A Remark on Finiteness and Duality of D-Modules

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The purpose of this paper is to prove a theorem on finite dimensionality of the cohomology groups of analytic differential complexes on compact real manifolds. This generalizes the classical finiteness theorem for elliptic differential complexes.

In this paper, a manifold is always assumed to be paracompact. For sheaves and functors, we follow the notations of [KS]. For a sheaf $\mathcal{F}$ on a topological manifold $X$, $\Gamma(\mathcal{F})$ denotes the set of global sections of $\mathcal{F}$, and $\Gamma_c(\mathcal{F})$ of global sections with compact support. $R\Gamma$ and $R\Gamma_c$ denote their right derived functors.

1. Main Result

Let $X$ be a complex manifold, $n = \dim_{\mathbb{C}} X$. Let $\mathcal{O}$ denote the sheaf of holomorphic functions on $X$, $\Omega^n$ the sheaf of holomorphic $n$-forms on $X$, and $\mathcal{D}$ the sheaf of rings of differential operators on $X$ (of finite order).

Let $T^*X$ denote the cotangent bundle of $X$.

For a coherent right $\mathcal{D}$-module $\mathcal{M}$ on $X$, $\text{Ch}(\mathcal{M})$ denotes its characteristic variety; $\text{Ch}(\mathcal{M})$ is a $\mathbb{C}^\times$-invariant closed analytic subset of $T^*X$ and $\dim(\text{Ch}(\mathcal{M})) \geq n$ (see [SKK]). Let $\text{Mod}(\mathcal{D}^\circ)$ be the abelian category of right $\mathcal{D}$-modules, $\mathbb{D}^b(\mathcal{D}^\circ)$ its derived category with bounded cohomology. Let $\mathbb{D}^b_{g,\text{coh}}(\mathcal{D}^\circ)$ be the full triangulated subcategory of $\mathbb{D}^b(\mathcal{D}^\circ)$ consisting of bounded complexes with good coherent cohomology groups. [We say that a $\mathcal{D}$-module $\mathcal{M}$ is good coherent if, for any relatively compact open subset $U$ of $X$, there exists a finite filtration $G$ of $\mathcal{M}|_U$ by $\mathcal{D}$-modules such that

\[ G_{-n} \supseteq G_{-n+1} \supseteq \ldots \supseteq G_{-1} \supseteq G_0 = \mathcal{M}|_U \]
For an object $\mathcal{M}^\bullet$ of $\text{D}_{g,\text{coh}}^b(\mathcal{D}^\infty)$, $\text{Ch}(\mathcal{M}^\bullet)$ denotes the union of $\text{Ch}(H^k\mathcal{M}^\bullet), k \in \mathbb{Z}$.

Let $\text{D}^b(\mathcal{X})$ denote the derived category with bounded cohomology of the abelian category of $\mathcal{C}_{\mathcal{X}}$-modules, $\text{D}_{\mathcal{R}_{-c}}^b(\mathcal{X})$ the full triangulated subcategory of $\text{D}^b(\mathcal{X})$ consisting of $\mathcal{R}$-constructible objects [KS, Sect. 8.4]. For an object $\mathcal{F}$ of $\text{D}^b(\mathcal{X})$, $\text{SS}(\mathcal{F})$ denotes the micro-support of $\mathcal{F}$ [KS, Sect. 5.1]. If $\mathcal{F}$ is $\mathcal{R}$-constructible, $\text{SS}(\mathcal{F})$ is then an $\mathcal{R}_4$-invariant closed subanalytic subset of $T^*\mathcal{X}$. (But we do not need this fact in this paper.)

For an object $(\mathcal{M}^\bullet, \mathcal{F})$ of $\text{D}_{g,\text{coh}}^b(\mathcal{D}^\infty) \times \text{D}_{\mathcal{R}_{-c}}^b(\mathcal{X})$, $\mathcal{M}^\bullet \otimes \mathcal{F}$ denotes the tensor product over $\mathcal{C}$ and is an object of $\text{D}^b(\mathcal{D}^\infty)$.

**Theorem 1.** Let $(\mathcal{M}^\bullet, \mathcal{F})$ be an object of $\text{D}_{g,\text{coh}}^b(\mathcal{D}^\infty) \times \text{D}_{\mathcal{R}_{-c}}^b(\mathcal{X})$. Assume that, for any irreducible component $\mathcal{V}$ of $\text{Ch}(\mathcal{M}^\bullet)$, $\mathcal{V} \cap \text{SS}(\mathcal{F})$ is contained in the zero section $T^*_X \mathcal{X}$ if $\dim \mathcal{V} \neq n$. Suppose $\text{Supp}(\mathcal{M}^\bullet) \cap \text{Supp}(\mathcal{F})$ is compact. Then every cohomology group of $\text{R}\Gamma(\mathcal{M}^\bullet \otimes \mathcal{F} \otimes^L_{\mathcal{D}} \mathcal{O})$ and $\text{R}\text{Hom}_{\mathcal{D}}(\mathcal{X}; \mathcal{M}^\bullet \otimes \mathcal{F}, \Omega^n)$ is finite dimensional and

\[
\text{R}\text{Hom}_{\mathcal{D}}(\mathcal{X}; \mathcal{M}^\bullet \otimes \mathcal{F}, \Omega^n)[n] \longrightarrow \text{Hom}_\mathcal{C}(\text{R}\Gamma(\mathcal{M}^\bullet \otimes \mathcal{F} \otimes^L_{\mathcal{D}} \mathcal{O}), \mathcal{C})
\]

is an isomorphism in $\text{D}^b(\mathcal{C})$. Hence, for any $k \in \mathbb{Z}$,

\[
\text{Tor}^\mathcal{D}_k(\mathcal{M}^\bullet \otimes \mathcal{F}, \mathcal{C}) \quad \text{and} \quad \text{Ext}^{k+n}_\mathcal{D}(\mathcal{X}; \mathcal{M}^\bullet \otimes \mathcal{F}, \Omega^n)
\]

are vector spaces of finite dimension and dual to each other.

**Remark.** We say that $(\mathcal{M}^\bullet, \mathcal{F})$ is an elliptic pair if $\text{Ch}(\mathcal{M}^\bullet) \cap \text{SS}(\mathcal{F}) \subset T^*_X \mathcal{X}$ [SS]. In that case, Theorem 1 is proved in [SS]. On the other hand, if $\mathcal{M}^\bullet$ is holonomic, $(\mathcal{M}^\bullet, \mathcal{F})$ satisfies the hypothesis of Theorem 1 for any object $\mathcal{F}$ of $\text{D}_{\mathcal{R}_{-c}}^b(\mathcal{X})$.

Let $\mathcal{M}$ be a real analytic manifold of dimension $n$, $\mathcal{X}$ a complex neighborhood of $\mathcal{M}$. Let $T^*_M \mathcal{X}$ denote the conormal bundle of $\mathcal{M}$. $\mathcal{A}_M$ denotes the sheaf of real analytic functions on $\mathcal{M}$, and $\mathcal{B}_M$ of hyperfunctions; $\mathcal{A}_M$ and $\mathcal{B}_M$ are $\mathcal{D}_{|\mathcal{M}}$-modules. Let

\[
\mathcal{B}^{(n)}_M = \mathcal{B}_M \otimes_{\mathcal{A}} (\Omega^n \otimes \mathcal{O}_{\mathcal{M}/\mathcal{X}}),
\]

where $\mathcal{O}_{\mathcal{M}/\mathcal{X}}$ is the relative orientation sheaf of $\mathcal{M}$ in $\mathcal{X}$; $\mathcal{B}^{(n)}_M$ is a right $\mathcal{D}_{|\mathcal{M}}$-module.

As an immediate corollary of Theorem 1, we have the following finiteness and duality theorem of analytic differential complexes on compact real manifolds.
Corollary 2. Let $M$ be a compact real analytic manifold of dimension $n$. Let $\mathcal{M}^\bullet$ be an object of $\text{D}_{\text{g}, \text{h}}^b(\text{C}O\text{D}^o)$. Assume that, for any irreducible component $V$ of $\text{Ch}(\mathcal{M}^\bullet)$, $V \cap T^*_M X$ is contained in the zero section if $\dim V \neq n$. Then, for any $k \in \mathbb{Z}$, $\text{Tor}_k^D(\mathcal{M}^\bullet, A_M)$ and $\text{Ext}_D^k(M; \mathcal{M}^\bullet, B_M^{(n)})$ are vector spaces of finite dimension and dual to each other.

Let $E^k$, $0 \leq k \leq k_0$, be holomorphic vector bundles over $X$ and let

\begin{equation}
\mathcal{O}(E^0) \xrightarrow{L_0} \mathcal{O}(E^1) \xrightarrow{L_1} \cdots \xrightarrow{L_{k_0}} \mathcal{O}(E^{k_0})
\end{equation}

be a differential complex of vector bundles, where $L_k$ is a differential operator mapping $\Gamma(\mathcal{O}(E^k))$ to $\Gamma(\mathcal{O}(E^{k+1}))$. [For a holomorphic vector bundle $E$, $\mathcal{O}(E)$ denotes the sheaf of holomorphic sections of $E$.]

Let $\mathcal{M}^k = \mathcal{O}(E^k) \otimes_{\mathcal{O}} D$ and

\begin{equation}
\mathcal{M}^\bullet = [0 \longrightarrow \mathcal{M}^0 \xrightarrow{L_0} \mathcal{M}^1 \xrightarrow{L_1} \cdots \xrightarrow{L_{k_0}} \mathcal{M}^{k_0} \longrightarrow 0],
\end{equation}

where $L_k$ acts on $\mathcal{M}^k$ by left multiplication; $\mathcal{M}^\bullet$ is then an object of $\text{D}_{\text{g}, \text{coh}}^b(\mathcal{D}^o)$. Then $\text{R} \Gamma(\mathcal{M}^\bullet \otimes_D^L A_M)$ is represented by a differential complex

\begin{equation}
0 \longrightarrow \Gamma(M, E^0) \xrightarrow{L_0} \Gamma(M, E^1) \xrightarrow{L_1} \cdots \xrightarrow{L_{k_0}} \Gamma(M, E^{k_0}) \longrightarrow 0
\end{equation}

and $\text{Tor}_k^D(\mathcal{M}^\bullet, A_M)$ is its $k$-th cohomology group, where $\Gamma(M, E^k)$ denotes the space of analytic sections of $E^k$ on $M$. For a vector bundle $E$, let us set $B^{(n)}(E) = \mathcal{O}(E) \otimes_{\mathcal{O}} B_M^{(n)}$. $\text{RHom}_D(M; \mathcal{M}^\bullet, B_M^{(n)})$ is represented by

\begin{equation}
0 \leftarrow \Gamma(M, B^{(n)}(E^0_0)) \xleftarrow{L_0} \Gamma(M, B^{(n)}(E^1_0)) \xleftarrow{L_1} \cdots \xleftarrow{L_{k_0}} \Gamma(M, B^{(n)}(E^*_{k_0})) \leftarrow 0,
\end{equation}

where $E^*_k$ is the dual bundle of $E^k$ and $L_k$ acts on $B^{(n)}(E^*_k)$ by right multiplication; $\text{Ext}_D^{-k}(M; \mathcal{M}^\bullet, B_M^{(n)})$ is its $k$-th homology group. The pairing of $\text{Tor}_k^D(\mathcal{M}^\bullet, A_M)$ and $\text{Ext}_D^{-k}(M; \mathcal{M}^\bullet, B_M^{(n)})$ is induced from

\begin{equation}
\Gamma(M, E^k) \times \Gamma(M, B^{(n)}(E^*_k)) \to \mathbb{C}, \quad (u, v) \mapsto \int_M \langle u, v \rangle,
\end{equation}

$\langle u, v \rangle$ being the pairing of $E^k$ and $E^*_k$.

Remark. If (1.3) is an elliptic complex of vector bundles on $M$, for $\mathcal{M}^\bullet$ given by (1.2), $\text{Ch}(\mathcal{M}^\bullet) \cap T^*_M X$ is contained in the zero section. The converse is not true in general.
2. Proof of Theorem 1

We can assume that $H^k\mathcal{M}^\bullet = 0$ for any $k \neq 0$; in what follows, $\mathcal{M}$ denotes a coherent right $D$-module on $X$.

Let $\mathcal{M}^\bullet = \mathcal{E}xt^n_{D}(\mathcal{M}, D)$; then $\mathcal{M}^\bullet$ is a holonomic left $D$-module, and we have an injective $D$ homomorphism $\mathcal{E}xt^n_{D}(\mathcal{M}^\bullet, D) \to \mathcal{M}$. Let $\mathcal{M}^{**} = \mathcal{E}xt^n_{D}(\mathcal{M}^\bullet, D)$, and $\mathcal{N} = \mathcal{M}/\mathcal{M}^{**}$; then $\mathcal{M}^{**}$ is a holonomic $D$-module, and the sequence

(2.0) \[ 0 \to \mathcal{M}^{**} \to \mathcal{M} \to \mathcal{N} \to 0 \]

is exact. Since $\mathcal{E}xt^n_{D}^{-1}(\mathcal{M}^{**}, D) = 0$ and $\mathcal{E}xt^n_{D}(\mathcal{M}, D) \to \mathcal{E}xt^n_{D}(\mathcal{M}^{**}, D)$ is an isomorphism, we see that $\mathcal{E}xt^n_{D}(\mathcal{N}, D) = 0$. Hence, by [K2, 2.11], $\text{Ch}(\mathcal{N})$ has no irreducible components of codimension $n$. Since $\text{Ch}(\mathcal{N}) \subset \text{Ch}(\mathcal{M})$, by the hypothesis of the theorem, $\text{Ch}(\mathcal{N}) \cap \text{SS}(F)$ is contained in the zero section; therefore $(\mathcal{N}, F)$ is elliptic in the sense of [SS]. Moreover, by the definition of $\mathcal{N}$, if $\mathcal{M}$ is a good coherent $D$-module, $\mathcal{N}$ is also good coherent.

Since Theorem 1 is proved for elliptic pairs in [SS, Part 1], by exact sequence (2.0), we may assume from the beginning $\mathcal{M}$ to be holonomic. If $\mathcal{M}$ is holonomic, by Kashiwara's theorem [K1], $\mathcal{M} \otimes_{D}^L \mathcal{O}$ is $\mathbb{C}$-constructible. Hence $(\mathcal{M} \otimes_{D}^L \mathcal{O}) \otimes F$ is an $\mathbb{R}$-constructible sheaf on $X$. Its support being compact by assumption, by [KS, Prop.8.4.8], $H^k\Gamma(\mathcal{M} \otimes_{D}^L \mathcal{O} \otimes F)$ is finite dimensional for all $k \in \mathbb{Z}$. In the same way, the $\mathbb{C}$-constructibility of $R\text{Hom}_{D}(\mathcal{M}, \Omega^n)$ yields the finite dimensionality of $H^k\Gamma(R\text{Hom}_{D}(\mathcal{M} \otimes F, \Omega^n))$. This completes the proof of the finiteness part.

We now prove (1.0) to be an isomorphism for a holonomic $D$-module $\mathcal{M}$, assuming $\text{Supp}(\mathcal{M}) \cap \text{Supp}(F)$ is compact. Let $D^b_{h}(D^o)$ denote the full triangulated subcategory of $D^b(D^o)$ consisting of bounded complexes with holonomic cohomology groups. Letting $\text{DR}(\mathcal{M}) = \mathcal{M} \otimes_{D}^L \mathcal{O}[-n]$ for an object $\mathcal{M}$ of $D^b(D^o)$, we have first:

**Lemma 2.1.** Let $\mathcal{M}$ be an object of $D^b_{h}(D^o)$, $F$ of $D^b_{R,c}(X)$. Then there is an isomorphism

(2.1) $\text{DR}(\mathcal{M}) \otimes F \cong D' R\text{Hom}_{D}(\mathcal{M} \otimes F, \Omega^n),$

where $D' = R\text{Hom}_{\mathbb{C}}(\bullet, C_X)$.

The proof will be given later.
Since $R\mathcal{H}om_D(M \otimes F, \Omega^n)$ is $R$-constructible, from (2.1), we have

$$D'(DR(M) \otimes F) \cong R\mathcal{H}om_D(M \otimes F, \Omega^n).$$

By the Verdier duality, we get

$$R\Gamma (R\mathcal{H}om_D(M \otimes F, \Omega^n)) [n] \cong R\mathcal{H}om_C(R\Gamma_c(DR(M) \otimes F), C)[-n]$$

$$= R\mathcal{H}om_C(R\Gamma_c(M \otimes^D \mathcal{O} \otimes F), C).$$

This completes the proof of Theorem 1. QED

**Proof of Lemma 2.1.** If $F = C_X$, this duality formula is contained in [KK] and [M2], and we have

(2.2) \hspace{1cm} DR(M) \cong D' R\mathcal{H}om_D(M, \Omega^n).

Let $C = R\mathcal{H}om_D(M, \Omega^n)$; then, by Kashiwara’s theorem [K1], $C$ is $C$-constructible. Hence

$$D'(D' C \otimes F) \cong R\mathcal{H}om_C(F, C)$$

(see [KS, 3.4.6]), and, since $D' C \otimes F$ is $R$-constructible,

$$D' C \otimes F \cong D' R\mathcal{H}om_C(F, C).$$

By (2.2), we get

$$DR(M) \otimes F \cong D' R\mathcal{H}om_C(F, C).$$

QED

**References**


