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Kyoto University
On Semicontinuity of Marginal Functions

\[
\sup_{y \in F(x)} f(y) \text{ and } \inf_{y \in F(x)} f(y)
\]

1. Introduction

Collective function's semicontinuity problem in research, this paper employs a research by ten years many researchers. Recently, with Luc [6] and Tanaka and Seino [11] vector value problem, collective function problem concerning semicontinuity interesting results are obtained. Moreover, Ferro [4] and Tan, Yu, and Yuan [9] have studied these concepts in collective function's semicontinuity problem. In this manner, collective function's semicontinuity problem is considered. Furthermore, semicontinuity of maximum theorem is discussed. Therefore, the purpose of this paper is to study these concepts. Consequently, with vector cone-semicontinuity, the following concept is introduced: cone-semicontinuity. Therefore, cone-semicontinuity is defined. Furthermore, cone-semicontinuity is defined. Finally, the result of this paper is introduced in this type of marginal function (i.e., \( \sup_{y \in F(x)} f(y), \inf_{y \in F(x)} f(y) \)) semicontinuity problem.

2. Preliminaries

\( X \) is a space, \( Y \) is a vector space, \( C \) is a pointed set (i.e., \( C \cap (-C) = \{ \theta \} \)) and \( \mathrm{int} C \) is not empty. \( \theta \) is a zero element. \( \mathrm{int} C \) is the interior of \( C \) and \( \mathrm{cl} C \) is the closure of \( C \).

\( F \) is a function and \( F : X \to Y \) is a vector-valued function. \( d_Y : Y \to R \) is the distance function. \( d_Y (y, A) = \inf_{a \in A} d(y, a) \) is defined.

\( F \) is an upper semicontinuous function and \( F : X \to Y \) is an upper semicontinuous function. Therefore, \( F \) is an upper semicontinuous function. Furthermore, \( F \) is an upper semicontinuous function. Consequently, \( F \) is an upper semicontinuous function. Therefore, \( F \) is an upper semicontinuous function.

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(2.1) で定義される。

$$\text{Graph}(F) := \{(x, y) \in X \times Y | y \in F(x)\}.$$  \hfill (2.1)

また、$F$ の定義域とは $F(x)$ が空でない $x \in X$ 全体の集合、つまり、

$$\text{Dom}F := \{x \in X | F(x) \neq \emptyset\}$$ \hfill (2.2)

であり、$F$ の値域は

$$\text{Im}F := \bigcup_{x \in X} F(x)$$ \hfill (2.3)

で定義される。

$X$ と $Y$ を線形位相空間とし、$F$ を $X$ から $Y$ への集合値写像としたとき、$F$ が $x_0$ で
equally weak upper semicontinuous (ewusc for short) [11] であるとは、$\theta_Y \in Y$ での任意
の開近傍 $G$ に対して、$x_0$ での近傍 $U$ が存在して、

$$F(x) \subset F(x_0) + G \text{ for all } x \in U \cap \text{Dom}F,$$ \hfill (2.4)

が成り立つことである。また、$F$ が $x_0$ で equally lower semicontinuous (elsc for short)
[11] であるとは、$\theta_Y \in Y$ での任意の開近傍 $G$ に対して、$x_0$ での近傍 $U$ が存在して、

$$F(x_0) \subset F(x) + G \text{ for all } x \in U \cap \text{Dom}F,$$ \hfill (2.5)

が成り立つことをいう。

次に、Marginal function に関する定理を挙げる。これは Maximum theorem [1, Th.1.4.16]
と呼ばれ詳細は [1, Chapter1] で述べられている。

Proposition 1. Let $X$ and $Y$ be metric spaces, respectively. For a set-valued map $F : X \sim Y$ and a real-valued function $f : \text{Graph}(F) \to \mathbb{R}$, we have the following statements.

(i) If $f$ and $F$ are lower semicontinuous in the sense of each definition so is the marginal
function $g$ is also a lower semicontinuous function.

(ii) If $f$ and $F$ are upper semicontinuous in the sense of each definition and if $F(x)$ is a
compact set for each $x \in X$, the marginal function $g$ is also an upper semicontinuous function.

3. Cone-Semicontinuity for Set-Valued Maps

ここで、集合値写像 $F : X \sim Y$ に対する cone-semicontinuity を定義し、更にそれらの
関係について論じていくが、はじめに集合値写像の古典的な upper semicontinuity の定義
を挙げ、次に、upper semicontinuity の拡張である cone-upper semicontinuity を定義する。

Definition 1. Let $X$ and $Y$ be topological spaces, respectively. A set-valued map $F : X \sim Y$ is said to be upper semicontinuous (u.s.c. for short) at $x_0$ if for any open set $V$ with $F(x_0) \subset V$, there exists a neighborhood $U$ of $x_0$ such that

$$F(x) \subset V \text{ for all } x \in U.$$ \hfill (3.1)
定義 2. Let $X$ and $Y$ be a topological space and an ordered topological vector space with a convex cone $C$, respectively. A set-valued map $F : X \rightrightarrows Y$ is said to be:

(u1) $C$-upper semicontinuous at $x_0$ ($C$-usc) if for any open neighborhood $V$ of $F(x_0)$, there exists an open neighborhood $U$ of $x_0$ such that $F(x) \subset V + C$ for all $x \in U \cap \text{Dom} F$ ([6, Def.7.1(p.33)]);

(u2) $C$-weak upper semicontinuous at $x_0$ ($C$-wusc) if for any open neighborhood $V$ of $	ext{cl} F(x_0)$, there exists an open neighborhood $U$ of $x_0$ such that $F(x) \subset V + C$ for all $x \in U \cap \text{Dom} F$;

(u3) $C$-equally weak upper semicontinuous at $x_0$ ($C$-ewusc) if for any open neighborhood $G$ of $\theta_Y \in Y$, there exists an open neighborhood $U$ of $x_0$ such that $F(x) \subset F(x_0) + G + C$ for all $x \in U \cap \text{Dom} F$.

上述の 3 つの集合値写像における cone-upper semicontinuitologies は [11] で述べられている集合値写像の upper semicontinuity や、weak upper semicontinuity、equally upper semicontinuity の一般化である。もちろん、古典的な upper semicontinuity であれば cone-upper semicontinuity であり、また cone-upper semicontinuity は実数値関数の一般の下半連続性や、また、ベクトル値関数の下半連続性的拡張になっている [10, Def.2.1].

記号 1. In [4], Ferro denote condition (u1) above the terminology “upper $C$-continuity”. When $C = \{\theta_Y\}$ in Definition 2., a set-valued map $F : X \rightrightarrows Y$ is $C$-usc at $x_0$ if and only if $F$ is u.s.c. at $x_0$.

定理 1. Let $X$ and $Y$ be a topological space and an ordered topological vector space with a convex cone $C$, respectively. A set-valued map $F : X \rightrightarrows Y$ satisfies the condition (u3) at $x_0$ if and only if $F$ satisfies the following condition:

(u3) For any $d \in \text{int} C$, there exists an open neighborhood $U$ of $x_0$ such that $F(x) \subset F(x_0) - d + \text{int} C$ for all $x \in U$.

(u1), (u2), そして (u3) の関係について次の Proposition 2. が成立する。

定理 2. Let $X$ and $Y$ be a topological space and an ordered topological vector space with a convex cone $C$, respectively. In the above definition, we have (u1) $\Rightarrow$ (u2) $\Rightarrow$ (u3).

例 1. ( (u2) であるが (u1) ではない例 ) Let $X = Y = \mathbb{R}$ and $C = \mathbb{R}_+$. We consider the following set-valued map $F$ from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$F(x) = \{y \in \mathbb{R} | -x^2 < y \leq 1\}. \quad (3.2)$$

We can verify that $F$ is $\mathbb{R}_+$-wusc at $x = 0$ but not $\mathbb{R}_+$-usc at the point, where $\mathbb{R}_+ = \{r \in \mathbb{R} | r \geq 0\}$.

例 2. ( (u3) であるが (u2) ではない例 ) Let $X = \mathbb{R}_+, Y = \mathbb{R}^2$ and $C = \mathbb{R}_+^2$. We consider the following set-valued map $F$ from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$F(x) = \{(z_1, z_2) \in \mathbb{R}^2 \mid z_2 > \frac{1}{z_1 + x}, z_1 \geq 0\}. \quad (3.3)$$

We can verify that $F$ is $\mathbb{R}_+^2$-ewusc at $x = 0$ but not $\mathbb{R}_+^2$-wusc at the point.
Proposition 3. Let $X$ and $Y$ be a topological space and an ordered topological vector space with a convex cone $C$, respectively. A set-valued map $F : X \rightsquigarrow Y$ satisfies the condition (u1) $x_0$ if and only if $F$ satisfies the following condition:

(u1)' For any open set $V$ with $F(x_0) \subset V + C$, there exists an open neighborhood $U$ of $x_0$ such that $F(x) \subset V + C$ for all $x \in U \cap \text{Dom}F$;

Also, a set-valued map $F : X \rightsquigarrow Y$ satisfies the condition (u2) at $x_0$ if and only if $F$ satisfies the following condition:

(u2)' For any open set $V$ with $\text{cl} F(x_0) \subset V + C$, there exists an open neighborhood $U$ of $x_0$ such that $F(x) \subset V + C$ for all $x \in U \cap \text{Dom}F$;

Proposition 4. Let $X$ and $Y$ be a topological space and an ordered metric and vector space with a convex cone $C$, respectively, where the metric of $Y$ is denoted by $d_Y$. A set-valued map $F : X \rightsquigarrow Y$ satisfies the condition (u3) at $x_0$ if and only if $F$ satisfies the following condition:

(u3)" For any $\varepsilon > 0$, there exists an open neighborhood $U$ of $x_0$ such that

$$F(x) \subset B_Y (F(x_0), \varepsilon) + C, \quad \forall x \in U \cap \text{Dom}F,$$

where $B_Y (A, \varepsilon) := \{y \in Y \mid d_Y (y, A) < \varepsilon\}$.

Proposition 5. Let $X$ and $Y$ be a topological space and an ordered topological vector space with a convex cone $C$, respectively. In the above definition, if $F(x_0)$ is closed then (u2) $\Rightarrow$ (u1). Also, $\text{cl} F(x_0)$ is compact in $Y$, then (u3) $\Rightarrow$ (u2).

次に，集合値写像に対する古典的な lower semicontinuity の定義を挙げ，更に cone-lower semicontinuity を次に定義する。

Definition 3. Let $X$ and $Y$ be topological spaces. A set-valued map $F : X \rightsquigarrow Y$ is said to be lower semicontinuous (l.s.c. for short) at $x_0$ if for any open set $V$ with $F(x_0) \cap V \neq \emptyset$, there exists an open neighborhood $U$ of $x_0$ such that

$$F(x) \cap V \neq \emptyset \quad \text{for all} \quad x \in U. \quad (3.4)$$

Definition 4. Let $X$ and $Y$ be a topological space and an ordered topological vector space with a convex cone $C$, respectively. A set-valued map $F : X \rightsquigarrow Y$ is said to be:

(11) $C$-equally lower semicontinuous at $x_0$ ($C$-elsc) if for any neighborhood $G$ of $\theta_Y \in Y$, there exists a neighborhood $U$ of $x_0$ such that $F(x_0) \subset F(x) + G - C$ for all $x \in U \cap \text{Dom}F$;

(12) $C$-lower semicontinuous at $x_0$ ($C$-lsc) if for any $y_0 \in F(x_0)$ and any neighborhood $G$ of $\theta_Y \in Y$, there exists a neighborhood $U$ of $x_0$ with $F(x) \cap (y_0 + G + C) \neq \emptyset$ for any $x \in U \cap \text{Dom}F$.

この2つの集合値写像における cone-lower semicontinuitities は，[11] で述べられている集合値写像の equally lower semicontinuity や, lower semicontinuity の一般化である．もちろん，古典的な lower semicontinuity であれば，$C$-lower semicontinuity である．
Remark 2. In [4], Ferro denote condition (11) above by the terminology "lower C-semicontinuity". When $C = \{\theta_Y\}$ in Definition def-C-lsc, a set-valued map $F : X \leadsto Y$ is C-lsc at $x_0$ if and only if $F$ is l.s.c. at $x_0$.

Proposition 6. Let $X$ and $Y$ be a topological space and an ordered topological vector space with a convex cone $C$, respectively. In the above definition, $(11) \Rightarrow (12)$. If $\text{cl} F(x_0)$ is compact, the converse is true. See Ferro [4] in detail.

4. Cone-Semicontinuity of Composite Maps and Marginal Functions

実数値関数 $f : Y \to R$ が $x_0 \in Y$ で upper semicontinuous (u.s.c. for short) であるとは、任意の正の実数 $\varepsilon > 0$ に対して、$x_0$ での近傍 $U$ が存在し、すべての $x \in U$ で $f(x) - f(x_0) < \varepsilon$ が成り立つことをいう。$f$ が $Y$ 上で u.s.c. であるための必要十分条件は、任意の実数 $a \in R$ に対して、$Y$ での部分集合 $\{x \in Y | f(x) < a\}$ が開集合となることである。また、$-f$ が $x_0$ で u.s.c. であるとき $f$ は $x_0$ で lower semicontinuous (l.s.c. for short) という。

次の Theorem 1 を示すために、実数値関数における上記の upper semicontinuity や lower semicontinuity よりも強い概念を導入する。

Definition 1. Let $Y$ be a topological vector space. A real-valued function $f : Y \to R$ is called monotonically u.s.c. (resp., monotonically l.s.c.) if for any $\varepsilon > 0$, there exists a neighborhood $G$ of $\theta_Y \in Y$ such that $f^{-1}(V + (-\varepsilon, \varepsilon) + R_-)$ is open and $f^{-1}(V) + G \subset f^{-1}(V + (-\varepsilon, \varepsilon) + R_-)$ for all $V \subset R$ (resp., by replacing $R_+$ by $R_-$, where $R_- = \{r \in R | r \leq 0\}$).

実数値関数と集合値関数の合成写像 $\varphi : \text{Dom} F \leadsto R$ を以下で定義する。

$$ \varphi(x) := f \circ F(x) = \bigcup_{y \in F(x)} \{f(y)\}. \quad (4.1) $$

また、以後凸錐 $C$ は $C = R_+$ または $C = R_-$ で考える。

Theorem 1. Let $X$ and $Y$ be a topological space and an ordered topological vector space with a convex cone $C$, respectively. For $F : X \leadsto Y$ with $\text{Dom} F \neq \emptyset$ and $f : Y \to R$, we have the following:

1. if $F$ is u.s.c. and $f$ is u.s.c. then $\varphi$ is $R_-$-ewusc;
2. if $F$ is ewusc and $f$ is monotonically u.s.c. then $\varphi$ is $R_-$-ewusc;
3. if $F$ is u.s.c. and $f$ is l.s.c. then $\varphi$ is $R_+$-ewusc;
4. if $F$ is ewusc and $f$ is monotonically l.s.c. then $\varphi$ is $R_+$-ewusc;
(5) if $F$ is l.s.c. and $f$ is u.s.c. then $\varphi$ is $R_{-}$-lsc;
(6) if $F$ is l.s.c. and $f$ is l.s.c. then $\varphi$ is $R_{+}$-elsc.

ここで、次の2つのタイプの marginal function を定義する。

$$\sup \varphi(x) := \sup_{y \in F(x)} f(y),$$
$$\inf \varphi(x) := \inf_{y \in F(x)} f(y),$$

ただし、$F : X \sim Y$ は集合値写像であり $f : Y \to \mathbb{R}$ は実数値関数である。

Lemma 1. Let $X$ be a topological space For a set-valued map $\varphi : X \sim \mathbb{R}$ is $R_{-}$-ewusc (resp. $R_{-}$-elsc, $R_{-}$-lsc) if and only if $-\varphi$ is $R_{+}$-ewusc (resp. $R_{+}$-elsc, $R_{+}$-lsc).

Theorem 2. Let $X$ be a topological space. For a set-valued map $\varphi : X \sim \mathbb{R}$, we have the following:

(1) if $\varphi$ is $R_{-}$-ewusc then $\sup \varphi$ is u.s.c.;
(2) if $\varphi$ is $R_{+}$-ewusc then $\varphi$ is l.s.c.;
(3) if $\varphi$ is $R_{+}$-elsc then $\sup \varphi$ is l.s.c.;
(4) if $\varphi$ is $R_{-}$-elsc then $\inf \varphi$ is u.s.c.;
(5) if $\varphi$ is $R_{+}$-lsc then $\inf \varphi$ is u.s.c.;
(6) if $\varphi$ is $R_{+}$-lsc then $\sup \varphi$ is l.s.c..

Theorem 1., Theorem 2. から次の Corollary 1. が得られる.

Corollary 1. Let $X$ and $Y$ be a topological space and an ordered topological vector space with a convex cone $C$, respectively. Let $F : X \sim Y$ be a set-valued map with $\text{Dom} F \neq \emptyset$ and $f : Y \to \mathbb{R}$. For the marginal function is defined by (4.2) and (4.3), we have the following:

(1a) if $F$ is u.s.c. and $f$ is u.s.c. then $\sup \varphi$ is u.s.c.;
(1b) if $F$ is ewusc and $f$ is monotonically u.s.c. then $\sup \varphi$ is u.s.c.;
(2a) if $F$ is u.s.c. and $f$ is l.s.c. then $\inf \varphi$ is l.s.c.;
(2b) if $F$ is ewusc and $f$ is monotonically l.s.c. then $\inf \varphi$ is l.s.c.;
(3) if $F$ is elsc and $f$ is monotonically u.s.c. then $\sup \varphi$ is l.s.c.;
(4) if $F$ is elsc and $f$ is monotonically l.s.c. then $\inf \varphi$ is u.s.c.;
(5) if $F$ is l.s.c. and $f$ is u.s.c. then $\inf \varphi$ is u.s.c.;
(6) if $F$ is l.s.c. and $f$ is l.s.c. then $\sup \varphi$ is l.s.c..
References


