数理解析研究所講究録
984巻 1997年 56-70

On a duck solution and delay in the FitzHugh-Nagumo equation

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Abstract

The computing results of N.Kakiuchi on the FitzHugh-Nagumo equation (FHN) suggested that the delayed phenomenon proved by J.Su (Journal of Differential equations 105, 1993, 180-215) may occur with a duck solution. E.Benoit (Societe Mathematique de France Asterisque 109-110, 1983, 159-191) proved the existence of a duck solution under the condition that a pseudo singular point is "saddle". In this paper, using the result of E.Benoit, a rigorous proof of the existence of a duck solution in the FHN equation is provided.

On the FHN equation, considering a necessary condition for the existence of a solution which is jumping, it can be found that the coefficient b in the FHN equation is $O(\varepsilon)$ where ε is a very small positive number. This fact is ensured via the experiment by R.FitzHugh and S.M.Baer etc.. The FHN equation is newly formulated under keeping a qualitative feature of the solutions when $I=I_0+\varepsilon t$ and $b=c\varepsilon$. Here c is any constant. Then, it can be shown that there exists a duck solution with a jumping solution in the FHN equation.

As a result, the FHN equation gives us a new bifurcation system with an injected control or a bifurcation parameter I.

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1. Introduction and delayed phenomena

In 1961, FitzHugh[2] and in 1962, Nagumo et al.[3] proposed a simplified system which contains the main qualitative features of the original Hodgkin-Huxley system in 1952[1]. These systems describe the generation and propagation of the nerve impulse along the giant axon of the squid. The above system so-called the FitzHugh-Nagumo equation (FHN) for the space clamped segment of the axon have the following autonomous form (1.1):

$$dv/dt=-\rho(v)-w+I, \tag{1.1a}$$

$$dw/dt=b(v-\gamma w), \tag{1.1b}$$

$$\rho(v)=v(v-1)(v-a), \tag{1.1c}$$

where a $(0<a<1/2)$, b, γ are positive constants.

Here v(t) denotes the potential difference at the time t across the membrane of the axon and w(t) represents a recovery current which is often taken to be the sum of all ion flows [1]. Furthermore, I is an injected electric current on the membrane, a control or bifurcation parameter. Equation (1.1a) expresses Kirchhoff's law applied to the membrane; (1.1b) relates the recovery current with the potential. From biophysical considerations, it is reasonable to restrict γ so that

$$\gamma_1 \equiv 1/\gamma-(1-a+a^2)/3>0, \tag{1.2a}$$

$$\gamma_2 \equiv (1-a+a^2)/3-b\gamma>0. \tag{1.2b}$$

If the current I is kept constant, equation (1.2a) ensures that the system (1.1) has a unique steady state solution $(v_0(I), w_0(I))$. This solution called the frame solution is determined by the equation

$$F(v,w)=\begin{pmatrix}-\rho(v)-w\\ bv-b\gamma w\end{pmatrix}=\begin{pmatrix}-I\\ 0\end{pmatrix}. \tag{1.3}$$

Its components $v_0(I)$ and $w_0(I)$ increase as I increases. The linearized system of (1.1) for the frame solution is the following:

$$d\tilde{v}/dt=-\rho'(v_0(I))\tilde{v}-\tilde{w}, \tag{1.4a}$$

$$d\tilde{w}/dt=b\tilde{v}-b\gamma\tilde{w}, \tag{1.4b}$$

$$(\tilde{v}=v-v_0(I),\ \tilde{w}=w-w_0(I)).$$

The stability of the system is determined by the two eigenvalues of the Jacobian matrix M;

$$M = \begin{pmatrix} -\rho'(v_0(I)) & -1 \\ b & -b\gamma \end{pmatrix} \quad (\rho'(v)=d\rho/dv) . \tag{1.5}$$

There exist I_- and I_+ where $I_-<(a+1)/3<I_+$ such that whenever $I<I_-$ or $I>I_+$, the eigenvalues of M have negative real parts, i.e., the frame equation (1.4) is stable and the eigenvalues of M have positive real parts if $I_-<I<I_+$, i.e., the equation is unstable.

Assume that the current I which is treated as a bifurcation parameter varies very slowly as the time goes by. Moreover, for simplicity, assume that the current I(t) has the form of

$$I=I(t)=I_0+\varepsilon t, \tag{1.6}$$

where $\varepsilon>0$ is a very small parameter and $I_0<I_-$. The frame solution $(v_0(I), w_0(I))$ is uniquely determined by the bifurcation parameter I, since F in (1.3) is diffeomorphic.

Using I as an independent variable, the system (1.1) becomes the <u>non-autonomous form</u> (1.7):

$$\varepsilon\, dv/dI=-\rho(v)-w+I, \tag{1.7a}$$

$$\varepsilon\, dw/dI=bv-b\gamma w \quad . \tag{1.7b}$$

Note that the conditions of ρ in (1.1c) and γ_1, γ_2 in (1.2) are still satisfied.

If I is an independent bifurcation parameter, the following phenomenon will occur. The solution of (1.7) with the initial conditions,

$$v(I_0)\equiv v_0(I_0),\ W(I_0)\equiv w_0(I_0),\ I(0)=I_0, \tag{1.8}$$

stays close to the frame solution until I reaches I_-, and then jumps away from the frame solution shortly after I increases and passes I_-. However, if I is as in (1.6), this phenomenon does not occur.

From a Hopf bifurcation structure, as I increases through I_-, the solution of (1.7) would turn to the large amplitude oscillations. Such a critical point is observed, but the value I_a of I at which it occurs is considerably delayed beyond the value I_-. In 1987, 1988, Neistadt[9],[10] and in 1989, Baer, Erneux and Rinzel[5] proceeded an extensive computational experiment of the FHN equation for the <u>delayed phenomena</u> and they began to consider the corresponding mathematical problem.

In 1993, Su[6] provided a rigorous proof of the results conjecturred in [4], [5] by considering the Taylor expansion of the system (1.7) for the frame solution $(v(I_0),w(I_0))$. He showed that the solution of (1.7) starting from any point near the frame solution at $I_0<I_-$ stays near the frame solution until I reaches $I_q>I_-$. Furthermore, in case that I_0 is close to I_-, a description of how the solution moves from the frame solution to become a large amplitude solution after $I>I_q$ was given.

Suppose that $\lambda(z(I))$ is the eigenvalue of the linearization at $I_-<I<I_q$;

$$\lambda(z)=(3z^2+\gamma_1)(-3z^2+3z_-^2+((3z^2-3z_-^2-2b\gamma)^2-4b)^{1/2})/2, \tag{1.9a}$$

$$z=z(I),\quad z_-=z(I_-)=v_0(I_-)-(a+1)/3=-(3\gamma_2)^{1/2}/3. \tag{1.9b}$$

In the complex domain, on a path L, there exists the level curve of the below function ϕ satisfying $\mathrm{Re}(\lambda(z(I))(dz/dI))=0$:

$$\phi(z)=\mathrm{Re}\int_L \lambda(\tau)d\tau,\quad \tau,\ z\in C, \tag{1.10a}$$

$$\lambda(z)=\partial\phi/\partial z_1-i\,\partial\phi/\partial z_2=0,\quad z=z_1+iz_2. \tag{1.10b}$$

To seek the path L leads us to draw a conclusion that the solution has a delayed phenomena. If $\lambda(z)$ is analytic, the Cauchy-Riemann condition ensures that $\phi(z)$ determined by $\lambda(z)$ of (1.9) is well defined.

FitzHugh[2] and Baer etc.[5] have already pointed out that the very small constant b in (1.7) keeps the qualitative features of (1.7) such as

$$b=c\varepsilon \tag{1.11}$$

where $\varepsilon>0$ is a very small parameter in (1.6) and $c>0$ is any constant. As the bifurcation parameter I varies very slowly and b satisfies (1.11), the system (1.7) becomes the non-autonomous form (1.12):

$$\varepsilon\, dv/dI=-\rho(v)-w+I, \tag{1.12a}$$

$$dw/dI=c(v-\gamma w). \tag{1.12b}$$

Furthermore, by changing the coordinate $I=X$, $w=Y$ and $v=Z$, equation (1.12) becomes the following autonomous form (1.13):

$$dX/dI=1, \tag{1.13a}$$

$$dY/dI=c(Z-\gamma Y), \tag{1.13b}$$

$$dZ/dI=(-\rho(Z)-Y+X)/\varepsilon, \tag{1.13c}$$

where the condition (1.1c) and (1.2) are still satisfied.

In the following section, it will be seen that the system of (1.13) gives us a new type of bifurcations.

2. Benoit's theorem concerning constrained systems

2.1 Standard ducks

First we consider a constrained system (2.1):

$dx/dt=f(x,y,z)$, (2.1a)

$dy/dt=g(x,y,z)$, (2.1b)

$h(x,y,z)=0$, (2.1c)

where f, g and h are defined in R^3 and satisfy the following conditions (B1), (B2) and (B3):

(B1) f and g are of class C^1, and h is of class C^2,

(B2) the set $S=\{(p=(x,y,z)\in R^3 \mid h(x,y,z)=0\}$ is a 2-dimensional differentiable manifold and the set $PL=\{p=(x,y,z)\in S \mid \partial h(x,y,z)/\partial z=0\}$ is an 1-dimensional differentiable manifold,

(B3) either the value of f or that of g is nonzero at the same point $p=(x,y,z)\in PL$.

Let $(x(t), y(t), z(t))$ be a solution of (2.1), then the following equation (2.1c') holds by differentiating (2.1c) with respect to the time t:

$h_x(x,y,z)f(x,y,z)+h_y(x,y,z)g(x,y,z)+h_z(x,y,z)dz/dt=0$, (2.1c')

where $h_\alpha(x,y,z)=\partial h(x,y,z)/\partial\alpha$ $(\alpha=x,y,z)$.

The above system (2.1) becomes the system (2.2):

$dx/dt=f(x,y,z)$, (2.2a)

$dy/dt=g(x,y,z)$, (2.2b)

$dz/dt=-(h_x(x,y,z)f(x,y,z)+h_y(x,y,z)g(x,y,z))/h_z(x,y,z)$, (2.2c)

where $(x,y,z)\in S\setminus PL$.

Remark

The system (2.1) coincides with the system (2.2) at any point $p\in S\setminus PL$.

Secondly in order to study the system (2.2), we consider the newly revised system (2.3):

$dx/dt= -h_z(x,y,z)f(x,y,z)$, (2.3a)

$dy/dt= -h_z(x,y,z)g(x,y,z)$, (2.3b)

$dz/dt= h_x(x,y,z)f(x,y,z)+h_y(x,y,z)g(x,y,z)$. (2.3c)

Note that equation (2.3) is well defined at any point of R^3. Therefore, (2.3) is well defined indeed on any point of PL.

Compare the solutions of (2.3) with those of (2.1) on $S \setminus PL$. The solutions of (2.3) coincide with those of (2.1) except the velocity (+,-) and the orientation when they start from the same initial points. Thus, each phase path is quite the same, that is, they have the very same solution curves.

<u>Definition 2.1</u>

A singular point of (2.3) is called a <u>pseudo singular point</u> of (2.1) and a <u>set of the pseudo singular points</u> (<u>PS</u>) is denoted as follows:

$$PS=\{(x,y,z)\in PL \mid h_x(x,y,z)f(x,y,z)+h_y(x,y,z)g(x,y,z)=0\}. \qquad (2.4)$$

Moreover, here, the next conditions (B4) and (B5) are assumed:

(B4) The surface S can be expressed as $y=\phi(x,z)$, (or $x=\phi(y,z)$) in the neighborhood of PL, i.e., for any $(x,y,z)\in S$, $h_y(x,y,z)\neq 0$ or $h_x(x,y,z)\neq 0$ hold.

When $y=\phi(x,z)$, the following system (2.5), which restricts the system (2.3) on the surface S, is obtained using (B4):

$$dx/dt=-h_z(x,\phi(x,z),z)f(x,\phi(x,z),z), \qquad (2.5a)$$

$$dz/dt=h_x(x,\phi(x,z),z)f(x,\phi(x,z),z)+h_y(x,\phi(x,z),z)g(x,\phi(x,z),z). \qquad (2.5b)$$

(B5) All the singular points of (2.5) are nondegenerate, i.e., the matrix induced from the linearized system of (2.5) at a singular point has two nonzero eigenvalues.

Note that all the points contained in PS are the singular points of (2.5). When $x=\phi(y,z)$, a similar equation is obtained in the same manner.

<u>Definition 2.2</u>

If the two eigenvalues λ_1, λ_2 mentioned in (B5) have the property that $\lambda_1<0<\lambda_2$, a pseudo singular point of (2.1) is called as a <u>pseudo singular saddle point</u>.

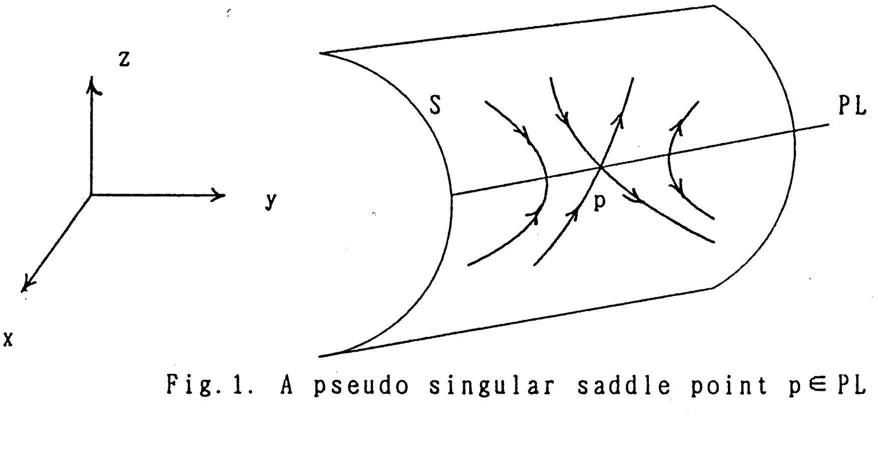

Fig.1. A pseudo singular saddle point $p\in PL$

Thirdly we consider the following new systems $E_n (n=1, 2, \cdots)$ (2.6):

$$dx/dt=f(x, y, z), \tag{2.6a}$$

$$(E_n) \quad dy/dt=g(x, y, z), \tag{2.6b}$$

$$\varepsilon_n dz/dt=h(x, y, z), \tag{2.6c}$$

where f, g and h are the same as the system (2.1) and $\varepsilon_n (>0) \to 0$ as $n \to \infty$. For a fixed sufficiently large n, E_n devides the surface $S \setminus PL$ into two parts: one consists of an attractive region and the other consists of a repulsive region.

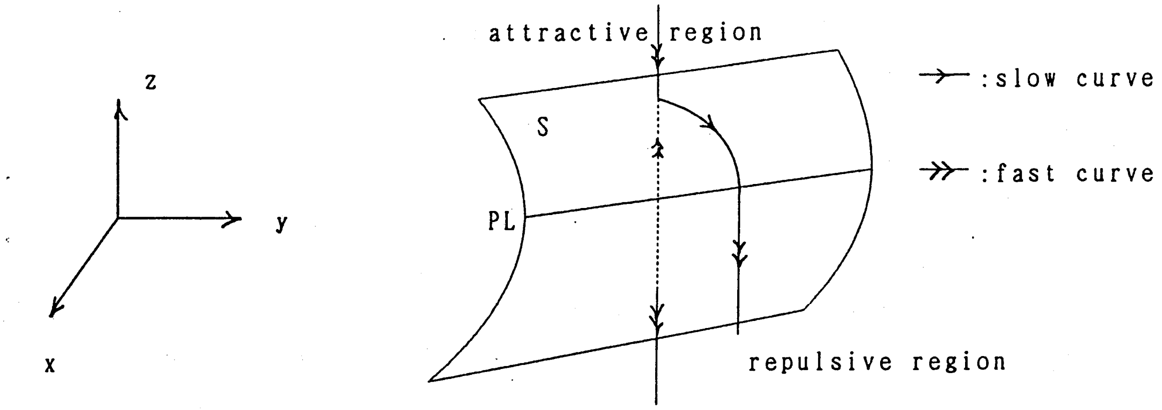

Fig. 2. Attractive, repulsive region

A solution which starts from a neighborhood of the attractive part goes rapidly toward S perpendicularly and then goes slowly along S. A solution which starts from a neighborhood of repulsive part leaves from S rapidly unless it starts at $p \in S \setminus PL$. Now let us define a duck solution on a set of the systems E_n $(n=1, 2, \cdots)$.

<u>Definition 2.3</u> (a <u>standard duck</u>)

A <u>standard duck solution</u> (or simply <u>standard duck</u>) on the set of equations $\{E_n \mid n=1, 2, \cdots\}$ is defined as follows: it is a sequence $(x_n(t), y_n(t), z_n(t))$ consisting of the solutions of E_n $(n=1, 2, \cdots)$ such that

(1) the sequence $(x_n(t), y_n(t), z_n(t))$ is defined for $t \in (c_n, d_n)$,

(2) there are two closed disjoint subintervals $[c_n', d_n']$ and $[c_n'', d_n'']$ of (c_n, d_n) in which for any $t \in [c_n', d_n']$, $(x_n(t), y_n(t), z_n(t))$ lies in the attractive part of S, and for any $t \in [c_n'', d_n'']$, $(x_n(t), y_n(t), z_n(t))$ lies in the repulsive part of S,

(3) as $n \to \infty$, the curves $(x_n(t), y_n(t), z_n(t))$ $(t \in (c_n, d_n))$ converge to a curve C of the finite length in S, and the curve C devides into two parts

C', C''. C' belongs to the attractive part of S, C'' belongs to the repulsive part and the lengths of C', C'' are not zeroes.

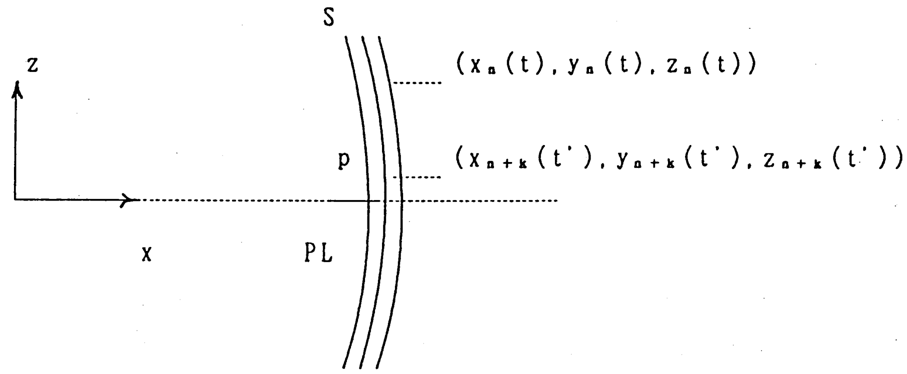

Fig. 3. A duck solution on $\{E_n\}$

2.2 <u>Non-standard ducks</u>

Let the set N be of all natural numbers. Let $\mathcal{U}_0$ be the filter on N consisting of all subsets of N whose complement is finite. Then we can extend $\mathcal{U}_0$ to an ω-incomplete ultrafilter $\mathcal{U}$ and define on N a probability measure in the sense of [11], such that $P(A)=1$ for $A \in \mathcal{U}$ and $P(A)=0$ for $A \notin \mathcal{U}$. In what follows the measure P is fixed.

Let us introduce the set $^*R$ of a random numbers, which consists of equivalence classes of maps $N \to R$, whose two maps x', $x'': N \to R$ are regarded as equivalent if $x'(i)=x''(i)$ $(\in \mathcal{U})$ almost everywhere (or briefly a.e.) on N. If $x \in {}^*R$, then any of the maps $x': N \to R$ defining x is called a <u>version</u> of the element x. There is a natural embedding $R \subset {}^*R$ under which to every number $\alpha \in R$ there corresponds the class of the constant function α on N. It is easy to see that $^*R$ has the natural structure of an ordered field, that is, the algebraic operations (addition and multiplication) are introduced by means of the corresponding operations over versions, which is carried out for each $i \in N$. The order is introduced as follows; if $x, y \in {}^*R$ and x', y' are versions of x and y, then N is devided into the subsets:

$$N = N_+ \cup N_0 \cup N_-, \tag{2.7a}$$

$$N_+ = \{i \mid x'(i) > y'(i)\},\ N_0 = \{i \mid x'(i) = y'(i)\},\ N_- = \{i \mid x'(i) < y'(i)\}. \tag{2.7b}$$

Owing to the property of P, one of these subsets N_+, N_0 and N_- has measure 1 and the other two have measure 0. In accordance with this, we define either a.e. $x'(i)>y'(i)$ $(x>y)$ or a.e. $x'(i)=y'(i)$ $(x=y)$ or a.e. $x'(i)<y'(i)$ $(x<y)$. This is an order relation in $^*R$.

Similarly, it is verified that if $x \in {}^*R \setminus \{0\}$, then there is an inverse element $1/x$. Put $\varepsilon_n = 1/n$ $(n=1, 2, \cdots)$ and let ε be the corresponding equivalence class in $^*R$. It is clear that $0 < \varepsilon < \varepsilon_0$, $\varepsilon_0 \in R$, since finite sets have measure 0 in N. Thus, $\varepsilon \in {}^*R \setminus R \neq \phi$, that is, $^*R$ is actually a non-trivial extension of R. The number $\varepsilon \in {}^*R$ just constructed above is <u>infinitesimally small</u>. Having one such number, by means of algebraic operations, we can construct arbitrarily many other numbers belonging to $^*R \setminus R$. For instance, if $\alpha \in R \setminus \{0\}$, $\alpha\varepsilon$ is also infinitesimally small and differs from ε for $\alpha \neq 1$. The number $1/\varepsilon$ is infinitely large. A <u>halo of α</u> is defined by adding to $\alpha \in R$ all infinitesimally small numbers. The haloes of any two distinct numbers α_1, $\alpha_2 \in R$ do not intersect. An element $x \in {}^*R$ is called <u>limited</u> if there is a number $c \in R$ such that $-c < x < c$. Each <u>limited</u> $x \in {}^*R$ has a <u>shadow $^\circ x$</u> i.e., a number $^\circ x \in R$ such that $x - {}^\circ x$ is infinitesimally small. It is well defined and the following formula holds,

$$^\circ x = \inf\{\alpha \mid \alpha \in R,\ \alpha > x\}. \tag{2.8}$$

Let $x \in {}^*R \setminus R$ be unlimited (not limited), then $|x| > c$ for any standard $c > 0$, from which it follows that x is infinitely large and $1/x$ is infinitesimally small.

Let W be an infinite countable set. Fixing a numbering of its points, it can be identified with the set N. For any infinite set X on W, a map $f: X \to Y$ is given. Then it generates a natural map $^*f: {}^*X \to {}^*Y$ such that if $x \in {}^*X$ and $x' = x'(w)$ is a version of x, then $^*f(x)$ must be put equal to the class of the map $w \to f(x'(w))$ from W to Y. Namely, $^*f$ is defined by applying f to each $w \in W$ on versions. It induces functors on many natural subcategories, for example, the categories of groups, rings, fields, partialy ordered sets, e.t.c..
If a version $w \to A'(w)$ of any subset $A \subset X$ $({}^*A \subset {}^*X)$ is given, then for any $x \in X$ with $x' = x'(w)$, $x \in A$ means that a.e. $x'(w) \in A'(w)$. Thus, $^*P(X)$ consisting of any subset of $^*X$ is included in $P({}^*X)$ of all subsets of $^*X$. Elements of $^*P(X)$ are called <u>internal subsets</u> of $^*X$. Elements of $P({}^*X) \setminus {}^*P(X)$ are called <u>external subsets</u> of $^*X$. For general statements restricted within set theory, it can be allowed to use any sets, functions, quantifiers, the implication sign, parentheses, the signs of equality, substitution of an argument in a function, e.t.c.. Seee Davis[12], where the rigorous mathematical description is given. For given $A \in P(X)$, applying the <u>asterisk operation</u>, we get the statements referring only to internal sets and functions.

In this way, the operation can be applied not only to R but any set X restricted within set theory, and a set $^*X$ is obtained like as R^* from R. Considering the existence of the duck solution, the internal sets and functions make an important role.

Now we consider the following system E:

$$\text{(E)}\quad \begin{aligned} dx/dt&=f(x,y,z),\\ dy/dt&=g(x,y,z),\\ \varepsilon\, dz/dt&=h(x,y,z), \end{aligned} \tag{2.9}$$

where ε is infinitesimally small.

Here a definition of a non-standard duck solution and the transfer principle are given as follows.

Definition 2.4 (a non-standard duck)

A solution $(x(t),y(t),z(t))$ of the system E on which $\varepsilon_n=\varepsilon$ (ε_n is in E_n and $\varepsilon\in{}^*R$) holds is called a non-standard duck solution (or simply non-standard duck),

if there are standard $t_1<t_0<t_2$ such that

(1) $^\circ(x(t_0),y(t_0),z(t_0))\in S$,

(2) for $t\in(t_1,t_0)$ the segment of the trajectory $(x(t),y(t),z(t))$ is infinitesimally close to the attracting part of the slow curve,

(3) for $t\in(t_0,t_2)$ it is infinitesimally close to the repelling part of the slow curve,

(4) the attracting and repelling pieces which the solutin covers are not infinitesimally small.

Transfer principle:

Any statement $\mathcal{Y}$ restricted within set theory is equivalent to the corresponding statement $^*\mathcal{Y}$; it is determined that their statements are "true" or "false" simultaneously.

Theorem 2.1

There exists a standard duck on the systems $\{En\}$ $(n=1,2,\cdots)$ if and only if there exists a non-standard duck on the system E.

(proof)

If $(x_n(t),y_n(t),z_n(t))$ is a standard duck, then taking an infinitely large natural number M and putting $(x(t),y(t),z(t))=(x_n(t),y_n(t),z_n(t))$, there is a non-standard duck corresponding to the parameter values $\varepsilon = \varepsilon_M$.

Conversely, let $(x(t), y(t), z(t))$ be a non-standard duck. Two compact connected pieces of the attracting and repelling parts of the slow curve denoted by Γ_1 and Γ_2, respectively, where $\Gamma = \Gamma_1 \cup \Gamma_2$ belongs to the shadow of the duck. By choosing any standard $\delta > 0$, there are positive number $\varepsilon \in {}^*R$ and a solution $(x(t), y(t), z(t))$ of the system E such that $\varepsilon < \delta$, and the compact set Γ lies in the δ-neighborhood of the solution curve. In fact, for equations in ${}^*R^3$, we can take the duck itself. By the transfer principle, it is true also in the standard sense. Taking $\delta = 1/n$ and denoting the resulting solutions by $(x_n(t), y_n(t), z_n(t))$, a standard duck is obtained.

□

Benoit[7] investigated the relations between the system (2.1) and the systems E_n $(n=1, 2, \cdots)$ by introducing a method of a non-standard analysis (<u>Nelson's version</u>) and got a result, which is essentially same as the following theorem.

<u>Theorem 2.2</u> (Benoit)

If the system (2.1) has a pseudo singular saddle point, then the systems $\{E_n \mid n=1, 2, \cdots\}$ has a duck solution.

(Outline of proof)

Choose n sufficiently large so that ε_n is very small. Let P be the pseudo singular point in PL. AP, PB are separatrices of P and PC is a vertical line in Fig. 4. In this situation, there exists a segment [X, Y] above S such that the solution curve of E_n which starts from X is very close to the curve A'APC and it starting from Y is very close to the curve A'APB, where A'A is vertical to x-y plane and the length of [X, Y] is very small. Using the assumption (B5), it can be proved that any solution of E_n which starts from a neighborhood of the point P does not go along PL. From this fact and by the continuity of the solution of E_n with respect to an initial condition, it is ensured that there exists a point $Q \in [X, Y]$ such that the solution curve of E_n which starts from the point Q is very close to the curve A'APD.

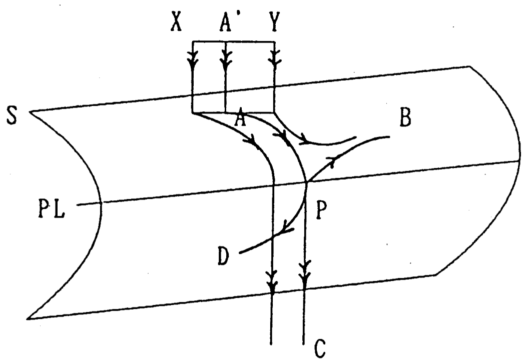

Fig. 4.

3. A duck solution in the FitzHugh-Nagumo equation

Now we return to the system (1.1). Let ε be any constant appeared in the equation (1.6) and let $I=I_0+\varepsilon_n t$, $b=c\varepsilon_n$, for each integer n (≥ 1), then by using I as an independent variable, the system (1.1) becomes the new systems D_n (n=1, 2, …) (3.1):

$$(D_n) \quad \varepsilon_n dv/dI = -\rho(v)-w+I, \tag{3.1a}$$

$$dw/dI=c(v-\gamma w), \tag{3.1b}$$

where $0<\varepsilon_n<\varepsilon$ for each n.

Suppose that ε be very small such that the conditions (1.2) are satisfied, then delayed oscillation phenomena occur in each system D_n (n=1, 2, …) proved by Su[6]. By changing the coordinate I=X, w=Y and v=Z, the system D_n becomes the system E_n (3.2):

$$dX/dI=1, \tag{3.2a}$$

$$(E_n) \quad dY/dI=c(Z-\gamma Y), \tag{3.2b}$$

$$dZ/dI=(-\rho(Z)-Y+X)/\varepsilon_n. \tag{3.2c}$$

Consider the constrained system (3.3) induced from the system (3.2):

$$dX/dI=1, \tag{3.3a}$$

$$dY/dI=c(Z-\gamma Y), \tag{3.3b}$$

$$-\rho(Z)-Y+X=0. \tag{3.3c}$$

The conditons (B1)-(B5) in the section2 are satisfied on the system (3.3). The condition (B1) holds, since $f(X,Y,Z)=1$, $g(X,Y,Z)=c(Z-\gamma Y)$ and $h(X,Y,Z)=-\rho(Z)-Y+X$ are all analytic. As the set S is expressed as

$$S=\{(X,Y,Z)\in R^3 \mid Y=-\rho(Z)+X\}, \tag{3.4}$$

the condition (B4) holds. It is obvious that S is a 2-dimensional, differentiable manifold. Put a set U:

$$U=\{(X,Y,Z)\in R^3 \mid h_Z=3Z^2-2(a+1)Z+a=0\}, \tag{3.5}$$

then the set U is a differentiable manifold and S intersects U transversely. The condition (B2) holds, since $PL=S\cap U$. Also the condition (B3) holds, since $dX/dl=1\neq 0$ at any point in R^3 and the condition (B5) holds, since the eigenvalues associated with the linearized equation (2.3) are not zeroes. As the system (3.3) is restricted to the surface S, the system (2.3) in the section2 is described by choosing a local coordinate (X, Z) as equation (3.6):

$$dX/dl=\rho'(Z), \tag{3.6a}$$

$$dZ/dl=1-c\gamma X-c(Z+\gamma\rho(Z)). \tag{3.6b}$$

Let (X_0, Z_0) be a singular point in the system (3.6), then the following equation (3.7) holds:

$$\rho'(Z_0)=0, \tag{3.7a}$$

$$1+c\gamma X_0+c(Z_0+\gamma\rho(Z_0))=0. \tag{3.7b}$$

Consider the linearized system (3.8) for (X_0, Z_0) in the system (3.6):

$$d\tilde{X}/dl=\rho''(Z_0)\tilde{Z}, \tag{3.8a}$$

$$d\tilde{Z}/dl=c\gamma\tilde{X}-c\tilde{Z}, \tag{3.8b}$$

where $\tilde{X}=X-X_0$, $\tilde{Z}=Z-Z_0$.

As $\rho'(Z)=0$ has two solutions $Z_{0\pm}=(a+1\pm(a^2-a+1)^{1/2})/3$, the system (3.6) has the two singular points; (X_{0+}, Z_{0+}) and (X_{0-}, Z_{0-}). The linearized system for (X_{0+}, Z_{0+}) in the system (3.6) is given by equation (3.9):

$$d\tilde{X}/dl=2(a^2-a+1)^{1/2}\tilde{Z}, \tag{3.9a}$$

$$d\tilde{Z}/dl=c\gamma\tilde{X}-c\tilde{Z}. \tag{3.9b}$$

Similarly, the linearized system for (X_{0-}, Z_{0-}) is given by equation (3.10):

$$d\tilde{X}/dl=-2(a^2-a+1)^{1/2}\tilde{Z}, \tag{3.10a}$$

$$d\tilde{Z}/dl=c\gamma\tilde{X}-c\tilde{Z}. \tag{3.10b}$$

<u>Theorem 3.1</u>

There exists a duck solution on the system (3.2).

(proof)

Our purpose is to show that the system (3.9) has a saddle point. By the Benoit's results, it follows that there is a duck solution on the set of the systems E_n $(n=1, 2, \cdots)$ in this situation. The characteristic equation for the system (3.9) is given by

$$\lambda^2+c\lambda-c\gamma(a^2-a+1)^{1/2}=0. \tag{3.11}$$

The solutions λ_1, λ_2 of (3.11), which are the eigenvalues associated with (3.10) are

$$\lambda_i=(1/2)(-c+(-1)^i(c^2+8c\gamma(a^2-a+1)^{1/2})^{1/2}) \quad (i=1, 2). \tag{3.12}$$

It follows immediately $\lambda_1<0<\lambda_2$. This implies that the system (3.9) has a saddle point. This completes the proof.

□

Remark

In the system (3.10), the characteristic equation is given by

$$\lambda^2+c\lambda+2c\gamma(a^2-a+1)^{1/2}=0. \tag{3.13}$$

Then the two solutions of (3.13) are

$$\lambda_i=(1/2)(-c+(-1)^i(c^2-8c\gamma(a^2-a+1)^{1/2})^{1/2}) \quad (i=1,2). \tag{3.14}$$

In this situation, there exist only two cases;

(i) $0<c<8\gamma(a^2-a+1)^{1/2}$, (3.15)

(ii) $c\geqq 8\gamma(a^2-a+1)^{1/2}$. (3.16)

In the case (i), the two solutions are complex. It can be easily proved that the system $\{E_n \mid n=1,2,\cdots\}$ does not have a duck solution. See Benoit[7]. In the case (ii), the two solutions are negative. It remains unproved whether the system $\{E_n \mid n=1,2,\cdots\}$ has a duck solution or not.

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