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A Remark on Finiteness and Duality of D-Modules

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The purpose of this paper is to prove a theorem on finite dimensionality of the cohomology groups of analytic differential complexes on compact real manifolds. This generalizes the classical finiteness theorem for elliptic differential complexes.

In this paper, a manifold is always assumed to be paracompact. For sheaves and functors, we follow the notations of [KS]. For a sheaf \( \mathcal{F} \) on a topological manifold \( X \), \( \Gamma(\mathcal{F}) \) denotes the set of global sections of \( \mathcal{F} \), and \( \Gamma_{c}(\mathcal{F}) \) of global sections with compact support. \( R\Gamma \) and \( R\Gamma_{c} \) denote their right derived functors.

1. Main Result

Let \( X \) be a complex manifold, \( n = \dim_{\mathbb{C}} X \). Let \( \mathcal{O} \) denote the sheaf of holomorphic functions on \( X \), \( \Omega^{n} \) the sheaf of holomorphic \( n \)-forms on \( X \), and \( D \) the sheaf of rings of differential operators on \( X \) (of finite order).

Let \( T^{*}X \) denote the cotangent bundle of \( X \).

For a coherent right \( D \)-module \( \mathcal{M} \) on \( X \), \( \text{Ch}(\mathcal{M}) \) denotes its characteristic variety; \( \text{Ch}(\mathcal{M}) \) is a \( \mathbb{C}^{\times} \)-invariant closed analytic subset of \( T^{*}X \) and \( \dim(\text{Ch}(\mathcal{M})) \geq n \) (see [SKK]). Let \( \text{Mod}(\mathcal{D}^{o}) \) be the abelian category of right \( D \)-modules, \( \mathbf{D}^{b}(\mathcal{D}^{o}) \) its derived category with bounded cohomology. Let \( \mathbf{D}^{b}_{g,\text{coh}}(\mathcal{D}^{o}) \) be the full triangulated subcategory of \( \mathbf{D}^{b}(\mathcal{D}^{o}) \) consisting of bounded complexes with good coherent cohomology groups. [We say that a \( D \)-module \( \mathcal{M} \) is good coherent if, for any relatively compact open subset \( U \) of \( X \), there exists a finite filtration \( G \) of \( \mathcal{M}|_{U} \) by \( D \)-modules such that...
For an object $\mathcal{M}^{\bullet}$ of $\mathcal{D}^{b,\text{coh}}(\mathcal{D}^{o})$, $\text{Ch}(\mathcal{M}^{\bullet})$ denotes the union of $\text{Ch}(H^{k}\mathcal{M}^{\bullet})$, $k \in \mathbb{Z}$.

Let $\mathcal{D}^{b}(X)$ denote the derived category with bounded cohomology of the abelian category of $\mathcal{C}_{X}$-modules, $\mathcal{D}^{b}_{\mathcal{R}-c}(X)$ the full triangulated subcategory of $\mathcal{D}^{b}(X)$ consisting of $\mathcal{R}$-constructible objects [KS, Sect.8.4]. For an object $F$ of $\mathcal{D}^{b}(X)$, $\text{SS}(F)$ denotes the micro-support of $F$ [KS, Sect.5.1]. If $F$ is $\mathcal{R}$-constructible, $\text{SS}(F)$ is then an $\mathbb{R}_{+}$-invariant closed subanalytic subset of $T^{*}X$. (But we do not need this fact in this paper.)

For an object $(\mathcal{M}^{\bullet}, F)$ of $\mathcal{D}^{b,\text{coh}}(\mathcal{D}^{o}) \times \mathcal{D}^{b}_{\mathcal{R}-c}(X)$, $\mathcal{M}^{\bullet} \otimes F$ denotes the tensor product over $\mathcal{C}$ and is an object of $\mathcal{D}^{b}(\mathcal{D}^{o})$.

**Theorem 1.** Let $(\mathcal{M}^{\bullet}, F)$ be an object of $\mathcal{D}^{b,\text{coh}}(\mathcal{D}^{o}) \times \mathcal{D}^{b}_{\mathcal{R}-c}(X)$. Assume that, for any irreducible component $V$ of $\text{Ch}(\mathcal{M}^{\bullet})$, $V \cap \text{SS}(F)$ is contained in the zero section $T^{*}_{X}X$ if $\dim V \neq n$. Suppose $\text{Supp}(\mathcal{M}^{\bullet}) \cap \text{Supp}(F)$ is compact. Then every cohomology group of $\text{R}\Gamma(\mathcal{M}^{\bullet} \otimes F \otimes_{\mathcal{D}} \mathcal{O})$ and $\text{RHom}_{\mathcal{D}}(X; \mathcal{M}^{\bullet} \otimes F, \Omega^{n})$ is finite dimensional and

\[(1.0) \quad \text{RHom}_{\mathcal{D}}(X; \mathcal{M}^{\bullet} \otimes F, \Omega^{n})[n] \longrightarrow \text{Hom}_{\mathcal{C}}(\text{R}\Gamma(\mathcal{M}^{\bullet} \otimes F \otimes_{\mathcal{D}} \mathcal{O}), \mathcal{C})\]

is an isomorphism in $\mathcal{D}^{b}(\mathcal{C})$. Hence, for any $k \in \mathbb{Z}$,

\[\text{Tor}_{k}^{\mathcal{D}}(\mathcal{M}^{\bullet} \otimes F, \mathcal{O}) \quad \text{and} \quad \text{Ext}_{\mathcal{D}}^{k+n}(X; \mathcal{M}^{\bullet} \otimes F, \Omega^{n})\]

are vector spaces of finite dimension and dual to each other.

**Remark.** We say that $(\mathcal{M}^{\bullet}, F)$ is an elliptic pair if $\text{Ch}(\mathcal{M}^{\bullet}) \cap \text{SS}(F) \subset T^{*}_{X}X$ [SS]. In that case, Theorem 1 is proved in [SS]. On the other hand, if $\mathcal{M}^{\bullet}$ is holonomic, $(\mathcal{M}^{\bullet}, F)$ satisfies the hypothesis of Theorem 1 for any object $F$ of $\mathcal{D}^{b}_{\mathcal{R}-c}(X)$.

Let $M$ be a real analytic manifold of dimension $n$, $X$ a complex neighborhood of $M$. Let $T^{*}_{M}X$ denote the conormal bundle of $M$. $\mathcal{A}_{M}$ denotes the sheaf of real analytic functions on $M$, and $\mathcal{B}_{M}$ of hyperfunctions; $\mathcal{A}_{M}$ and $\mathcal{B}_{M}$ are $\mathcal{D}|_{M}$-modules. Let

\[\mathcal{B}_{M}^{(n)} = \mathcal{B}_{M} \otimes_{\mathcal{A}} (\Omega^{n} \otimes \text{or}_{M/X}),\]

where $\text{or}_{M/X}$ is the relative orientation sheaf of $M$ in $X$; $\mathcal{B}_{M}^{(n)}$ is a right $\mathcal{D}|_{M}$-module.

As an immediate corollary of Theorem 1, we have the following finiteness and duality theorem of analytic differential complexes on compact real manifolds.
Corollary 2. Let $M$ be a compact real analytic manifold of dimension $n$. Let $\mathcal{M}^\bullet$ be an object of $\mathcal{D}_{g, h}^b(\text{COD})$. Assume that, for any irreducible component $V$ of $\text{Ch}(\mathcal{M}^\bullet)$, $V \cap T^*_M X$ is contained in the zero section if $\dim V \neq n$. Then, for any $k \in \mathbb{Z}$, $\text{Tor}_k^D(\mathcal{M}^\bullet, A_M)$ and $\text{Ext}_D^k(M; \mathcal{M}^\bullet, B_M^{(n)})$ are vector spaces of finite dimension and dual to each other.

Let $E^k, 0 \leq k \leq k_0$, be holomorphic vector bundles over $X$ and let

$$0 \longrightarrow \mathcal{M}^0 \xrightarrow{L_0} \mathcal{M}^1 \xrightarrow{L_1} \cdots \xrightarrow{} \mathcal{M}^{k_0} \longrightarrow 0,$$

be a differential complex of vector bundles, where $L_k$ is a differential operator mapping $\Gamma(\mathcal{O}(E^k))$ to $\Gamma(\mathcal{O}(E^{k+1}))$. [For a holomorphic vector bundle $E$, $\mathcal{O}(E)$ denotes the sheaf of holomorphic sections of $E$.]

Let $\mathcal{M}^k = \mathcal{O}(E^k) \otimes_{\mathcal{O}} \mathcal{D}$ and

$$\mathcal{M}^\bullet = \left[ \begin{array}{cccc} 0 & \mathcal{M}^0 & \mathcal{M}^1 & \mathcal{M}^{k_0} \\ L_0 & L_1 & \cdots & \end{array} \right],$$

where $L_k$ acts on $\mathcal{M}^k$ by left multiplication; $\mathcal{M}^\bullet$ is then an object of $\mathcal{D}_{g, \text{coh}}^b(D^\circ)$. Then $\mathcal{R}\Gamma(\mathcal{M}^\bullet \otimes_D A_M)$ is represented by a differential complex

$$0 \longrightarrow \Gamma(M, E^0) \xrightarrow{L_0} \Gamma(M, E^1) \xrightarrow{L_1} \cdots \xrightarrow{} \Gamma(M, E^{k_0}) \longrightarrow 0$$

and $\text{Tor}_k^D(\mathcal{M}^\bullet, A_M)$ is its $k$-th cohomology group, where $\Gamma(M, E^k)$ denotes the space of analytic sections of $E^k$ on $M$. For a vector bundle $E$, let us set $B^{(n)}(E) = \mathcal{O}(E) \otimes_{\mathcal{O}} B_M^{(n)}$. $\mathcal{R}\text{Hom}_D(M; \mathcal{M}^\bullet, B_M^{(n)})$ is represented by

$$0 \longleftarrow \Gamma(M, B^{(n)}(E^*_0)) \xleftarrow{L_0} \Gamma(M, B^{(n)}(E^*_1)) \xleftarrow{L_1} \cdots \xleftarrow{} \Gamma(M, B^{(n)}(E^*_{k_0})) \longleftarrow 0,$$

where $E^*_k$ is the dual bundle of $E^k$ and $L_k$ acts on $B^{(n)}(E^*_k)$ by right multiplication; $\text{Ext}_D^{-k}(M; \mathcal{M}^\bullet, B_M^{(n)})$ is its $k$-th homology group. The pairing of $\text{Tor}_k^D(\mathcal{M}^\bullet, A_M)$ and $\text{Ext}_D^{-k}(M; \mathcal{M}^\bullet, B_M^{(n)})$ is induced from

$$\Gamma(M, E^k) \times \Gamma(M, B^{(n)}(E^*_k)) \rightarrow \mathbb{C}, \quad (u, v) \mapsto \int_M \langle u, v \rangle,$$

$\langle u, v \rangle$ being the pairing of $E^k$ and $E^*_k$.

Remark. If (1.3) is an elliptic complex of vector bundles on $M$, for $\mathcal{M}^\bullet$ given by (1.2), $\text{Ch}(\mathcal{M}^\bullet) \cap T^*_M X$ is contained in the zero section. The converse is not true in general.
2. Proof of Theorem 1

We can assume that \( H^k \mathcal{M}^\bullet = 0 \) for any \( k \neq 0 \); in what follows, \( \mathcal{M} \) denotes a coherent right \( \mathcal{D} \)-module on \( X \).

Let \( \mathcal{M}^* = \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \); then \( \mathcal{M}^* \) is a holonomic left \( \mathcal{D} \)-module, and we have an injective \( \mathcal{D} \) homomorphism \( \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M}^*, \mathcal{D}) \to \mathcal{M} \). Let \( \mathcal{M}^{**} = \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M}^*, \mathcal{D}) \), and \( \mathcal{N} = \mathcal{M}/\mathcal{M}^{**} \); then \( \mathcal{M}^{**} \) is a holonomic \( \mathcal{D} \)-module, and the sequence

\[
0 \to \mathcal{M}^{**} \to \mathcal{M} \to \mathcal{N} \to 0
\]

is exact. Since \( \mathcal{E}xt^{n-1}_{\mathcal{D}}(\mathcal{M}^{**}, \mathcal{D}) = 0 \) and \( \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M}, \mathcal{D}) \to \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{M}^{**}, \mathcal{D}) \) is an isomorphism, we see that \( \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{N}, \mathcal{D}) = 0 \). Hence, by [K2, 2.11], \( \text{Ch}(\mathcal{N}) \) has no irreducible components of codimension \( n \). Since \( \text{Ch}(\mathcal{N}) \subset \text{Ch}(\mathcal{M}) \), by the hypothesis of the theorem, \( \text{Ch}(\mathcal{N}) \cap \text{SS}(\mathcal{F}) \) is contained in the zero section; therefore \( (\mathcal{N}, \mathcal{F}) \) is elliptic in the sense of [SS]. Moreover, by the definition of \( \mathcal{N} \), if \( \mathcal{M} \) is a good coherent \( \mathcal{D} \)-module, \( \mathcal{N} \) is also good coherent.

Since Theorem 1 is proved for elliptic pairs in [SS, Part 1], by exact sequence (2.0), we may assume from the beginning \( \mathcal{M} \) to be holonomic. If \( \mathcal{M} \) is holonomic, by Kashiwara's theorem [K1], \( \mathcal{M} \otimes_\mathcal{D} \mathcal{O} \) is \( \mathbb{C} \)-constructible. Hence \( (\mathcal{M} \otimes_\mathcal{D} \mathcal{O}) \otimes \mathcal{F} \) is an \( \mathbb{R} \)-constructible sheaf on \( X \). Its support being compact by assumption, by [KS, Prop.8.4.8], \( H^k \mathcal{R}\Gamma((\mathcal{M} \otimes_\mathcal{D} \mathcal{O}) \otimes \mathcal{F}) \) is finite dimensional for all \( k \in \mathbb{Z} \). In the same way, the \( \mathbb{C} \)-constructibility of \( \mathcal{R}\Omega_{\mathcal{D}}(\mathcal{M}, \Omega^n) \) yields the finite dimensionality of \( H^k \mathcal{R}\Gamma(\mathcal{R}\Omega_{\mathcal{D}}(\mathcal{M} \otimes \mathcal{F}, \Omega^n)) \). This completes the proof of the finiteness part.

We now prove (1.0) to be an isomorphism for a holonomic \( \mathcal{D} \)-module \( \mathcal{M} \), assuming \( \text{Supp}(\mathcal{M}) \cap \text{Supp}(\mathcal{F}) \) is compact. Let \( D_h^b(\mathcal{O}) \) denote the full triangulated subcategory of \( D^b(\mathcal{O}) \) consisting of bounded complexes with holonomic cohomology groups. Letting \( \text{DR}(\mathcal{M}) = \mathcal{M} \otimes_\mathcal{D} \mathcal{O}[-n] \) for an object \( \mathcal{M} \) of \( D^b(\mathcal{O}) \), we have first:

**Lemma 2.1.** Let \( \mathcal{M} \) be an object of \( D^b_h(\mathcal{O}) \), \( F \) of \( D^b_{\mathcal{R}-\mathcal{c}}(X) \). Then there is an isomorphism

\[
(2.1) \quad \text{DR}(\mathcal{M}) \otimes \mathcal{F} \cong D' \mathcal{R}\Omega_{\mathcal{D}}(\mathcal{M} \otimes \mathcal{F}, \Omega^n),
\]

where \( D' = \mathcal{R}\Omega_{\mathcal{C}}(\bullet, \mathcal{C}_X) \).

The proof will be given later.
Since $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_\mathcal{D}(\mathcal{M} \otimes F, \Omega^n)$ is $\mathcal{R}$-constructible, from (2.1), we have

$$\mathcal{D}'(\mathcal{D}\mathcal{R}(\mathcal{M}) \otimes F) \cong \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_\mathcal{D}(\mathcal{M} \otimes F, \Omega^n).$$

By the Verdier duality, we get

$$\mathcal{R}\Gamma (\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_\mathcal{D}(\mathcal{M} \otimes F, \Omega^n))[n] \cong \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_\mathcal{C}(\mathcal{R}\Gamma_c(\mathcal{D}\mathcal{R}(\mathcal{M}) \otimes F)), \mathcal{C})[-n]$$

$$= \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_\mathcal{C}(\mathcal{R}\Gamma_c(\mathcal{M} \otimes \mathcal{O} \otimes F), \mathcal{C}).$$

This completes the proof of Theorem 1. QED

Proof of Lemma 2.1. If $F = \mathcal{C}_X$, this duality formula is contained in [KK] and [M2], and we have

$$\mathcal{D}\mathcal{R}(\mathcal{M}) \cong \mathcal{D}' \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_\mathcal{D}(\mathcal{M}, \Omega^n).$$

Let $C = \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_\mathcal{D}(\mathcal{M}, \Omega^n)$; then, by Kashiwara's theorem [K1], $C$ is $\mathcal{C}$-constructible. Hence

$$\mathcal{D}'(\mathcal{D}'C \otimes F) \cong \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_\mathcal{C}(F, C)$$

(see [KS, 3.4.6]), and, since $\mathcal{D}'C \otimes F$ is $\mathcal{R}$-constructible,

$$\mathcal{D}'C \otimes F \cong \mathcal{D}' \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_\mathcal{C}(F, C).$$

By (2.2), we get

$$\mathcal{D}\mathcal{R}(\mathcal{M}) \otimes F \cong \mathcal{D}' \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_\mathcal{C}(F, C).$$

QED

References


