

Some characterizations of uniformly non-square Banach spaces

Mikio KATO (加藤幹雄) and Yasuji TAKAHASHI (高橋泰嗣)

Kyushu Inst. Tech. and Okayama Prefect. Univ.

In this note we present a sequence of characterizations of uniformly non-square Banach spaces which was recently given by the authors [15]. Some of them are similar to the well-known homogeneous characterization of uniformly convex spaces. As direct consequences we have the following: (i) A Banach space X is uniformly non-square if and only if the von Neumann-Jordan (NJ-) constant for X is less than 2. (Note that X is super-reflexive if and only if X admits an equivalent norm with NJ-constant less than 2.) (ii) Uniform non-squareness is inherited by dual spaces; this seems not to have appeared in literature. (iii) $L_p(X)$ ($1 < p < \infty$) is uniformly non-square if and only if X is (Smith and Turett [14]).

Let X be a Banach space. Let B_X denote the closed unit ball of X . X is called *uniformly convex* if for any $\varepsilon > 0$ ($0 < \varepsilon < 2$) there exists a $\delta > 0$ such that $\|(x+y)/2\| < 1 - \delta$, whenever $\|x-y\| \geq \varepsilon$, $x, y \in B_X$. X is called *uniformly non-square* ([7]) if there exists a $\delta > 0$ such that $\|(x+y)/2\| \leq 1 - \delta$, whenever $\|(x-y)/2\| > 1 - \delta$, $x, y \in B_X$. The *von Neumann-Jordan (NJ-) constant* for X ([4]), we denote it by $C_{NJ}(X)$, is the smallest constant for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C \quad \forall (x, y) \neq (0, 0).$$

We refer the reader to [9], [11], [12], [15] for some recent results on NJ-constant.)

It is well known that uniformly convex spaces are uniformly non-square, and uniformly non-square spaces are super-reflexive, or equivalently uniformly convexifiable (cf. [1]).

Now, Recall the following well-known homogeneous characterization of uniformly convex spaces:

A. Proposition (cf. [1]). Let $1 < p < \infty$. A Banach space X is uniformly convex if and only if for any $\varepsilon > 0$ there exists $\delta = \delta_p(\varepsilon) > 0$ such that $\|x - y\| \geq 2(1 - \varepsilon)$, $x, y \in B_X$ implies

$$(1) \quad \left\| \frac{x + y}{2} \right\|^p \leq (1 - \delta) \frac{\|x\|^p + \|y\|^p}{2}.$$

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Let $l_r^2(X)$ denote the X -valued l_r^2 -space.

1. Theorem (Takahashi and Kato [15]). Let $1 < p < \infty$. For a Banach space X the following are equivalent:

(i) X ; uniformly non-square.

(ii) There exist ε and δ ($0 < \varepsilon, \delta < 1$) such that if $\|x - y\| \geq 2(1 - \varepsilon)$, $x, y \in B_X$, then

$$(2) \quad \left\| \frac{x + y}{2} \right\|^p \leq (1 - \delta) \frac{\|x\|^p + \|y\|^p}{2}.$$

(iii) There exists a δ ($0 < \delta < 1$) such that if $\|x - y\| \geq 2(1 - \delta)$, $x, y \in B_X$, then

$$(3) \quad \left\| \frac{x + y}{2} \right\|^p \leq (1 - \delta) \frac{\|x\|^p + \|y\|^p}{2}.$$

(iv) There exists a δ ($0 < \delta < 2$) such that for any $x, y \in X$,

$$(4) \quad \left\| \frac{x + y}{2} \right\|^p + \left\| \frac{x - y}{2} \right\|^p \leq (2 - \delta) \frac{\|x\|^p + \|y\|^p}{2}.$$

$$(v) \|A : l_p^2(X) \rightarrow l_p^2(X)\| < 2.$$

(vi) For any (resp. some) $1 < r \leq \infty$, $1 \leq s < \infty$,

$$\|A : l_r^2(X) \rightarrow l_s^2(X)\| < 2^{1/r' + 1/s},$$

where $1/r + 1/r' = 1$.

$$(vii) C_{NJ}(X) < 2.$$

2. Remarks. (i) For the case $p = 1$, the homogeneous characterization of uniform convexity stated in Proposition A fails to hold (put $y = 0$), whereas the corresponding characterizations of uniform non-squareness (ii) and (iii) given in Theorem 1 remain valid; the assertions (iv) and (v) are false for $p = 1$.

(ii) For any Banach space X and for any $1 \leq p \leq \infty$ it holds that

$$\|A : l_p^2(X) \rightarrow l_p^2(X)\| \leq 2.$$

For $X = L_p$ we have

$$\|A : l_p^2(L_p) \rightarrow l_p^2(L_p)\| = 2^{1/\min(p, p')},$$

which is equivalent to the following Clarkson's inequality:

$$(\|f+g\|_p^p + \|f-g\|_p^p)^{1/p} \leq 2^{1/\min(p, p')} (\|f\|_p^p + \|g\|_p^p)^{1/p} \quad (\forall f, g \in L_p)$$

where equality is attained (Clarkson [3]).

(iii) Smith and Turett [14; esp. Lemma 14] gave a characterization of uniformly non- $l_1(n)$ Banach spaces, which in particular implies the equivalence of (i) and (iv) of Theorem 1 as the case $n = 2$ (their proof differs from ours in [15]). On the other hand, J. J. Schäffer (cf. [13], esp. p. 131) introduced a different notion of uniform non-squareness, which is known to be equivalent to James' treated here.

(iv) For any Banach space X

$$\|A : l_r^2(X) \rightarrow l_s^2(X)\| \leq 2^{1/r' + 1/s} \quad \text{for all } 1 \leq r, s \leq \infty$$

and

$$\|A : l_r^2(X) \rightarrow l_s^2(X)\| = 2^{1/r' + 1/s} \quad \text{for } r = 1 \text{ or } s = \infty.$$

For $X = L_p$ ($1 \leq p \leq \infty$) we have

$$(5) \quad \|A : l_r^2(L_p) \rightarrow l_s^2(L_p)\| = 2^{c(r,s;p)} \quad \text{for all } 1 \leq r, s \leq \infty,$$

where $c(r,s;p) = \max\{1/r', 1/s, 1/r' + 1/s - 1/\max(p, p')\}$, which yields the following Clarkson-Boas-Koskela's inequality:

$$(6) \quad (\|f+g\|_p^s + \|f-g\|_p^s)^{1/s} \leq 2^{c(r,s;p)} (\|f\|_p^r + \|g\|_p^r)^{1/r}$$

for $\forall f, g \in L_p$

(see [10]).

As Boas [2] observed (cf. Clarkson [3]), the inequality (6), or (5), with $c(r,s;p) = 1/r'$ implies the uniform convexity of L_p . The same is clearly true for a general Banach space X ; that is, if

$$(7) \quad \|A : l_r^2(X) \rightarrow l_s^2(X)\| \leq 2^{1/r'}$$

with some $1 \leq r, s < \infty$, then X is uniformly convex. (Note that if (7) is valid, then (7) is in fact reduced to identity; (x, x) , $x \neq 0$, is norm-attaining.) As is seen in the example below, this fails to be valid if the above norm of A is greater than $2^{1/r'}$.

Theorem 1 enables us to understand difference between uniform convexity and uniform non-squareness via behavior of norms of the Littlewood matrix:

3. **Example.** Let $1 < p \leq 2$ and $1 < \lambda < 2^{1/p'}$. Let $X_{p, \lambda}$ be the space $l_{p'}$ equipped with the norm $\|x\|_{p, \lambda} := \max\{\|x\|_{p'}, \lambda \|x\|_{\infty}\}$, where $1/p + 1/p' = 1$. Then, in the same way as the proof of Proposition 1 in [15] we have for $p \leq r < \infty$

$$(8) \quad 2^{1/r'} < \|A: l_{r'}^2(X_{p, \lambda}) \rightarrow l_{p'}^2(X_{p, \lambda})\| = \lambda 2^{1/r'} < 2^{1/r' + 1/p'}.$$

By Proposition 1 of [15], $X_{p, \lambda}$ is not uniformly convex (nor strictly convex) for all $1 < \lambda < 2^{1/p'}$, whereas Theorem 1 asserts that $X_{p, \lambda}$ is uniformly non-square (compare (8) with (7)).

Now, Theorem 1 immediately yields the following results.

4. **Corollary.** (i) The dual space X' of X is uniformly non-square if and only if X is.

(ii) Let $1 < p < \infty$. Then, the Lebesgue-Bochner space $L_p(X)$ is uniformly non-square if and only if X is (Smith and Turett [14]; see also [15]).

Indeed the assertion (i) is a direct consequence of Theorem 1 (vi) since $C_{NJ}(X') = C_{NJ}(X)$, which is easily seen (cf. [12; Proposition B]). For the if-part of (ii) integrate the inequality (4) and use Theorem 1 (iv) (see [14]).

5. **Remark.** The above result (i) of Corollary 4 seems not to have appeared in literature. Note here that uniform convexity is not inherited by the dual (cf. [1]). Giesy [5] showed that the bidual X'' is uniformly non- $l_1(n)$ if and only if X is (this is evident for $n = 2$, namely for uniform non-squareness, since uniformly non-square spaces are reflexive; James [7]); and the dual space X' is B-convex (uniformly non- $l_1(n)$ for

some n) if and only if X is. It is known that for some Orlicz spaces uniform non-squareness coincides with reflexivity and also with B -convexity ([6]).

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