<table>
<thead>
<tr>
<th>Title</th>
<th>Laplace transform and Fourier-Sato transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>KASHIWARA, MASAKI; SCHAPIRA, PIERRE</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1997), 983: 33-35</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-03</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/60945">http://hdl.handle.net/2433/60945</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
Laplace tranform and Fourier-Sato tranform

MASAKI KASHIWARA AND PIERRE SCHAPIRA

Review on formal and moderate cohomology. Let $M$ be a real manifold, and let $\mathcal{R}-\text{Cons}(M)$ denote the category of $\mathcal{R}$-constructible sheaves on $M$, $\mathcal{D}^b_{\mathcal{R}-c}(\mathcal{C}_M)$ its derived category. Recall first the functors $\mathcal{T}hom(\cdot, \mathcal{D}b_M)$ of [4] and the dual functor $\otimes \mathcal{C}^\infty_M$ of [6], defined on the category $\mathcal{R}-\text{Cons}(M)$, with values in the category $\text{Mod}(\mathcal{D}_M)$ of $\mathcal{D}_M$-modules on $M$. (The first functor is contravariant). They are characterized as follows.

Denote by $\mathcal{D}b_M$ the sheaf of Schwartz's distributions on $M$ and by $\mathcal{C}^\infty_M$ the sheaf of $\mathcal{C}^\infty$ functions on $M$. Let $Z$ (resp. $U$) be a closed (resp. open) subanalytic subset of $M$. Then these two functors are exact and moreover:

$$\mathcal{T}hom(\mathcal{C}_Z, \mathcal{D}b_M) = \Gamma_Z(\mathcal{D}b_M),$$

$$C_U \otimes \mathcal{C}^\infty_M = \mathcal{I}^\infty_{M\setminus U},$$

where $\Gamma_Z(\mathcal{D}b_M)$ denotes as usual the subsheaf of $\mathcal{D}b_M$ of sections supported by $Z$ and $\mathcal{I}^\infty_{M\setminus U}$ denotes the ideal of $\mathcal{C}^\infty_M$ of sections vanishing up to order infinity on $M\setminus U$.

These functors being exact, they extend naturally to the derived category $\mathcal{D}^b_{\mathcal{R}-c}(\mathcal{C}_X)$. We keep the same notations to denote the derived functors.

Now let $X$ be a complex manifold and denote by $\overline{X}$ the complex conjugate manifold and by $X_\mathcal{R}$ the real underlying manifold. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X$, let $\mathcal{D}_X$ be the sheaf of finite order holomorphic differential operators on $X$. The functors of moderate and formal cohomology (see [4], [6]) are defined for $F \in \mathcal{D}^b_{\mathcal{R}-c}(\mathcal{C}_{X_\mathcal{R}})$ by:

$$\mathcal{T}hom(F, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_\overline{X}}(\mathcal{O}_{\overline{X}}, \mathcal{T}hom(F, \mathcal{D}b_{X_\mathcal{R}}))$$

$$F \otimes \mathcal{O}_X = R\mathcal{H}om_{\mathcal{D}_\overline{X}}(\mathcal{O}_{\overline{X}}, F \otimes \mathcal{C}^\infty_{X_\mathcal{R}}).$$

Laplace transform. Consider a complex vector space $\mathcal{E}$ of complex dimension $n$, and denote by $j : \mathcal{E} \hookrightarrow \mathcal{P}$ its projective compactification. Let $\mathcal{D}^b_{\mathcal{R}-c, \mathcal{R}^+}(\mathcal{C}_\mathcal{E})$ denote the full triangulated subcategory of $\mathcal{D}^b_{\mathcal{R}-c}(\mathcal{C}_\mathcal{E})$ consisting of $\mathcal{R}^+$-conic objects (i.e.

AMC classification: 32L25, 58G37
Laplace transform and Fourier-Sato transform

objects whose cohomology is R-constructible and locally constant on the orbits of the action of $R^+$ on E).

Let $F \in D^b_{R-c,R+}(C_E)$ and set for short

$$\text{THom}(F, \mathcal{O}_E) = R\Gamma(P; \text{Thom}(j_! F, \mathcal{O}_P))$$

$$\text{WTens}(F, \mathcal{O}_E) = R\Gamma(P; j_! F \otimes \mathcal{O}_P)$$

These are objects of the bounded derived category $D^b(W(E))$ of the category of modules over the Weyl algebra $W(E)$. Let $E^*$ denote the dual vector space to $E$. One denotes by $F^\wedge$ the Fourier-Sato transform of the sheaf $F$, an object of $D^b_{R-c,R^+}(C_{E^*})$. (see [5] for an exposition). One identifies $D^b(W(E^*))$ to $D^b(W(E))$ by the usual Fourier transform.

**Theorem 0.1.** The classical Laplace transform extends naturally as isomorphisms in $D^b(W(E))$:

$$L_E : \text{THom}(F, \mathcal{O}_E) \simeq \text{THom}(F^\wedge[n], \mathcal{O}_{E^*})$$  \hspace{1cm} (0.1)

$$L_E : \text{WTens}(F, \mathcal{O}_E) \simeq \text{WTens}(F^\wedge[n], \mathcal{O}_{E^*}).$$  \hspace{1cm} (0.2)

**Applications 1.** Let $M$ be a real vector space of dimension $n$ such that $E$ is a complexification of $M$. As a particular case of the theorem, we obtain a characterization of the Laplace transform of the space of distributions on $M$ supported by (not necessarily convex) cones. Let $\gamma$ denote a closed subanalytic cone in $M$ and let $\Gamma_\gamma S'_M$ denote the space of tempered distributions supported by $\gamma$. One has $\Gamma_\gamma S'_M \simeq \text{THom}(C_\gamma[-n], \mathcal{O}_E)$. Hence, we get that the Laplace transform of distributions induces an isomorphism:

$$L_E : \Gamma_\gamma S'_M \simeq \text{THom}((C_\gamma)^\wedge, \mathcal{O}_{E^*}).$$

When $\gamma$ is proper and convex, this result is well known, since $(C_\gamma)^\wedge \simeq C_U$ where $U$ is the open convex tube $\text{int} \gamma^0 \times \sqrt{-1}M^*$, the interior of the polar cone to $\gamma$, and the right hand side denotes the space of holomorphic functions in this tube, tempered up to infinity. When $\gamma = M$, one recovers the isomorphism between $S'_M$ and $S'_{\sqrt{-1}M^*}$.

Let us consider now the case where $\gamma$ is a non degenerate quadratic cone. Let $(x', x'')$ denote the coordinates on $M = R^n = R^p \times R^q$ with $p, q \geq 1$, and let $\gamma = \{(x; x'^2 - x''^2 \leq 0)\}$. Let $(u', u'')$ denote the dual coordinates on $M^*$, and let $\lambda = \{(u', u''); u'^2 - u''^2 \geq 0\}$. One checks easily that $(C_\gamma)^\wedge \simeq C_{\lambda}[-q]$. We get the isomorphism:

$$L_E : \Gamma_\gamma S'_M \simeq H^q \text{THom}(C_{\lambda \times \sqrt{-1}M^*}, \mathcal{O}_{E^*}).$$

This last result is essentially due to Faraut-Gindikin [2] (in a different language and with a different proof).

Masaki Kashiwara and Pierre Schapira
Applications 2. Denote by $\mathcal{O}_E^t$ and $\mathcal{O}_E^w$ the conic sheaves on $E$ associated to the presheaves $U \mapsto \text{THom}(C_U, \mathcal{O}_E)$ and $U \mapsto \text{WTens}(C_U, \mathcal{O}_E)$, respectively. One easily deduces from the main theorem that the Laplace transform induces isomorphisms:

\[
(\mathcal{O}_E^t)^{\wedge}[n] \cong \mathcal{O}_{E^*}^t,
\]

\[
(\mathcal{O}_E^w)^{\wedge}[n] \cong \mathcal{O}_{E^*}^w.
\]

This gives a new proof of a result of Hotta-Kashiwara [3] and Brylinski-Malgrange-Verdier [1] on the Fourier-Sato transform of the sheaf of holomorphic solutions of a monodromic $\mathcal{D}$-module.

References


M.K. RIMS, Kyoto University, Kyoto 606 Japan

P.S. Institut de Mathématiques; Analyse Algébrique; Université Pierre et Marie Curie; Case 247; 4, place Jussieu; F-75252 Paris Cedex 05; email: schapira@mathp6.jussieu.fr

Masaki Kashiwara and Pierre Schapira