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Laplace transform and Fourier-Sato transform

MASAKI KASHIWARA AND PIERRE SCHAPIRA

Review on formal and moderate cohomology. Let $M$ be a real manifold, and let $R$-$Cons(M)$ denote the category of $R$-constructible sheaves on $M$, $D_{R-c}^b(C_M)$ its derived category. Recall first the functors $Thom(\cdot, Db_M)$ of [4] and the dual functor $\otimes wC_M^\infty$ of [6], defined on the category $R$-$Cons(M)$, with values in the category $Mod(D_M)$ of $D_M$-modules on $M$. (The first functor is contravariant.)

They are characterized as follows. Denote by $Db_M$ the sheaf of Schwartz’s distributions on $M$ and by $C_M^\infty$ the sheaf of $C^\infty$ functions on $M$. Let $Z$ (resp. $U$) be a closed (resp. open) subanalytic subset of $M$. Then these two functors are exact and moreover:

$$Thom(C_Z, Db_M) = \Gamma_Z(Db_M),$$

$$C_U \otimes C_M^\infty = \mathcal{I}_{M\setminus U}^\infty,$$

where $\Gamma_Z(Db_M)$ denotes as usual the subsheaf of $Db_M$ of sections supported by $Z$ and $\mathcal{I}_{M\setminus U}^\infty$ denotes the ideal of $C_M^\infty$ of sections vanishing up to order infinity on $M \setminus U$.

These functors being exact, they extend naturally to the derived category $D_{R-c}^b(C_X)$. We keep the same notations to denote the derived functors.

Now let $X$ be a complex manifold and denote by $\overline{X}$ the complex conjugate manifold and by $X_R$ the underlying manifold. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X$, let $\mathcal{D}_X$ be the sheaf of finite order holomorphic differential operators on $X$. The functors of moderate and formal cohomology (see [4], [6]) are defined for $F \in D_{R-c}^b(C_{X_R})$ by:

$$Thom(F, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_X}^{\mathcal{O}_X}(\mathcal{O}_{\overline{X}}, Thom(F, Db_{X_R}))$$

$$F \otimes w \mathcal{O}_X = R\mathcal{H}om_{\mathcal{D}_X}^{\mathcal{O}_X}(O_{\overline{X}}, F \otimes w C_{X_R}^\infty).$$

Laplace transform. Consider a complex vector space $E$ of complex dimension $n$, and denote by $j : E \rightarrow P$ its projective compactification. Let $D_{R-c,R^+}(C_E)$ denote the full triangulated subcategory of $D_{R-c}^b(C_E)$ consisting of $R^+$-conic objects (i.e.

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objects whose cohomology is $R$-constructible and locally constant on the orbits of the action of $R^+$ on $E$.

Let $F \in D^{b}_{R-\epsilon,R^+}(C_E)$ and set for short

$$\mathcal{T}\text{Hom}(F, \mathcal{O}_E) = R\Gamma(P; \mathcal{T}\text{hom}(j_! F, \mathcal{O}_P))$$

$$\mathcal{W}\text{Tens}(F, \mathcal{O}_E) = R\Gamma(P; j_! F \otimes \mathcal{O}_P)$$

These are objects of the bounded derived category $D^b(W(E))$ of the category of modules over the Weyl algebra $W(E)$. Let $E^*$ denote the dual vector space to $E$. One denotes by $F^\wedge$ the Fourier-Sato transform of the sheaf $F$, an object of $D^b_{R-\epsilon,R^+}(C_{E^*})$. (see [5] for an exposition). One identifies $D^b(W(E^*))$ to $D^b(W(E))$ by the usual Fourier transform.

**Theorem 0.1.** The classical Laplace transform extends naturally as isomorphisms in $D^b(W(E))$:

$$L_E : \mathcal{T}\text{Hom}(F, \mathcal{O}_E) \simeq \mathcal{T}\text{Hom}(F^\wedge[n], \mathcal{O}_{E^*})$$ (0.1)

$$L_E : \mathcal{W}\text{Tens}(F, \mathcal{O}_E) \simeq \mathcal{W}\text{Tens}(F^\wedge[n], \mathcal{O}_{E^*}).$$ (0.2)

**Applications 1.** Let $M$ be a real vector space of dimension $n$ such that $E$ is a complexification of $M$. As a particular case of the theorem, we obtain a characterization of the Laplace transform of the space of distributions on $M$ supported by (not necessarily convex) cones. Let $\gamma$ denote a closed subanalytic cone in $M$ and let $\Gamma_\gamma S'_M$ denote the space of tempered distributions supported by $\gamma$. One has

$$\Gamma_\gamma S'_M \simeq \mathcal{T}\text{Hom}(C_\gamma[-n], \mathcal{O}_E).$$

Hence, we get that the Laplace transform of distributions induces an isomorphism:

$$L_E : \Gamma_\gamma S'_M \simeq \mathcal{T}\text{Hom}(\Gamma_\gamma S'_M, \mathcal{O}_E).$$

When $\gamma$ is proper and convex, this result is well known, since $(C_\gamma)^\wedge \simeq C_U$ where $U$ is the open convex tube $\text{int} \gamma^0 \times \sqrt{-1}M^*$, the interior of the polar cone to $\gamma$, and the right hand side denotes the space of holomorphic functions in this tube, tempered up to infinity. When $\gamma = M$, one recovers the isomorphism between $S'_M$ and $S'_\sqrt{-1}M^*$.

Let us consider now the case where $\gamma$ is a non degenerate quadratic cone. Let $(x', x'')$ denote the coordinates on $M = R^n = R^p \times R^q$ with $p, q \geq 1$, and let $\gamma = \{x; x'^2 - x''^2 \leq 0\}$. Let $(u', u'')$ denote the dual coordinates on $M^*$, and let $\lambda = \{(u', u''); u'^2 - u''^2 \geq 0\}$. One checks easily that $(C_\gamma)^\wedge \simeq C_\lambda[-q]$. We get the isomorphism sm:

$$L_E : \Gamma_\gamma S'_M \simeq H^q \mathcal{T}\text{Hom}(C_{\lambda \times \sqrt{-1}M^*}, \mathcal{O}_{E^*}).$$

This last result is essentially due to Faraut-Gindikin [2] (in a different language and with a different proof).

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Applications 2. Denote by $\mathcal{O}_E^t$ and $\mathcal{O}_E^w$ the conic sheaves on $E$ associated to the presheaves $U \mapsto \text{THom}(C_U, \mathcal{O}_E)$ and $U \mapsto \text{WTens}(C_U, \mathcal{O}_E)$, respectively. One easily deduces from the main theorem that the Laplace transform induces isomorphisms:

\[
\begin{align*}
(\mathcal{O}_E^t)^\wedge[n] & \simeq \mathcal{O}_E^{t,n}, \\
(\mathcal{O}_E^w)^\wedge[n] & \simeq \mathcal{O}_E^{w,n}.
\end{align*}
\]

This gives a new proof of a result of Hotta-Kashiwara [3] and Brylinski-Malgrange-Verdier [1] on the Fourier-Sato transform of the sheaf of holomorphic solutions of a monodromic $\mathcal{D}$-module.

References


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