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Kyoto University
Semicontinuity of set valued mappings and duality formulas of integral functionals

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§1 DUALITY FORMULAS

Let $X$ be a metric space, and let $f$ be a real valued function defined on $X \times \mathbb{R}^d$. Suppose that for each $x \in X$, $f_x(p) = f(x, p)$ is convex and positively homogeneous in $p \in \mathbb{R}^d$. By $K_x$, we denote the subdifferential of $f_x$ at 0;

$$K_x = \partial f_x(0)$$

$$= \{q \in \mathbb{R}^d \mid \langle q, p \rangle \leq f_x(p), \quad p \in \mathbb{R}^d\}$$

For every $x \in X$ the set $K_x$ is convex in $\mathbb{R}^d$, and since $f_x(p)$ is finite for all $p \in \mathbb{R}^d$, $K_x$ is compact. Let $\mu = (\mu_1, \cdots, \mu_n)$ be a $\mathbb{R}^d$-valued finite Borel regular measure on $X$. The finite Borel measure $f(x, \mu)$ on $X$ is defined by

$$\int_A f(x, \mu) = \int_A f(x, \mu(x))d|\mu| \quad \text{for a Borel set } A \subset X$$

where $|\mu|$ is the total variation measure of $\mu$ and $\mu(x) = \frac{d\mu}{d|\mu|}(x)$ is the Radon Nikodym derivative of $\mu$ with respect to $|\mu|$. The measure $f(x, \mu)$ is independent of the choice of a norm in $\mathbb{R}^d$. 

THEOREM 1. Suppose that $f$ satisfies

1. $f$ is lower semicontinuous (l.s.c.) on $X \times \mathbb{R}^d$,
2. for each $x \in X$, $f_x(p) = f(x, p)$ is convex, positively homogeneous in $p$,
3. $f(x, p) \leq c|p| \quad (x \in X, p \in \mathbb{R}^d)$ with some constant $c$.

Then for every bounded $|\mu|$-measurable function $\varphi \geq 0$ on $X$,

\[(F,1) \quad \int_X f(x, \mu)\varphi = \sup\{\int_X \langle \mu(x), v(x) \rangle > \varphi(x)d|\mu|(x) \mid v \in C(X, \mathbb{R}^d), v(x) \in K_x \text{ for all } x \in X \}.
\]

Next we consider the case when $f_x(\cdot)$ is only convex in $p \in \mathbb{R}^d$, and is not necessarily positively homogeneous. For defining the measure $f(x, \mu)$ in this case, we introduce the homogenization $F(x, p_0, p)$ of $f(x, p)$ defined by

\[
F(x, p_0, p) = \begin{cases} 
    f_\infty(x, p) & p_0 = 0 \\
    f(x, \frac{p}{p_0})p_0 & p_0 > 0 \\
    \infty & p_0 < 0
\end{cases}
\]

where $f_\infty$ is the recession function of $f$, i.e.,

\[
f_\infty(x, p) = \lim_{t \to 0} f(x, \frac{p}{t})t.
\]

If $f$ satisfies $f(x, p) \leq c(1 + |p|) \quad (x \in X, p \in \mathbb{R}^d)$ with some constant $c$, $F$ is well-defined real valued function on $X \times C$ with $C = [0, \infty) \times \mathbb{R}^d$ and $F = \infty$ on $X \times (\mathbb{R}^{d+1} \setminus C)$. Moreover, $F$ is convex and positively homogeneous in $(p_0, p) \in \mathbb{R}^{d+1}$. (See [8, §8])

Let $\alpha$ be a nonnegative finite Borel regular measure on $X$. We fix this measure and now define the measure $f(x, \mu)$ by

\[
f(x, \mu) = F(x, \alpha, \mu),
\]

where $F$ is the homogenization of $f$. Here $(\alpha, \mu)$ is a $C = [0, \infty) \times \mathbb{R}^d$ valued Borel regular measure, and since $F$ is positively homogeneous, $f(x, \mu)$ is a finite Borel regular measure.
It is easy to see that
\[
f(x, \mu) = F(x, \alpha, \mu) = F(x, 1, h(x)) + F(X, 0, \mu^s) = F(x, \alpha + f(x, \mu^s)) = f(x, h(x) \alpha) + f_{\infty}(x, \mu^s)\]

where \(h(x) \alpha\) is the absolutely continuous part of \(\mu\), and \(\mu^s\) is the singular part with respect to \(\alpha\).

**Theorem 2.** Suppose that \(f\) satisfies

1. for every \(x_0 \in X\) and \(\epsilon > 0\), there is \(\delta > 0\) such that \(d(x, x_0) < \delta\) implies
   \[f(x_0, p) - f(x, p) < \epsilon(1 + |p|),\]
2. for each \(x \in X\), \(f_x(p)\) is convex in \(p\),
3. \(f(x, p) \leq c(1 + |p|)\) \((x \in X, p \in \mathbb{R}^d)\) with some constant \(c\).

Then for every bounded \(|\mu|\)-measurable function \(\varphi \geq 0\) on \(X\),

\[
\int_X f(x, \mu) \varphi = \sup \{ \int_X <\mu(x), v(x)> \varphi(x) d\mu | (x) - \int_X \varphi(x)f^*(x, v(x)) d\alpha \mid v \in C(X, \mathbb{R}^d), f^*(x, v(x)) \in L^1(X, d\alpha) \}.
\]

Similar results can be seen in [2], [3], [6]. In the proof of Rockafellar [6], it is assumed that \(K_x\) has an interior point and the assumption on the regularity of \(f\) in \(x\) is slightly stronger than ours. In [2], it is assumed that \(f\) is continuous on \(X \times \mathbb{R}^d\). We have weakened these assumptions by some arguments of the continuous selection.

We consider the set valued mapping \(K\) which carries each \(x \in X\) to the compact convex set \(K_x \subset \mathbb{R}^d\). \(K\) is said to be lower semicontinuous (l.s.c.) if \(x_n \rightarrow x_0\) in \(X\) and \(q_0 \in K_{x_0}\) implies the existence of a sequence \(\{q_n\}\) such that \(q_n \in K_{x_n}\) and \(q_n \rightarrow q_0\). \(K_x\) is said to be upper semicontinuous (u.s.c.) if for any sequence \(\{x_n\}\) tends to \(x_0\) and \(\epsilon > 0\), \(K_{x_n} \subset K_{x_0} + \epsilon B\) holds for sufficiently large \(n\), where \(K_{x_0} + \epsilon B = \{q + q' \in \mathbb{R}^d \mid q \in K_{x_0}, |q'| \leq \epsilon\}\). Furthermore, when \(K_x\) is both l.s.c. and u.s.c., \(K\) is said to be continuous.

One can find some other definitions of this semicontinuity in [1], [5], and [6] for instance.
However, in our case, most of them are all equivalent because $K_x$ is always compact. The importance of the lower semicontinuity is that this allows us to take continuous selection of $K_x$. For example, In [6], the lower semicontinuity of $K_x$ and the continuous selection theorem ([5]) are applied to prove a type of duality formula. Also in [2], the conditions for the same formula are given in terms of the function $f(x, p)$. However, the relation between the conditions of these two theorems is unclear. In this note, we investigate the conditions of $f$ under which $K_x$ is lower semicontinuous. Moreover, we will consider the upper semicontinuity and derive some duality of these two notions.

§2 Semi continuity of $K_x$

Lemma 3. Let $f(x, p)$ be a function on $X \times \mathbb{R}^d$, and suppose that $f_x(p) = f(x, p)$ is convex and positively homogeneous in $p \in \mathbb{R}^d$. Put $K_x = \partial f_x(0)$, then the following conditions are equivalent.

$(l, 1)$ $f$ is l.s.c. on $X \times \mathbb{R}^d$.

$(l, 2)$ For every $x_0 \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d(x_0, x) < \delta$ implies

$$f(x_0, p) - f(x, p) < \epsilon |p|, \text{ for all } p \in \mathbb{R}^d.$$ 

$(l, 3)$ $K : x \rightarrow K_x$ is l.s.c. on $X$.

Remark: When $f$ is l.s.c. only in $x$, these conditions do not hold though $f$ is convex (and hence continuous) in $p$. This fact is the only thing that the symmetry of Lemma 3 and Proposition 6 fails. The space $\mathbb{R}^d$ in this theorem can be replaced by any closed convex cone in $\mathbb{R}^d$, but not by any infinite dimensional space. Moreover, positively homogeneity of $f$ is essential in this lemma even if $K_x$ can be defined as the subdifferential of $f$.

Proof: $(l, 1) \Rightarrow (l, 2)$

It suffices to show that $\{f(\cdot, p) \mid |p| = 1\}$ is equi l.s.c.. If not, there exists $x_0 \in X$, $\epsilon > 0$, and sequences $\{x_n\} \subset X$ and $\{p_n\} \subset \mathbb{R}^d$, such that $x_n \rightarrow x_0$, $|p_n| = 1$, and $f(x_0, p_n) - f(x_n, p_n) \geq \epsilon$ for every $n$. Since $\{p \in \mathbb{R}^d \mid |p| = 1\}$ is compact, we can assume
that \( p_n \to p_0 \) for some \(|p_0|\). By the convexity of \( f \) in \( p \), it is continuous in particular.

Hence it follows by \((l,1)\) that

\[
f(x_0, p_n) - f(x_n, p_n) = f(x_0, p_n) - f(x_0, p_0) + f(x_0, p_0) - f(x_n, p_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\]

for sufficiently large \( n \) and this contradicts the assumption.

\((l,2) \Rightarrow (l,3)\)

Suppose that \( K \) is not l.s.c. at \( x_0 \in X \). Then there exist a sequence \( \{x_n\} \) with \( x_n \to x_0 \), \( q_0 \in \text{It}^{-}x_0 \) and \( \epsilon > 0 \) such that

\[
K_{x_n \cap \mathcal{E}B(q_0)} = \phi,
\]

for every \( n \), where \( \mathcal{E}B(q_0) = \{q \in \mathbb{R}^d \mid d(q, q_0) \leq \epsilon\} \). By the condition \((l,2)\), we have for sufficiently large \( n \),

\[
f(x_0, p) - f(x_n, p) < \epsilon \quad \text{for all } p \in \mathbb{R}^d \text{ with } |p| = 1.
\]

We fix such \( n \), and by the separation theorem and \((1)\), there exists \( p_0 \in \mathbb{R}^d \) with \(|p_0| = 1\), such that

\[
\sup_{q \in K_{x_n}} < q, p_0 > \leq \inf_{q \in \mathcal{E}B(q_0)} < q, p_0 >.
\]

Now we take the supporting point \( \bar{q} \) of \( \mathcal{E}B(q_0) \) with respect to \( p_0 \), that is, \( \bar{q} \in \mathcal{E}B(q_0) \) and \( \inf_{q \in \mathcal{E}B(q_0)} < q, p_0 > = < \bar{q}, p_0 > \). Then,

\[
\inf_{q \in \mathcal{E}B(q_0)} < q, p_0 > = < q_0, p_0 > - < q_0 - \bar{q}, p_0 > \\
= < q_0, p_0 > - \epsilon \\
\leq \sup_{q \in K_{x_0}} < q, p_0 > - \epsilon \\
= f(x_0, p_0) - \epsilon.
\]

By \((3)\), we obtain

\[
f(x_n, p_0) \leq f(x_0, p_0) - \epsilon.
\]
Since $p$ in (2) is arbitrary, this is a contradiction.

$(l, 3) \Rightarrow (l, 1)$

Suppose that $x_n \rightarrow x_0$ in $X$ and $p_n \rightarrow p_0$ in $\mathbb{R}^d$. For every $\epsilon > 0$, we take $q_0 \in K_{x_0}$ such that

$$<q_0, p_0> \geq \sup_{q \in K_{x_0}} <q, p_0> - \epsilon$$

$$= f(x_0, p_0) - \epsilon.$$

By $(l, 3)$, there exists a sequence $\{q_n\}$ such that each $q_n$ belongs to $K_{x_n}$ and $q_n \rightarrow q_0$. Since $<q_n, p_n> \leq \sup_{q \in K_{x_n}} <q, p_n> = f(x_n, p_n)$, we have

$$f(x_0, p_0) - f(x_n, p_n) \leq <q_0, p_0> + \epsilon - <q_n, p_n>$$

$$< 2 \epsilon$$

for sufficiently large $n$. This implies that $f$ is l.s.c. on $X \times \mathbb{R}^d$.

**Corollary 4.** Suppose that $f$ satisfies one of three conditions in Theorem 3. Then for every $x_0 \in X$ and $p_0 \in \mathbb{R}^d$, there exists a continuous function $L$ on $X \times \mathbb{R}^d$ satisfying

1. for every $x \in X$, $L(x, p)$ is linear in $p \in \mathbb{R}^d$,
2. $L(x, p) \leq f(x, p)$ for all $x \in X$ and $p \in \mathbb{R}^d$,
3. $L(x_0, p_0) = f(x_0, p_0)$.

**Proof:** First we note that $L$ is continuous on $X \times \mathbb{R}^d$ if it satisfies (1) and is continuous with respect to each variable. By the separation theorem or Hahn Banach theorem, there exists $q_0 \in \mathbb{R}^d$ such that $<q_0, p> \leq f(x_0, p)$ for all $p \in \mathbb{R}^d$, and $<q_0, p_0> = f(x_0, p_0)$. Take a set valued mapping $K'$ defined by

$$K'_x = \begin{cases} K_x & x \neq x_0 \\ \{q_0\} & x = x_0. \end{cases}$$

Since $q_0 \in K_{x_0}$, it is easy to see that $K'$ is l.s.c., and hence we can take a continuous selection $q(x)$ of $K'_x$. Thus the function $L$ defined by $L(x, p) = <q(x), p>$, $x \in X, p \in \mathbb{R}^d$ is what we want.

By an analogy, one can also prove the following.
Corollary 5. Suppose that \( f \) satisfies one of the three conditions in Theorem 3. Let \( E \) be a closed subset of \( X \), and let \( L \) be a continuous function on \( E \times \mathbb{R}^d \) satisfying

1. for every \( x \in E \), \( L(x, p) \) is linear in \( p \in \mathbb{R}^d \),
2. \( L(x, p) \leq f(x, p) \) for all \( x \in E \) and \( p \in \mathbb{R}^d \).

Then \( L \) can be continuously extended to \( X \times \mathbb{R}^d \) such that (1) and (2) hold replacing \( E \) by \( X \).

Next we consider the upper semicontinuity of \( K_x \). We note that the following proposition and Lemma 3 have some symmetricity but it is not perfect.

Proposition 6. Under the hypotheses in Lemma 3, the following conditions are equivalent.

1. \( f(x, p) \) is u.s.c. in \( x \in X \).
2. \( f \) is u.s.c. on \( X \times \mathbb{R}^d \).
3. For every \( x_0 \in X \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( d(x_0, x) < \delta \) implies
   \[
   f(x, p) - f(x_0, p) < \varepsilon |p|,
   \]
   for all \( p \in \mathbb{R}^d \).
4. \( K : x \mapsto K_x \) is u.s.c. on \( X \).

Remark: A set valued mapping \( K \) is said to be closed if for any sequence \( \{x_n\} \) with \( x_n \to x_0 \) and \( \{q_n\} \) with \( q_n \in K_{x_n} \), \( q_n \to q_0 \) for some \( q_0 \in \mathbb{R}^d \) implies \( q_0 \in K_{x_0} \). This is also a notion of upper semicontinuity of set valued mappings. Since \( K_x \) is compact in our case, the upper semicontinuity of \( K \) implies the closedness. However, the converse is not true in general. The equivalence of \( (u, 0) \) and \( (u, 1) \) is still valid when \( f \) is only convex and not positively homogeneous in \( p \).

Proof: \( (u, 0) \Rightarrow (u, 1) \)

Suppose that \( x_n \to x_0 \) in \( X \) and \( p_n \to p_0 \) in \( \mathbb{R}^d \). Since \( f \) is continuous in \( p \), there exists \( \bar{p}_1, \cdots, \bar{p}_{d+1} \in \mathbb{R}^d \) such that

\[
f(x_0, \bar{p}_i) \leq f(x_0, p_0) + \frac{\varepsilon}{2} \quad (i = 1, \cdots, d + 1)
\]
and the convex hull $co\{\overline{p}_1, \cdots, \overline{p}_{d+1}\}$ forms a neighborhood of $p_0$. Moreover by the condition $(u, 0)$,

$$f(x_n, \overline{p}_i) \leq f(x_0, \overline{p}_i) + \frac{\varepsilon}{2} \quad (i = 1, \cdots, d+1)$$

holds for sufficiently large $n$. Since $p_n \in co\{\overline{p}_1, \cdots, \overline{p}_{d+1}\}$ for sufficiently large $n$, we obtain by the convexity of $f(x, \cdot)$ that

$$f(x_n, p_n) \leq \max_{1 \leq i \leq d+1} f(x_n, \overline{p}_i) \leq \max_{1 \leq i \leq d+1} f(x_0, \overline{p}_i) + \frac{\varepsilon}{2} \leq f(x_0, p_0) + \varepsilon.$$  

This proves that $(u, 1)$ holds.

$(u, 1) \Rightarrow (u, 2)$ 

we can prove this by the same way as in $(l, 1) \Rightarrow (l, 2)$ in Lemma 3.

$(u, 2) \Rightarrow (u, 3)$ 

Take $x_0 \in X$ and $\varepsilon > 0$ arbitrarily, and Suppose that $x_n \to x_0$ in $X$. By $(u, 2)$,

$$f(x_n, p) - f(x_0, p) \leq \varepsilon |p| \quad (p \in \mathbb{R}^d),$$

for sufficiently large $n$. Then $q \in K_{x_n}$ implies that

$$f(x_0, p) - \langle q, p \rangle \geq f(x_0, p) - f(x_n, p) \geq -\varepsilon |p| \text{ for all } p \in \mathbb{R}^d.$$ 

By the separation theorem, there exists $q_0 \in \mathbb{R}^d$ such that

$$-\varepsilon |p| \leq \langle q_0, p \rangle \leq f(x_0, p) - \langle q, p \rangle \quad (p \in \mathbb{R}^d).$$

This inequality implies that $|q_0| \leq \varepsilon$, and $q + q_0 \in K_{x_0}$. Hence we have $q \in K_{x_0} + \varepsilon B$ and this proves $(u, 3)$.

$(u, 3) \Rightarrow (u, 1)$

For the reason stated in the remark of this theorem, we can assume that $K$ is closed. Suppose that $(u, 1)$ does not hold, then there exist sequences $\{x_n\}$ with $x_n \to x_0$ for some $x_0$ in $X$, and $\{p_n\}$ with $p_n \to p_0$ for some $p_0$ in $\mathbb{R}^d$, and $\varepsilon > 0$ such that $f(x_n, p_n) >$
\[ f(x_0, p_0) + \epsilon \] for every \( n \). Since \( f(x_n, p_n) = \sup_{q \in K_{x_n}} < q, p_n > \), we can choose a sequence \( \{q_n\} \subset \mathbb{R}^d \) such that \( q_n \in K_{x_n} \) and

\[
| f(x_n, p_n) - < q_n, p_n > | \to 0 \quad (n \to \infty).
\]

By the definition of upper semicontinuity, \( K_{x_n} \) is uniformly bounded. Therefore the sequence \( \{q_n\} \) is bounded, and we can take a convergent subsequence \( \{q_m\} \) of \( \{q_n\} \) with \( q_m \to q_0 \) for some \( q_0 \in \mathbb{R}^d \). Hence it follows that

\[
< q_0, p_0 > \geq f(x_0, p_0) + \epsilon.
\]

On the other hand, by the closedness of \( K \), \( q_0 \) has to be an element of \( K_{x_0} \), and this is a contradiction. \( \blacksquare \)

Combining Lemma 3 and Proposition 6, we also obtain the following theorem. To see the equivalence between \((c, 0)\) and \((c, 1)\), refer to Theorem 1.1 in [3].

**Proposition 7.** Under the hypotheses in Lemma 3, the following conditions are equivalent.

\( (c, 0) \) For every \( p \in \mathbb{R}^d \), \( f(x, p) \) is continuous in \( x \in X \).
\( (c, 1) \) \( f \) is continuous on \( X \times \mathbb{R}^d \).
\( (c, 2) \) For every \( x_0 \in X \) and \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( d(x_0, x) < \delta \) implies

\[
| f(x, p) - f(x_0, p) | < \epsilon |p|, \quad \text{for all } p \in \mathbb{R}^d.
\]

\( (c, 3) \) \( K : x \to K_x \) is continuous on \( X \).

\section*{3 Proof of the Duality Formula}

For a subset \( U \subset \mathbb{R}^d \), we denote the inverse image of a set valued mapping \( K \) by

\[ K^{-1}(U) = \{ x \in X | K_x \cap U \neq \phi \}. \]
$\Gamma(x) = \{ p \in \mathbb{R}^d \mid <\mu(x), p> \geq f_x(\mu(x)) - \varepsilon \},$

$\Gamma_0(x) = \{ p \in K_x \mid <\mu(x), p> \geq f_x(\mu(x)) - \varepsilon \}.$

Since $f_x(\mu(x)) = \sup_{p \in K_x} <\mu(x), p>,$ $\Gamma(x)$ and $\Gamma_0(x)$ are nonempty closed convex sets in $\mathbb{R}^d,$ and $\Gamma(x) = \Gamma_0(x) \cap K_x.$ By the condition (1) and Lemma 3, $K$ is l.s.c. as a set valued mapping, and also measurable in particular. Hence by [7, Theorm 1M], $\Gamma$ is a $|\mu|$-measurable set valued mapping provided that so is $\Gamma_0.$ Let $U$ be an open set in $\mathbb{R}^d.$ Since $\Gamma_0(x)$ is an affine half space, $\Gamma_0(x) \cap U \neq \emptyset$ if and only if $\Gamma_0(x) \cap D \neq \emptyset$ where $D$ is an arbitrary countable dense subset of $U.$ Hence we have

$$\Gamma_0^{-1}(U) = \bigcup_{p \in D} A_p$$

where $A_p = \{ x \in X \mid <\mu(x), p> \geq f_x(\mu(x)) - \varepsilon \}.$ We note that $f_x(\mu(x))$ is $|\mu|$-measurable because of the lower semicontinuity of $f.$ Thus $\Gamma_0^{-1}(U)$ is $|\mu|$-measurable, and by the measurable selection theorem we can take a measurable function $w$ on $X$ such that $w(x) \in \Gamma(x).$ In other words

$$\int_X <\mu(x), w(x)> \varphi(x) d|\mu| \geq \int_X (f_x(\mu(x)) - \varepsilon)\varphi(x) d|\mu|$$

$$= \int_X f(x, \mu)\varphi - \varepsilon \int_X \varphi d|\mu| \quad (4)$$

Since $|\mu|$ is finite measure and $\varphi$ is bounded, this yields the duality formula of weaker version.
We next construct a desired continuous function $v : X \rightarrow \mathbb{R}^d$ from $w$ which has been obtained above. By Lusin's theorem, for arbitrary $\delta > 0$ there exists a closed set $Y \subset X$ such that $|\mu|(Y^c) < \delta$ and $w$ is continuous on $Y$. We define a set valued mapping $K'$ by

$$K'_x = \begin{cases} \{w(x)\} & x \in Y \\ K_x & x \notin Y \end{cases}$$

for $x \in X$. We see by [1, Corollary 9.1.3] (the closedness of $K$ is missing in the condition of this corollary) that $K'$ is also l.s.c. and have a continuous selection. In other words, there exists a continuous function $v : X \rightarrow \mathbb{R}^d$ such that $v(x) \in K_x$ on $X$ and $v(x) = w(x)$ on $Y$. Hence we have

$$\int_X <\overline{\mu(x)}, w(x)> \varphi d|\mu| = \int_X <\overline{\mu(x)}, v(x)> \varphi d|\mu| + \int_{Y^c} <\overline{\mu(x)}, w(x)> \varphi d|\mu|$$

$$- \int_{Y^c} <\overline{\mu(x)}, v(x)> \varphi d|\mu|$$

$$\leq \int_X <\overline{\mu(x)}, w(x)> \varphi d|\mu| + \int_{Y^c} f(x, \overline{\mu(x)}) \varphi d|\mu|$$

$$+ \|v\| \int_{Y^c} \varphi d|\mu|.$$

Since $f(x, p) \leq c$ for $x \in X$ and $|p| = 1$, we thus obtain from (4) that

$$\int_X f(x, \mu) \varphi \leq \int_X <\overline{\mu(x)}, v(x)> \varphi d|\mu| + (c + \|v\||\varphi||\mu|(Y^c))$$

$$+ \varepsilon\|\varphi\||\mu|(X).$$

We note that $v(x) \in K_x$ implies $\|v\| = \sup_{x \in X} |v(x)| \leq c$, which is independent of $\delta$ and $\varepsilon$. Since $\varepsilon$ and $\delta$ are arbitrary, this yields the desired formula (F,1).

The formula (F,1) is still valid in the case when the effective domain of $f_x(\cdot)$ is a closed convex cone $C \subset \mathbb{R}^d$. The proof can be done by a similar way except some standard arguments. Moreover, the formula (F,1) of this case is used for the proof of Theorem 2. Indeed, under the conditions in Theorem 2, the homogenization $F(x, p_0; p)$ satisfies the conditions in Theorem 1 by replacing $\mathbb{R}^d$ by the cone $C = [0, \infty) \times \mathbb{R}^d$, and we can apply Theorem 1 for $F$. To end this note, we show this fact in the following proposition.
Proposition 8. If $f$ satisfies (1),(2),(3) in Theorem 2, then the homogenization $F$ satisfies (1),(2),(3) in Theorem 1 by replacing $\mathbb{R}^d$ by $C = [0, \infty) \times \mathbb{R}^d$.

Proof: It is stated in §1 that $F$ satisfies (2). Moreover,

$$F(x, 0, p) = \lim_{t \downarrow 0} f(x, \frac{p}{t})t$$
$$\leq \lim_{t \downarrow 0} c(1 + \frac{|p|}{t})t$$
$$= c|p|,$$

$$F(x, p_0, p) = f(x, \frac{p}{p_0})p_0$$
$$\leq c(1 + \frac{|p|}{p_0})p_0$$
$$= c(|p_0| + |p|) \quad (p_0 \neq 0),$$

and this proves (3). Hence it remains to prove (1). It is easy to see that $F$ is l.s.c. in $(p_0, p) \in C = [0, \infty) \times \mathbb{R}^d$. Hence it follows from (1) in Theorem 2 that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|(p_0, p) - (q_0, q)| < \delta, d(x_0, x) < \delta, q_0 \neq 0$ implies

$$F(x_0, p_0, p) - F(x, q_0, q) = F(x_0, p_0, p) - F(x_0, q_0, q) + F(x_0, q_0, q) - F(x, q_0, q)$$
$$< \varepsilon + (f(x_0, \frac{q}{q_0}) - f(x, \frac{q}{q_0}))q_0$$
$$< \varepsilon + \varepsilon(1 + \frac{|q|}{q_0})q_0$$
$$= \varepsilon + \varepsilon(|q_0| + |q|).$$

It is similar in the case of $q_0 = 0$. So $F$ is l.s.c. on $X \times C$ and the proof is complete. ■

References


