, \gamma_2, \ldots, \gamma_n \in \mathbb{N}_0} A_{\gamma}(z, \tau) \partial_z^\gamma.$$

Then, applying the argument as in Bony-Schapira [Bo-Sc] regarding $\tau$ as holomorphic parameters, we find that

$$A_{\Sigma} f(z, \tau) := \sum_{\gamma_1, \ldots, \gamma_n \in \mathbb{N}_0} A_{\gamma}(z, \tau) \partial_z^\gamma f(z, \tau)$$

$$+ \sum_{\gamma_1 < 0, \gamma_2, \ldots, \gamma_n \in \mathbb{N}_0} A_{\gamma}(z, \tau) (\partial_{z_1}^{-\gamma_1})\Sigma \partial_z^\gamma f(z, \tau)$$

is holomorphic on $\Omega \times B(R)$. Let $s$ be a parameter with $0 < s < 1$. We fix a point $z_0 \in \Omega \cap \Sigma$ and set

$$\Omega_s := \{s(z - z_0) + z_0 \in \mathbb{C}^n; z \in \Omega\}$$

Consider the following condition:
[A-1]. There exist a positive constant $C > 0$ and a neighborhood $W$ of $[U]$ such that for any $(z; \zeta) \in [W]$ and $\beta = (\beta^{(1)}, \ldots, \beta^{(N)}) \in (\mathbb{N}_0^d)^N$ with $\beta^{(\nu)} \geq m^{(\nu)} - k^{(\nu)} (1 \leq \nu \leq N)$

$$|\mathcal{I}_P(z; \zeta; \beta)| \geq C \prod_{\nu=1}^{N} (\beta^{(\nu)} + 1_d)^{m^{(\nu)}}.$$ 

Note that if $N = 1$, then [A-1] is a natural generalization of Madi’s condition which is similar to the “Fuchsian ellipticity condition” due to Szmydt-Ziemian [Sz-Zi].

1.3 Theorem. Let $P$ be a matrix of microdifferential operators defined in a neighborhood of $[U \times B(R)]$. Assume that $P$ is of Fuchsian type with weight $(k, m)$ and satisfies [A-1]. Then, there exist constants $r_0 > 0$ and $R^\circ$ with $0 < R^\circ \leq R$ such that the following hold:

Take arbitrary $h$ and $r$ with $0 < h < h_0$ and $0 < r < r_0$ respectively. Let $\Omega$ be any $h$-$\Sigma$-flat open convex subset of $V$ with $\text{dia} \Omega \leq r$, where $\text{dia}$ denotes the diameter. Then, there exists a constant $\delta$ such that for any $\overline{R}$ with $0 < \overline{R} \leq R^\circ$ it follows that for any holomorphic functions $f(z, \tau) = \left( f_1(z, \tau), \ldots, f_N(z, \tau) \right)$ and $g(z, \tau) = \left( g_1(z, \tau), \ldots, g_N(z, \tau) \right)$ on $\Omega \times B(\overline{R})$, there exists a unique holomorphic solution

$$u(z, \tau) = \left( u_1(z, \tau), \ldots, u_N(z, \tau) \right)$$

of the Goursat problem

$$(G.P.) \quad \begin{cases} P_\Sigma u = f, \\ u - g = O(\tau^{m-k}), \end{cases}$$
and each $u_{\nu}(z, \tau)$ ($1 \leq \nu \leq N$) is holomorphic on

$$
\bigcup_{0<s<1} \left( \Omega_s \times \left\{ \tau \in B(\overline{R}); \prod_{j=1}^{d} |\tau_j| < \delta (1 - s)^{|m|} \right\} \right),
$$

where $|m| := \sum_{\nu=1}^{N} \sum_{j=1}^{d} m_j^{(\nu)}$.

We can prove Theorem 1.3 by applying techniques of [O1] and [W].

1.4 Remark. Assume that $P$ is a differential operator. Then Theorem 1.3 is (essentially) obtained by Madi [M] (cf. [La-MF]).

§2. Applications.

Let $M$ be $\mathbb{R}_x^{n} \times \mathbb{R}_t^{d}$ with its complexification $X := \mathbb{C}_x^{n} \times \mathbb{C}_t^{d} = Y \times \mathbb{C}^{d}$ and $\pi_M$ the canonical projection $T_M^*X \longrightarrow M$. Set $N := \mathbb{R}^{n} \cong M \cap \{t = 0\} \hookrightarrow M$, $L := X \cap \{\text{Im} z = 0\} = \mathbb{R}^{n} \times \mathbb{C}^{d}$, $\tilde{\Lambda} := T_L^*X \cong T_N^*Y \times \mathbb{C}^{d}$ and $\Lambda := T_M^*X \cap \tilde{\Lambda}$. We denote the sheaf of microfunctions on $T_M^*X$ (resp. $T_N^*Y$) by $\mathcal{C}_M$ (resp. $\mathcal{C}_N$) as usual. Further, let $\mathcal{CO}_L$ be the sheaf of microfunctions with holomorphic parameters on $\tilde{\Lambda}$; that is,

$$
\mathcal{CO}_L := \mu_L(\mathcal{O}_X) \otimes \text{or}_{N/Y}[n],
$$

where $\mu_L$ denotes Sato's microlocalization functor along $L$ and $\text{or}_{N/Y}$ denotes the relative orientation sheaf (see [K-Sc] and [S-K-K]). The sheaf $\mathcal{B}_M$ of hyperfunctions on $M$ and the sheaf $\mathcal{BO}_L$ of hyperfunctions with
holomorphic parameters on $L$ are defined by $\mathcal{B}_M := \mathcal{C}_M|_M$ and $\mathcal{B_0}_L := \mathcal{C}_0|_L$ respectively. Let $\rho$ be a natural mapping

$$N \times T^*_M X \ni (x, 0; \sqrt{-1} (\langle \xi, dx \rangle + \langle \eta, dt \rangle)) \longmapsto (x; \sqrt{-1} \langle \xi, dx \rangle) \in T^*_N Y.$$  

Then, we have the following canonical morphisms:

$$\mathcal{C}O_L|_A \rightarrow \mathcal{C}_M|_A, \quad \rho!(\mathcal{C}_M|_N \times T^*_M X) \rightarrow \mathcal{C}_N.$$  

Set $p_0 := (0; \sqrt{-1} dx_1) \in T^*_N Y$ and assume that $P(x, t; \partial_x, \partial_t)$ is a matrix of microdifferential operators of Fuchsian type with weight $(k, m)$ defined in some neighborhood of $\rho^{-1}(p_0)$, then the following morphism is induced:

$$P \colon \rho!(\mathcal{C}_M|_N \times T^*_M X)_{p_0} \rightarrow \rho!(\mathcal{C}_M|_N \times T^*_M X)_{p_0},$$

where $\rho!(\mathcal{C}_M|_N \times T^*_M X)_{p_0}$ denotes the stalk at $p_0$.

Consider the following condition:

[A-2]. $\det(\sigma_{m(\nu)}(P^{(\mu, \nu)})(z, \tau; \zeta, \eta)) = \tau^{\tilde{k}} \overline{P}(z, \tau; \zeta, \eta)$ for a function $\overline{P}$ ($\tilde{k} := \sum_{\nu=1}^N k^{(\nu)} \in \mathbb{N}_d$) which satisfies the following condition:

There exist positive constants $h_0$, $M$ and $\nu_i$ with $\nu_i \geq 1 (1 \leq i \leq d)$ such that $\overline{P}(z, t; \zeta, \eta)$ never vanishes on the set

$$\left\{(z, t; \zeta, \eta) \in \mathbb{C}^n \times \mathbb{R}^d \times \mathbb{C}^n \times \mathbb{C}^d; |z|, |t| < h_0, \right\},$$

$$|\zeta_j| < h_0 |\zeta_1| (2 \leq j \leq n), |\text{Im}(\eta_i/\zeta_1)| = \nu_i \lambda (1 \leq i \leq d)$$

for $\exists \lambda > M \left(\sum_{j=1}^n |\text{Im} z_j| + \sum_{j=2}^n |\text{Im}(\zeta_j/\zeta_1)|\right)$.
2.1 **Remark.** Condition [A-2] is satisfied if

\[ \tilde{P}(x, t; \xi, \eta) = \prod_{j=1}^{d} P_j(x, t; \xi, \eta_j) \]

and each \( P_j \) is of degree \( \sum_{i=1}^{N} m^{(j)}_i \)

and hyperbolic with respect to the direction \( t_j \) (cf. Kashiwara-Kawai [K-K]).

2.2 **Theorem.** Assume that \( P \) satisfies [A-1] and [A-2]. Then, for any microfunctions with holomorphic parameters

\[ f(x, t), \ g(x, t) \in \rho_\ast (\mathcal{O}_L|_{N \times T_M^\ast X})_{p_0}^{\oplus N}, \]

there exists a microfunction

\[ u(x, t) \in \rho_! (\mathcal{C}_M|_{N \times T_M^*X})_{p_0}^{\oplus N} \]

such that \( u \) is a solution of the Goursat problem

\[ \begin{cases} P(x, t; \partial_x, \partial_t) u(x, t) = f(x, t), \\ u(x, t) - g(x, t) = O(t^{m-k}). \end{cases} \]

Outline of Proof of Theorem 2.2 is as follows: First, choosing suitable defining functions, we can solve \((G.P.)\) in a complex open set by using Theorem 1.3. Next, we can apply the holomorphic continuation method due to Kashiwara-Kawai [K-K] by assumption [A-1].

2.3 **Remark.** The author does not know how to prove the uniqueness of \( u(x, t) \) in Theorem 2.2 (cf. [O2] and [O3]).
2.4 Corollary. Let $P$ be a matrix of an analytic differential operators of Fuchsian type defined on a neighborhood of $(x,t) = (0,0)$. Assume that [A-1] and [A-3]. Then, for any holomorphic hyperfunctions with holomorphic parameters

$$f(x,t), g(x,t) \in (\mathcal{B}_\mathcal{O}_L|_M)^{\oplus N},$$

there exists a hyperfunction

$$u(x,t) \in (\mathcal{B}_M)^{\oplus N}$$

such that $u$ has $t$ as real analytic parameters and is a solution of the Goursat problem

$$(G.P.) \begin{cases} P(x,t; \partial_x, \partial_t) u(x,t) = f(x,t), \\ u(x,t) - g(x,t) = O(t^{m-k}). \end{cases}$$

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