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PAYMENT SYSTEM AND SYSTEMIC RISK

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1 Introduction

Payment system is a set of arrangements made for the purpose of discharging obligations assumed by economic agents in their economic transactions. There are two basic ingredients of payments systems: they are settlement (or fund transfer) arrangements, and netting (or clearing) arrangements. Netting is an offsetting of a similar type of financial obligations, and only the net difference is settled. Thus arrangements for the binding netting of financial obligations provide a service that is a very close substitute for the function of money as a medium of payment. Netting between two parties is called bilateral netting, and netting among multiple parties is called multilateral netting. In the latter case netting is also referred to as clearing.

Foreign exchange transactions account for a large share of all payment flows in major financial centers. Because financial institutions have strong incentives to lower credit risk and payment flows, there have been active movements toward structural innovations in interbank clearing and settlement procedures and a number of proposals have been presented to establish multilateral foreign exchange netting arrangements in recent years.

In this paper we would like to introduce a formal model of foreign exchange contracts netting, and present an analysis of multilateral and bilateral netting from the view point of credit risk reduction. In particular, we are interested in comparing these two different forms of netting arrangements with respect to inherent systemic risks involved. In our analysis we will introduce a comparative statics type argument in intrinsically stochastic environments. The approach might be termed as a comparative contingencies. By looking at an event, given by a possible finite sequence of defaulting banks, we shall compare credit risks of a bank between the two different netting arrangements.

In the literature there have been very few studies that present formal models in discussing payment systems. It appears that serious studies on payment systems have just begun. Schoenmaker [6], and Rochet and Tirole [5] present a formal model of interbank settlement system. Eisenberg [3], Chakravorti [2], and Schoenmaker [7] present an analysis of systemic risks in settlement systems. Humphrey [4] gives a simulation study of systemic risk. We are not aware of formal models in which different netting arrangements are analyzed. However, the Angell report [1] gives a very nice discussion and a verbal analysis of risk structures in different netting schemes.

2 A Model of Foreign Exchange Contracts Netting

We shall consider a finite set I of financial institutions or *banks* that engage in cross-border or off-shore foreign exchange transactions over discrete points in time that may be referred to as *dates* also. A *foreign exchange contract or obligation* specifies a contractual duty to deliver a defined (foreign) currency in exchange for another (foreign) currency between two institutions on an agreed date or *value date.* The words — transactions, contracts, and obligations — are used synonymously in this paper. Institutions that engage in a financial contract are called *parties.* The opposite party to a financial contract is a *counterparty.* They are counterparties to each other. In this section we will build up a model of foreign exchange contracts netting based upon individual transactions of financial institutions.

Given a set of financial institutions I, typical elements of I will be written as

$$h, i, j \in I.$$

The number of financial institutions, $\sharp I$, to be considered for the purpose of our analysis is assumed to be at least four so that $4 \leq \sharp I < \infty$.

Now, let us consider foreign exchange transactions between banks $i, j \in I$ at time t_1, and collect all the transactions that take place on the same date t_1 with identical value date $t_2 (> t_1)$. Denote by

$$J_{ij}(t_1, t_2)$$

the set of all such transactions or contracts. With this notation above, we have

$$J_{ij}(t_1, t_2) = J_{ji}(t_1, t_2)$$

for any i and j and $J_{ij}(t_1, t_2) = \emptyset$ if $i = j$. For convenience, a contract $k \in J_{ij}(t_1, t_2)$ in this paper represents a "matched pair" of promises to deliver foreign currencies to counterparties. A *foreign exchange contract* $k \in J_{ij}(t_1, t_2)$ at t_1 obligates bank i to pay bank j at time t_2 the amount $y_{jik}(t_1, t_2, c')$ of currency c' and receives the amount $y_{ijk}(t_1, t_2, c)$ in currency c for some pair of currencies c and c'. For notational convenience we put $y_{ijk}(t_1, t_2, c'') = 0$ if the contract k does not involve an exchange of currency c''. In the notation $y_{ijk}(t_1, t_2, c)$, suffix i refers to a party, j to its counterparty, and k to a contract. If $y_{ijk}(t_1, t_2, c)$ is positive, then it means that party i is to receive the amount $y_{ijk}(t_1, t_2, c)$ from its counterparty j, and the counterparty is to pay the amount $-y_{jik}(t_1, t_2, c)$ to the party i. If $y_{ijk}(t_1, t_2, c)$ is negative, then i is to pay to j the amount $-y_{ijk}(t_1, t_2, c)$, and j is to receive the amount $y_{jik}(t_1, t_2, c)$ from i. Thus, for any contract

$k \in J_{ij}(t_1, t_2)$, we have

$$y_{ijk}(t_1, t_2, c) + y_{jik}(t_1, t_2, c) = 0 \tag{1}$$

which states that the amount to be received or paid $(= y_{ijk}(t_1, t_2, c))$ by a party is equal to the amount to be paid or received $(= -y_{jik}(t_1, t_2, k))$ by its counterparty respectively.

We assume that there is a finite set of (foreign) currencies traded in the markets. It is denoted by C. We assume $2 \leq \sharp C < \infty$. It is also assumed that there is a *"base currency"*, denoted by b , that is used to express values of all other currencies or that is the standard of value among different currencies.

Netting is an offsetting of receipts and payments to be made for a similar type of financial contracts. *Bilateral netting* is a netting between two parties. It is considered to be a most natural form of netting in foreign exchange transactions as large number of "matched trades" with same currency, same value date and same counterparties exist.

Consider two banks i and j and the set of contracts $J_{ij}(t_1, t_2)$ between them. Without a netting of contracts, banks i and j face payments of

$$\sum_{k \in J_{ij}(t_1,t_2)} y_{ijk}(t_1, t_2, c)^- \quad \text{and}$$

$$\sum_{k \in J_{ji}(t_1,t_2)} y_{jik}(t_1, t_2, c)^-$$

respectively at time t_2 as a result of contracts in $J_{ij}(t_1, t_2) = J_{ji}(t_1, t_2)$.

We shall define, for any t_1, t_2, c with $t_1 < t_2$,

$$y_{ij}(t_1, t_2, c) \equiv \sum_{k \in J_{ij}(t_1,t_2)} y_{ijk}(t_1, t_2, c).$$

Here, summing the amounts $y_{ijk}(t_1, t_2, c)$ of payments and/or receipts instead of the amounts $y_{ijk}(t_1, t_2, c)^-$ of payments and the amounts $y_{ijk}(t_1, t_2, c)^+$ of receipts separately over all the contracts in $J_{ij}(t_1, t_2)$ means that the various payments $y_{ijk}(t_1, t_2, c)^-$ are offset against various receipts $y_{ijk}(t_1, t_2, c)^+$ for $k \in J_{ij}(t_1, t_2)$ to arrive at one net amount $y_{ij}(t_1, t_2, c)$. With this notation, it follows from the equality in (1) that we have for each t_1, t_2, c

$$y_{ij}(t_1, t_2, c) + y_{ji}(t_1, t_2, c) = 0, \tag{2}$$

which means that the amount that i is to receive from j is equal to the amount that j is to pay to i for each currency c under bilateral netting.

For notational convenience, if there are no transactions in $J_{ij}(t_1, t_2)$ for which $y_{ijk}(t_1, t_2, c) \neq 0$, then we put

$$y_{ij}(t_1, t_2, c) = 0. \tag{3}$$

In particular, if $J_{ij}(t_1, t_2) = \emptyset$, we have

$$y_{ij}(t_1, t_2, c) = 0$$

for all $c \in C$.

We assume that all the forward foreign exchange contracts take place during a definite time span that is given by time periods between T_1 and T_2. Let us define for each t, t_2, c with $t \leq t_2$

$$x_{ij}(t, t_2, c) \equiv \sum_{t_0 \leq t_1 \leq t} y_{ij}(t_1, t_2, c),$$

where t_0 is the date at which an initial transaction of currency c with value at t_2 took place between i and j. The net amount, as of time t, that i is to pay to or receive from j in currency c at time t_2 is expressed by $x_{ij}(t, t_2, c)$. There are two basic properties relating to $x_{ij}(t, t_2, c)$.

First one is that it is the "accumulated" bilateral position at time t of bank i with respect to j for contracts in currency c with value at t_2, that is, for any t, t_2, c with $t < t_2$, we have

$$x_{ij}(t, t_2, c) = x_{ij}(t-1, t_2, c) + y_{ij}(t, t_2, c).$$

Second one is that we have again a fundamental "mirror image" equality for any t, t_2, c with $t < t_2$ due to (2) that

$$x_{ij}(t, t_2, c) + x_{ji}(t, t_2, c) = 0. \tag{4}$$

Let $q(t, t_2, c)$ denote forward exchange rates of currencies $c \in C$ quoted in terms of the base currency b at time t with value at time t_2. Now, for a foreign exchange transaction $k \in J_{ij}(t_1, t_2)$ which took place at date t_1, there is a pair of currencies (c,c') for which we have $y_{ijk}(t_1, t_2, c) \neq 0$, $y_{ijk}(t_1, t_2, c') \neq 0$ and $y_{ijk}(t_1, t_2, c'') = 0$ for $c'' \neq c, c'$. For the pair (c,c') one must have

$$q(t_1, t_2, c) y_{ijk}(t_1, t_2, c) + q(t_1, t_2, c') y_{ijk}(t_1, t_2, c') = 0.$$

It thus follows that for any contract $k \in J_{ij}(t_1, t_2)$

$$\sum_{c \in C} q(t_1, t_2, c) y_{ij}(t_1, t_2, c) = 0.$$

However, for $t_1 < t \leq t_2$, one may have

$$\sum_{c \in C} q(t, t_2, c) y_{ij}(t_1, t_2, c) \neq 0$$

due to fluctuations in forward exchange rates $q(t, t_2, c)$. If this amount is negative, bank i's forward book shows a loss concerning the transactions at t_1 with value at t_2. The amount $\sum_c q(t,t_2,c)y_{ij}(t_1,t_2,c)$ represents the *mark-to-market value* or *net present value* at time t of the transactions at time t_1 between banks i and j with value at time t_2. We denote this value by

$$v_{ij}(t_1,t_2)(t) \equiv \sum_{c\in C} q(t,t_2,c)y_{ij}(t_1,t_2,c).$$

We now calculate the *mark-to-market value* or *net present value* of the forward book of bank i with respect to bank j as of time t which will be written as $v_{ij}(t)$. It is given by

$$v_{ij}(t) \equiv \sum_{\substack{t_1\le t\\ t<t_2\le T_2}} v_{ij}(t_1,t_2)(t)$$

for each t. Note that by our notational convention in (3) we have $v_{ij}(t)=0$ if $i=j$ or if $J_{ij}(t_1,t_2)=\emptyset$ for all $t_1\le t$.

$v_{ij}(t)$ is the *bilateral position* of bank i with bank j at time t concerning i's forward book. We note that we have

$$v_{ij}(t)+v_{ji}(t)=0. \tag{5}$$

Since $v_{ij}(t)$ is the net present value of all forward transactions of i with j evaluated by forward prices at time t, the equality (5) is interpreted to say that in forward transactions both two parties cannot win at the same time — a party or its counterparty must loose between the two.

3 Risk Assessment

3.1 Bilateral Credit Exposures

We first consider bilateral netting by novation between two banks i and j. The amount $x_{ij}(t,t,c)$, $c\in C$, exhibits the bilateral settlement position of bank i with respect to bank j in the sense that it represents the amount of currency $c\in C$ that bank i is to receive from bank j at time t. Potential loss of bank i may arise during the course of business day t from bank j's failure to make settlements. It will be called the *bilateral settlement exposure* in currency $c\in C$ of bank i with respect to j.

Let us denote the bilateral settlement exposure at time t by $\xi_{ij}(t,c)$. Then, by definition we have

$$\xi_{ij}(t,c)\equiv x_{ij}(t,t,c)^+$$

for i, $j\in I$ and $c\in C$. Note that settlement exposures at time t derive from contracts that are due to be settled on that day. Thus, it is natural for exposures to be defined currency by currency.

But in the event of a default of a counterparty, there is another potential loss. It arises from the couterparty's failure to honor its forward obligations. When forward obligations are dishonored, it is natural to assume that each bank replaces them by similar forward obligations in current markets to resume its forward positions in each currency. Thus, the idea of close-out netting, which is invoked in case of a default, is to compute replacement costs of dishonored forward obligations. Loss arising from the failure of a counterparty to honor its forward obligations will be termed as *actual exposure* of a party with respect to its counterparty. It may also be called *mark-to-market exposure* or *net present value exposure*. Following the notation of our model, the *(actual) bilateral exposure* at time t of bank i with respect to bank j, denoted by $\varepsilon_{ij}(t)$, is thus defined by

$$\begin{aligned}\varepsilon_{ij}(t) &\equiv v_{ij}(t)^+\\ &= \left(\sum_{t<t_2\le T_2}\sum_c q(t,t_2,c)x_{ij}(t,t_2,c)\right)^+.\end{aligned}$$

3.2 Multilateral Credit Exposures

Let the set of financial institutions I represent the group of participating banks to an arrangement of multilateral netting by novation and substitution for clearing and settlement, and that of multilateral close-out netting in case of a default. The clearing house will be denoted by the letter H. Define for each $i \in I$ and t, t_2, c

$$\begin{aligned}x_{iH}(t,t_2,c) &\equiv \sum_{j\in I} x_{ij}(t,t_2,c),\\ x_{Hi}(t,t_2,c) &\equiv \sum_{j\in I} x_{ji}(t,t_2,c).\end{aligned}$$

Other notations with respect to H are introduced similarly. For example, for $i \in I$ and t,

$$\begin{aligned}v_{iH}(t) &\equiv \sum_{j\in I} v_{ij}(t),\\ v_{Hi}(t) &\equiv \sum_{j\in I} v_{ji}(t).\end{aligned}$$

In essence suffix H indicates that sum is taken over all participating banks to netting and clearing arrangements.

Under a multilateral netting arrangement by novation and substitution, *multilateral settlement position* in currency c at time t is given by

$$\begin{aligned}x_{iH}(t,t,c) &= \sum_{j\in I} x_{ij}(t,t,c)\\ &= \sum_{j\in I}\sum_{t_1\le t}\sum_{k\in J_{ij}(t_1,t)} y_{ijk}(t_1,t,c).\end{aligned}$$

for each participating bank $i \in I$. Here, for every contract $k \in J_{ij}(t_1, t)$, $t_1 \leq t$, the clearing house is substituted for the counterparty j so that the bilateral settlement position $x_{ij}(t,t,c)$ of i with respect to j becomes only a part of settlement position of i with respect to the clearing house H. Taking the sum of $x_{ij}(t,t,c)$'s over $j \in I$ means totalling all those parts of bilateral settlement positions of counterparties of bank i for which the clearing house is substituted. Thus, the *multilateral* settlement position of i given by $x_{iH}(t,t,c)$ is the *bilateral* settlement position of i with respect to the clearing house after substitutions and novations of obligations are effected.

We are now concerned with the amount of forward book positions of bank i that is exposed to a default risk. If bank $j(\neq i)$ will default at time t, potential loss of bank i is *not* necessarily given by the amount $v_{iH}(t)^+$ *nor* $v_{ij}(t)^+$. It is because original contracts of bank i with bank j are replaced by those with the clearing house. Hence, possible losses that each bank must face in multilateral netting depend upon how losses are allocated among participating banks in netting arrangements.

Bilateral position under a multilateral netting arrangement will be called *notional bilateral position.* The *notional bilateral forward position* of bank i with bank j at time t is

$$v_{ij}(t) \quad = \sum_{t<t_2\leq T_2} \sum_{c\in C} q(t,t_2,c)x_{ij}(t,t_2,c).$$

The *multilateral (forward) position* of bank i at time t is

$$v_{iH}(t) = \sum_{j\in I} v_{ij}(t).$$

The amount $v_{iH}(t)$ is the bilateral position of bank i with respect to the clearing house at time t under close-out netting. We say bank i at time t has a *multilateral profit position* if $v_{iH}(t) > 0$ and a *multilateral loss position* if $v_{iH}(t) < 0$.

Let us introduce a few more concepts before going into definitions of forward book credit exposures. We say that bank i's forward position is *perfectly hedged* or that bank i's *foreign exchange risk* is *perfectly hedged* at time t if

$$(\forall c \in C, \forall t_2 \leq T_2) \quad \sum_{j\in I} x_{ij}(t,t_2,c) = 0.$$

If bank i's forward position is perfectly hedged, then, for all $T_1 \leq t \leq T_2$, we have

$$\begin{aligned} v_{iH}(t) &= \sum_{j\in I} v_{ij}(t) \\ &= \sum_{j\in I} \sum_{t<t_2\le T_2} \sum_{c\in C} q(t,t_2,c)x_{ij}(t,t_2,c) \\ &= \sum_{t<t_2\le T_2} \sum_{c\in C} q(t,t_2,c) \sum_{j\in I} x_{ij}(t,t_2,c) \\ &= 0 \end{aligned}$$

so that for all t

$$v_{iH}(t) = 0.$$

That is, if the forward position of a bank is perfectly hedged, then its forward position with the clearing house is nil. From a traditional banker's point of view it represents an ideal situation for foreign exchange transactions and, moreover, it will correspond to the most ideal case for contracts netting efficiency as all the credits exactly offset all the debits in each currency at every due date.

Fact 1 [Clearing Efficiency] *If a participating bank's forward position is perfectly hedged, then no settlements are needed at due dates as credits and debits are fully matched.*

Even if bank i's foreign exchange risk is perfectly hedged so that $v_{iH}(t) = 0$, it faces default risks of counterparties. Assume that one of the counterparties, say bank j, of forward contracts that bank i had, defaulted at time t. Then, a general rule of the clearing house is to allocate default induced losses among the participants, called *concerned* participants or *concerned* banks, who have contracts with the defaulting bank maturing at time t. Losses are allocated pro rata to the profit levels of the concerned participants. By assuming this loss allocation rule, we define *multilaterally netted bilateral exposure or indirect bilateral exposure* of bank i with respect to bank j at time t, denoted by $\eta_{ij}(t)$, to be

$$\eta_{ij}(t) \equiv \left(\frac{v_{ij}(t)^+}{\sum_{h\in I} v_{hj}(t)^+}\right) v_{Hj}(t)^+.$$

It simply says that the loss $v_{Hj}(t)^+$ of the clearing house caused by the default of bank j is allocated among those participants i having notional profit position with bank j, i.e., $v_{ij}(t)^+ > 0$, according to the ratio

$$\left(\frac{v_{ij}(t)^+}{\sum_{h\in I} v_{hj}(t)^+}\right)$$

of its profit level to the total of profits made by concerned banks in transaction with the defaulting bank j.

Now, assume bank j's forward position is not perfectly hedged so that at time t

$$v_{jH}(t) \neq 0$$

and assume that the net present value of its forward books show a great deal of losses, a typical situation for a defaulting bank. Then, the *actual credit exposure of the clearing house with respect to bank* j at time t is

$$\begin{aligned} v_{Hj}(t)^+ &= \left(\sum_{i\in I} v_{ji}(t)^+ - \sum_{i\in I} v_{ji}(t)^-\right)^- \\ &= \max\left\{0, \sum_{i\in I} v_{ij}(t)^+ - \sum_{i\in I} v_{ij}(t)^-\right\}. \end{aligned} \tag{6}$$

We thus obtain

$$\sum_{i\in I} v_{ij}(t)^+ - v_{Hj}(t)^+ = \min\left\{\sum_{i\in I} v_{ij}(t)^+, \sum_{i\in I} v_{ij}(t)^-\right\}. \tag{7}$$

Since we assumed $v_{jH}(t) < 0$ so that

$$\sum_{i\in I} v_{ji}(t)^+ < \sum_{i\in I} v_{ji}(t)^-,$$

we obtain

$$v_{Hj}(t)^+ = \sum_{i\in I} v_{ij}(t)^+ - \sum_{i\in I} v_{ij}(t)^-.$$

Proposition 1 [Credit Exposure of the Clearing House] *Suppose that bank j has a multilateral loss position at time t. Then, the sum of the bilateral credit exposures of other participants with respect to the bank j at time t exceeds the actual credit exposure of the clearing house with respect to j by the amount of the sum of losses that the participants are making in transactions with j, i.e.*

$$\sum_{i\in I} v_{ij}(t)^+ - v_{Hj}(t)^+ = \sum_{i\in I} v_{ij}(t)^-. \tag{8}$$

One can immediately compare direct bilateral exposure with indirect bilateral exposure. Using the equality (8), one obtains

$$\varepsilon_{ij}(t) - \eta_{ij}(t) = \frac{v_{ij}(t)^+}{\sum_{h\in I} v_{hj}(t)^+}\left(\sum_{h\in I} v_{hj}(t)^-\right). \tag{9}$$

Proposition 2 [Bilateral Exposure vs. Multilaterally Netted Bilateral Exposure] *The difference between direct bilateral exposure and indirect bilateral exposure of a bank i with respect to any other bank $j \neq i$ having a multilateral loss position at time t is exactly equal to the sum of losses of* *individual banks in transaction with the bank j multiplied by the proportion of bank i's profits to the total profits of all the participating banks in transactions with the bank j, i.e.*

$$\frac{v_{ij}(t)^+}{\sum_{h\in I} v_{hj}(t)^+}\left(\sum_{h\in I} v_{hj}(t)^-\right).$$

Let us introduce a notion which indicates the extremity of market risk. We say that the *foreign exchange risk* of *bank i at time t* is *extreme* if for all $h \neq i$

$$v_{ih}(t)^+ = 0.$$

and for some $h \neq i$

$$v_{ih}(t)^- \neq 0.$$

One may decompose the reduction of total credit exposures in (8) into that of individual banks. We define (apparent) *bilateral benefits of multilateral netting*, denoted by $\beta_{ij}(t)$,[1] as the residual of direct bilateral exposures over indirect bilateral exposures, that is,

$$\begin{aligned}\beta_{ij}(t) &\equiv \varepsilon_{ij}(t) - \eta_{ij}(t)\\ &= v_{ij}(t)^+ - \left(\frac{v_{ij}(t)^+}{\sum_{h\in I} v_{hj}(t)^+}\right) v_{Hj}(t)^+.\end{aligned}$$

It follows from the definition of the bilateral benefits of multilateral netting and the equation (9) that

$$\begin{aligned}\beta_{ij}(t) &= \frac{v_{ij}(t)^+}{\sum_{h\in I} v_{hj}(t)^+}\left(\sum_{h\in I} v_{hj}(t)^-\right)\\ &= \frac{v_{ij}(t)^+}{\sum_{h\in I} v_{hj}(t)^+}\left(\sum_{h\in I} v_{jh}(t)^+\right).\end{aligned}$$

Thus, if the foreign exchange risk of bank j is not extreme, there will be positive (apparent) bilateral benefits (*i.e.*, $\beta_{ij}(t) > 0$) of multilateral netting on the part of concerned participant i, having a positive bilateral exposure, *i.e.*, $v_{ij}(t)^+ > 0$. But if the foreign exchange risk of bank j is extreme, then the above equation for $\beta_{ij}(t)$ shows that $\beta_{ij}(t) = 0$, that is, there are no bilateral benefits of multilateral netting over bilateral netting. Thus one obtains the following proposition.

Proposition 3 [Bilateral Benefits/No–Benefits of Multilateral Netting] *Suppose that bank j is making a loss on its forward book at time t. If its foreign*

[1] One should not be misled by the term benefit. This need not be an actual benefit when one considers a possibility of systemic risk.

exchange risk is not extreme, then for any concerned participant having a positive (notional) bilateral credit exposure, the bilateral benefit of multilateral netting is positive. But if bank j's foreign exchange risk is extreme, then there will be no benefits of multilateral netting.

The implication of this proposition is that as the loss-making bank's foreign exchange risk increases the bilateral benefit of multilateral netting declines. It has a policy implication as well: in order for a multilateral netting system to have increased efficiency for reducing credit risks, extreme foreign exchange contract positions should be controlled.

4 Systemic Risk Assessment

We now consider a possibility of chain reactions of multiple defaults of participating banks to a multilateral netting system. If a finite sequence of defaulting banks $\{j_n\}_{n=1,\ldots,N}$ at time t is given, a bank $j \in \{j_n\}_{n=1,\ldots,N}$ is called a *defaulter* or a *defaulting* bank and a bank $j \notin \{j_n\}_{n=1,\ldots,N}$ is called a *survivor* or a *surviving bank*. Given a defaulting bank j, a surviving bank i is called *positively concerned* if

$$v_{ij}(t)^+ > 0,$$

and is called *negatively concerned* if

$$v_{ij}(t)^- > 0.$$

Let a finite sequence of banks $\{j_n\}_{n=1,\ldots,N}$ with $1 \leq N \leq \sharp I - 1$ is a possible set of defaulters at time t. For simplicity we assume that all the defaults at time t occurs in a time stream matching the order in the sequence $\{j_n\}_n$ during the date t but before the beginning of the next date $t+1$.

Our assumed loss share rule is to distribute the losses of the clearing house, due to a default of a participant, among positively concerned surviving banks according to the ratio of their notional bilateral profit levels with respect to the defaulter at each round of default. Let us first define

$$r_{ij_n}^m(t) \equiv \frac{v_{ij_n}(t)^+}{\sum_{h \neq j_1,\ldots,j_m} v_{hj_n}(t)^+}$$

for $i \neq j_1, \ldots, j_m$ and $n \leq m \leq N$. $r_{ij_n}^m(t)$ represents the proportion of the loss, $v_{Hj_n}(t)^+$, caused by the n-th defaulter j_n at time t, which the bank i has to share *immediately after* the m-th default. Coefficients, $r_{ij_n}^m(t)$'s may be called *direct bilateral loss share* coefficients at m-th round. We will need notation to express proportions of losses shared by subsequent defaulters for their preceding defaulters.

Given a finite sequence of banks $\{j_n\}_{n=1,\dots,N}$ at time t, define for $m = 2,\dots,N, n = 1,\dots,N-1, m \geq n+1$

$$r_{mn}(t) \equiv \sum_{s=n}^{m-1} r^s_{j_m j_n}(t) r_{sn}(t) \tag{10}$$

with a convention that

$$r_{nn}(t) \equiv 1$$

for $n = 1,\dots,N-1$. In particular, we have

$$\begin{aligned} r_{21}(t) &= r^1_{j_2 j_1}(t) \\ r_{31}(t) &= r^1_{j_3 j_1}(t) + r^2_{j_3 j_1} r_{21}(t) \\ &\vdots \\ r_{N1}(t) &= r^1_{j_N j_1}(t) + r^2_{j_N j_1}(t) r_{21}(t) + \dots + r^{N-1}_{j_N j_1}(t) r_{(N-1)1}. \end{aligned}$$

$r_{mn}(t)$ is the total sum of the direct and indirect proportional loss shares of the m-th defaulter for the loss of the n-th defaulter accumulated as the sequential defaults continued up to the $(m-1)$-st round. In this sense, coefficients $r_{mn}(t)$'s may be called *total bilateral loss share coefficients* among defaulters.

Proposition 4 [Property of Loss Share Coefficients] *For any $n = 1,\dots,N$, the following equality holds:*

$$\sum_{s=n}^{k} r^s_{j_m j_n}(t) r_{sn}(t) = r^k_{j_m j_n}(t) \tag{11}$$

for $m = n+1,\dots,N$ and $k = n,\dots,m-1$.

As an immediate corollary to Proposition 4 one obtains the following proposition:

Proposition 5 [Loss Share Coefficients and Direct-Indirect Loss Share] *Given two defaulters j_m and j_n with j_n preceding j_m in the default sequence at time t, the total sum of proportional shares of loss, due to the default of j_n, that the bank j_m has to share directly or indirectly, is exactly equal to the direct proportional loss share of j_m excluding the first $m-1$ defaulters, that is,*

$$r_{mn}(t) = r^{m-1}_{j_m j_n}(t) \ .$$

The following proposition shows the similar property as in Proposition 4 for an arbitrary participating bank to multilateral netting arrangements.

Proposition 6 [Loss Share Coefficients in General] *For any participating bank $i \in I$ to the multilateral netting, the direct loss share coefficients at the m-th round of defaults with respect to the loss of the clearing house due to the n-th defaulter ($n \leq m$), $r^m_{ij_n}(t)$, is exactly equal to its accumulated direct and indirect loss shares up to the m-th round of defaults, that is, for any $i \neq j_1, \ldots, j_n, n = 1, \ldots, N$ or for any $i = j_{n'}$, with $n' \geq m \geq n$, $n' \neq n$, one has*

$$r^m_{ij_n}(t) = \sum_{s=n}^{m} r^s_{ij_n}(t) r_{sn}(t) \tag{12}$$

for $m = n, \ldots, N$.

Let us introduce further notation to express surviving banks' total direct and indirect loss shares. For each $i \neq j_1, \ldots, j_N$ and for each $n = 1, \ldots, N$, define

$$l_{in}(t) \equiv \sum_{m=n}^{N} r^m_{ij_n}(t) r_{mn}(t).$$

$l_{in}(t)$ is the total sum of proportional shares by bank i of the loss caused by the n-th defaulter, accumulated at the end of the default sequence $\{j_n\}_{n=1,\ldots,N}$ at time t. As an immediate corollary to Proposition 6, we obtain the following:

Proposition 7 [Survivors' Total Loss Share Coefficients] *For any surviving bank i, the total sum of direct and indirect proportional shares of the loss of the clearing house due to the n-th defaulting bank is identical to the direct proportional share of the loss of the clearing house due to the n-th defaulter when all the N defaulters are excluded in the calculation of its proportional share, that is,*

$$l_{in}(t) = r^N_{ij_n}(t).$$

Given a finite sequence of banks $\{j_n\}_{n=1,\ldots,N}$ at time t with $1 \leq N \leq \sharp I - 1$, the *systemic credit exposure* of bank i at time t with respect to the sequence $\{j_n\}_{n=1,\ldots,N}$, written $\sigma_{ij_1\ldots j_N}(t)$, is defined by

$$\sigma_{ij_1\ldots j_N}(t) \equiv \sum_{n=1}^{N} \left(\sum_{m=n}^{N} r^m_{ij_n}(t) r_{mn}(t) \right) v_{Hj_n}(t)^+ \tag{13}$$

for $i \neq j_1, \ldots, j_N$ and $n \leq m \leq N$. In the expression (13), n-th term is composed of $N - n + 1$ parts. The first part $r^n_{ij_n}(t) v_{Hj_n}(t)^+$ represents the bank i's direct share of the loss of the clearing house due to the default of n-th bank. The m-th part $r^m_{ij_n}(t) r_{mn}(t) v_{Hj_n}(t)^+$, $n < m \leq N$, represents the bank i's indirect share of the loss that the m-th defaulting bank j_m had to share for the loss caused by the n-th defaulting bank j_n.

With the interpretation of each term in the expression (13) as above, the systemic credit exposure of a survived bank i represents the sum of its share of losses that are directly or indirectly related to its forward book profits position with respect to defaulting banks. In order to see that a participant may have to share a part of the losses that are *indirectly* related to its profit position, let us take for example the term $r^3_{ij_1}(t)r_{31}(t)v_{Hj_1}(t)^+$ in a sequence of three-bank defaults. It represents the part of the losses of the initial defaulting bank j_1 which came to be shared by the surviving bank i because it was originally due to be shared by bank j_3 that subsequently defaulted.

We are ready to define the *benefit of multilateral netting* of bank i with respect to a given finite sequence $\{j_n\}_{n=1,\ldots,N}$ of banks at time t. It is defined by

$$\beta_{ij_1\ldots j_N}(t) \equiv \sum_{n=1}^{N} \varepsilon_{ij_n}(t) - \sigma_{ij_1\ldots j_N}(t) \tag{14}$$

The benefit of multilateral netting is thus the difference between the sum of notional bilateral credit exposures and the systemic credit exposures with respect to a given sequence of banks. So to speak, our approach is that of the *comparison by contingencies* in assessing the possible benefit of multilateral netting *vis-à-vis* bilateral netting.

One can prove the following proposition.

Proposition 8 [General Decreasing Benefits of Multilateral Netting] *Suppose that there is a possibility of multiple defaults among participants to a multilateral netting system with the initial defaulting bank having a multilateral loss position. Then, for any participating bank i to the netting system, the sum of bilateral benefits of multilateral netting always overstates actual benefits of multilateral netting, i.e.* $\beta_{ij_1\cdots j_n}(t) < \sum_{n=1}^{N} \beta_{ij_n}(t)$, *if, in a sequence of possible multiple defaulters at time t,*

- *there is a subsequent defaulter with whom the participant i is positively concerned and who is positively concerned with the immediately preceding defaulter, or*

- *there are at least two defaulters with the preceding defaulter having a multilateral loss position such that the participant i is positively concerned with the preceding defaulter and that the succeeding defaulter is positively concerned with the preceding defaulter.*

In comparing merits of multilateral netting with those of bilateral netting from a view point of credit risk reduction one might be tempted to compare multilaterally netted bilateral exposures with (actual) bilateral credit exposures. However, if one does compare bilateral netting with multilateral netting using the sum of bilateral benefits of multilateral netting in the similar spirits as above, then Proposition 8 above warns us that in possible events of multiple bank failures one overestimates the benefits of multilateral netting because the actual benefit of multilateral netting is strictly less than the sum of bilateral benefits of multilateral netting due to a possibility of an indirect sharing of losses of defaulting banks.

One should probably stress the fact that there are possibilities of indirect sharing of losses in face of a chain reaction of multiple defaults that could reduce the attractiveness of multilateral netting system over that of bilateral netting system. Actual situation for multilateral netting may be worse in the sense that a reversal of relative attractiveness of netting between multilateral and bilateral netting may occur when one allows for a consideration of multiple defaults. In order to show that there are possibilities of negative benefits of multilateral netting over bilateral netting, we wish to proceed to exhibit conditions under which we would have

$$\beta_{ij_1\cdots j_N}(t) < 0$$

in general.

Recall that bank i has a *perfectly hedged* position at time t if

$$(\forall c \in C, \forall t_2 \leq T_2) \quad \sum_{j\in I} x_{ij}(t, t_2, c) = 0.$$

If bank i has a perfectly hedged position at time t, then

$$v_{iH}(t) = 0.$$

Even if a bank has a perfectly hedged position, it cannot avoid default risk of a counterparty. This motivates us to introduce a further concept of perfect hedging. Let us say that bank i has a *pairwise perfectly hedged* position with respect to bank $j \neq i$ at time t if

$$(\forall c \in C \text{ and } \forall t_2 \leq T_2) \; x_{ij}(t, t_2, c) = 0. \tag{15}$$

A perfect pairwise hedging with respect to all $j \neq i$ clearly implies a perfect hedging. But note that if a party has a pairwise perfectly hedged position with respect to its counterparty, then it will not face default risk of its counterparty. Since our focus is the reduction of credit risk rather than that of market or foreign exchange risk in comparing different netting arrangements, we shall analyze in this subsection participants who have a perfectly hedged position but *not* a pairwise perfectly hedged position with an initial defaulter for, otherwise, it will not face credit risk. However, in order to minimize the complexity of analysis we analyze cases of a chain of bank defaults where a participant has pairwise perfectly hedged positions with secondary and subsequent defaulters.

A scenario of systemic risk we consider in this subsection is as follows:

- The original defaulter $j_1 \in I$ fails to meet its obligations at time t. The bank j_1 faces foreign exchange risk at time t and its multilateral position shows a huge loss by an amount $L_1 > 0$ so that

$$v_{Hj_1}(t)^+ = v_{j_1H}(t)^- = L_1 > 0\;.$$

- Bank i has a perfectly hedged forward position so that

$$v_{iH}(t) = 0,$$

  but it has bilateral profit position with the initial defaulter j_1. The bank i, however, has pairwise perfectly hedged positions with the remaining defaulting banks $j_2, \ldots, j_N$.

Then, we have

$$v_{ij_n}(t)^+ = 0 \text{ for } n = 2, \ldots, N \ .$$

To ease our notation, put

$$\begin{aligned}
\pi &\equiv v_{ij_1}(t)^+ > 0 \ , \\
l &\equiv \sum_{h \in I} v_{hj_1}(t)^-, \\
\Pi &\equiv \sum_{h \in I} v_{hj_1}(t)^+ \ , \text{ and} \\
\delta &\equiv \frac{\sum_{h=j_2,\ldots,j_N} v_{hj_1}(t)^+}{\sum_{h \in I} v_{hj_1}(t)^+} \ .
\end{aligned}$$

π is the notional bilateral profit of the bank i with respect to the initial defaulter j_1 and Π is the total sum of notional bilateral profits of participants with respect to the defaulter j_1. l, on the other hand, is the total sum of notional bilateral losses of participants with respect to j_1. We have

$$L_1 = \Pi - l$$

and since we are assuming that the initial defaulter j_1 is making a huge loss, put

$$\Pi = Ml$$

for some "large" number M. Then,

$$L_1 = (M-1)l$$

Now, it follows from the equality (??) that we have

$$\begin{aligned}
&\beta_{ij_1 \ldots j_N}(t) \\
&= r^N_{ij_1}(t) \min \left\{ \sum_{h \in I} v_{hj_1}(t)^- - \sum_{h=j_1,\ldots,j_N} v_{hj_1}(t)^+, \sum_{h \neq j_1,\ldots,j_N} v_{hj_1}(t)^+ \right\} \\
&= r^N_{ij_1}(t) \left(\sum_{h \in I} v_{hj_1}(t)^- - \sum_{j_2,\ldots,j_N} v_{hj_1}(t)^+ \right) \qquad (16)
\end{aligned}$$

where the first and second equalities follow from the assumed properties that i has pairwise perfectly hedged positions with respect to banks $j_2, \ldots, j_N$ so that $v_{ij_n}(t)^+ = 0$ for $n = 2, \ldots, N$, and that

$$v_{Hj_1}(t)^+ = \sum_{h \in I} v_{hj_1}(t)^+ - \sum_{h \in I} v_{hj_1}(t)^- > 0.$$

Using simplified notation, one obtains

$$\beta_{ij_1 \ldots j_N}(t) = r^N_{ij_1}(t)(l - \delta\Pi) \ . \tag{17}$$

Since $\delta\Pi = \delta M l$ and since i is positively concerned with j_1 so that $r^N_{ij_1}(t) > 0$, we have

$$\beta_{ij_1 \ldots j_N}(t) < 0 \Longleftrightarrow \delta M > 1 \ . \tag{18}$$

This gives a necessary and sufficient condition under which one has

$$\beta_{ij_1 \ldots j_N}(t) < 0 \ .$$

Recall that M indicates that the total of the notional bilateral losses of the initial defaulter with its counterparties is M times the total of its profits with its counterparties. Thus, the systemic credit exposure under a multilateral netting system is always greater than the sum of bilateral credit exposures with respect to the possible sequence of defaulting banks if and only if M times the proportion of the sum of notional bilateral profits of all the remaining defaulters to the total of notional bilateral profits with the initial defaulter is greater than 1, that is, $\delta M > 1$; in other words, if and only if M is large enough so that its inverse is strictly less than the proportion of the sum of notional bilateral profits of the remaining defaulters to the total of notional bilateral profits of all the participants with respect to the initial defaulter. This condition will be met if the total of bilateral losses of the initial defaulter is very large relative to its profits at time t.

Since one has

$$\begin{aligned} \delta = \frac{\sum_{h=j_2,\ldots,j_N} v_{hj_1}(t)^+}{\sum_{h \in I} v_{hj_1}(t)^+} \quad &> \quad \frac{1}{M} = \frac{\sum_{h \in I} v_{hj_1}(t)^-}{\sum_{h \in I} v_{hj_1}(t)^+} \\ \Longleftrightarrow \quad \sum_{h=j_2,\ldots,j_N} v_{hj_1}(t)^+ \quad &> \quad \sum_{h \in I} v_{hj_1}(t)^- \ , \end{aligned}$$

still another way of stating the condition above is that the total of notional bilateral profits of the remaining defaulters, with respect to the initial defaulter, is strictly greater than the total of notional bilateral losses of all the participants with respect to the initial defaulter.

We thus obtain the following proposition which we regard as the main result of the present paper:

Proposition 9 [Multilateral vs. Bilateral Netting] *Consider a possible arbitrary finite sequence of defaulting banks* $\{j_n\}_{n=1,\dots,N}$, $1 < N < \sharp I$ *at time* t. *Assume that the initial defaulting bank* j_1 *faced with high foreign exchange risk suffers a huge mark-to-market gross loss by an amount* L_1 *which is* M *times the amount of the sum of mark-to-market notional bilateral losses of other participants with respect to the initial defaulter* j_1.

Consider any participating bank that has a net bilateral profit position with the initial defaulter. And assume that at time t *it has a multilateral perfectly hedged position and pairwise perfectly hedged positions with respect to each of the defaulting banks except for the initial defaulter. Then, its systemic credit exposure under multilateral netting is strictly greater than the sum of bilateral credit exposures with respect to each of the banks along the sequence of possible defaulters under bilateral netting if and only if the total of notional bilateral profits of the remaining defaulters, with respect to the initial defaulter, is strictly greater than the total of notional bilateral losses of all the participants with respect to the initial defaulter, or differently put, if and only if* M *is large enough so that its inverse is strictly less than the proportion of the sum of notional bilateral profits of the remaining defaulters to the total of notional bilateral profits of all the participants with respect to the initial defaulter.*

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