数理解析研究所講究録
988巻 1997年 182-187

ON THE FATOU COMPONENT IN $P^k(C)$

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1. Preliminaries Let P^k be the complex projective space of k dimension, and $f : P^k \to P^k$ be a holomorphic self map on P^k. Taking the homogeneous coordinates $[z_0 : z_1 : ... : z_k]$ of a point z of P^k, we can denote f by

$$f(z) = [f_0(z) : f_1(z) : ... : f_k(z)],$$

where $f_j(z)$ is a homogeneous polynomial of $z = (z_0, ..., z_k)$ and $f_j(z)$ has no common zeros. Corresponding to f, we have a nondegenerate homogeneous holomorphic map $\tilde{f}$ on C^{k+1}, which is given by

$$\tilde{f}(z) = (f_0(z), f_1(z), ..., f_k(z)).$$

If π is the canonical projection $C^{k+1} \setminus \{0\} \to P^k$, then $\pi \circ \tilde{f} = f \circ \pi$. The homogeneous polynomials f_j have the same degree, for example d, hence f is called a holomorphic map of degree d. Let $H_d(P^k)$ or simply H_d denote the space of all holomorphic self maps on P^k of degree d. Hereafter we assume $d \geq 2$. As usual f^n denotes the n fold iterate of a map f. About the fundamental properties of the space H_d, see [FS2].

According to [FS4], we shall give the definition of l-th Julia set of f. (It was firstly given in [HP], see also [U2].) Let $f \in H_d$ $(d \geq 2)$, and $\tilde{f}$ be a lifting of f to C^{k+1}. Since $\tilde{f}$ is a homogeneous polynomial map with $\tilde{f}^{-1}(o) = \{o\}$, an attracting basin of f,

$$A = \{z \in C^{k+1}; lim_{n\to\infty}\tilde{f}^n(z) = o\}$$

is a complete circular domain. For A we have the Green function

$$G(z) = lim_{n\to\infty}d^{-n}log\left|\tilde{f}^n(z)\right|$$

which vanishes on A, is plurisubharmonic on C^{k+1}, and satisfies

$$G(\lambda z) = log\,|\lambda|+G(z), \text{for any } \lambda \in C,\ G(\tilde{f}(z)) = d{\cdot}G(z).$$

We can define a $(1,1)$ positive closed current T on P^k by the relation $\pi^*T = dd^cG$. For the theory of current which is used in the theory of complex dynamics, see [HP], [FS 3], [FS 4].

Since T has a continuous potential function on any chart in P^k, there exists a closed positive current T^l of bidegree (l,l) such that $\pi^*T^l = (dd^cG)^l$ for $l = 1, ..., k$.

Definition 1.1. For $l = 1, ..., k$, $J_l = supp(T^l)$ is called the l-th Julia set of f. Setting $F_l = P^k \setminus J_l$, we call it the l-th Fatou set of f.

Fornaess-Sibony showed the following([FS4]):

(1.1) J_l is a nonempty totally f-invariant set .

(1.2) We call simply $J_1 = J$ and $F_1 = F$ the Julia set of f and the Fatou set of f respectively.

(1.3) F is a domain of holomorpy, furthermore, for $l = 1, ..., k$, F_l is $(k-l)$-pseudoconvex.

Also Ueda showed (see[U]);

(1.4) The Fatou set F is Kobayashi hyperbolic.

This means that each component of the Fatou set of f is Kobayashi hyperbolic.

2. Limit maps and Fatou components For a map $f \in H_d(P^2)$, let Ω be a forward invariant Fatou component of f. A map $\varphi : P^2 \to P^2$ is said to be a limit map on Ω if there exists a subsequence $\{f^{n_j}\}$ which locally uniformly converge to φ in Ω. Let $L(\Omega)$ denote the set of all limit maps on Ω. It is clear that $L(\Omega)$ is a commutative semigroup. If $L(\Omega)$ contains the identity map, we call Ω a rotation domain. Then $L(\Omega)$ is a group and f is holomorphic automorphim of Ω. In 1 dimensional case, the rotation domain is Siegel disc or Herman ring. Furthermore if $L(\Omega)$ contains only a constant map then Ω is an attractive component or parabolic component, and if $L(\Omega)$ contains a nonconstant map then Ω is a rotation domain. We try to extend these facts to the 2 dimensional case.

When $\{f^n\}$ is nonrecurrent on Ω, that is, for any compact set $K \subset \Omega$, $f^n(K) \cap K = \emptyset$ for all but finite set of n, we write $f^n \to \partial\Omega$, where $\partial\Omega$ is the boundary of Ω. Also if a limit map is a constant map with value ζ, we denote it by ζ^*, i.e. $\zeta^*(z) = \zeta$ on Ω.

We may assume a forward invariant component Ω of f is in the chart $(z_0 \neq 0)$ of P^2 and (z_1, z_2) is an inhomogeneous coordinate there.

We start with the result of Bedford [Bed]. Let f, Ω be the same as the above.

Lemma 2.1. For any map $\varphi \in L(\Omega)$ with $\varphi(\Omega) \subset \bar{\Omega}$, there exists a complex submanifold V in Ω, a holomorphic retraction R and a map $\phi \in Aut(V)$ such that $\varphi = \phi R$. Furthermore $\dim V$ depends only on f.

When $dimV = 0$, $L(\Omega)$ contains only constant map.

Theorem 2.2. Let Ω be a forward invariant Fatou component of f. Suppose $L(\Omega)$ contains only constant maps. Then there is exactly one point $\zeta \in \bar{\Omega}$, which is attractive or parabolic fixed point of f and $f^n \to \zeta^*$ locally uniformly on Ω.

V is the set of fixed points of R. We assume $\varphi(\Omega) \subset \Omega$. If $\dim V = 2$, then $V = \Omega$ and $R = I$ (the identity map on Ω), thus $\varphi \in Aut(\Omega)$ and $f \in Aut(\Omega)$. Therefore Ω is a rotation domain.

Next we consider the case $\dim V = 1$. If V is simply connected, then V is conformally equivalent to the disc Δ and each fiber $R^{-1}(z)$ for $z \in V$ is one of the Fatou components of dimension 1. If V is multiple connected, V must be doubly connected, since there is an analytic automorphism ϕ on V. Hence we have the following theorem:

Theorem 2.3. Let f belong to $H_d(P^2)$ and Ω be a forward invariant Fatou component of f. Suppose Ω is recurrent and $L(\Omega)$ contains a nonconstant map φ such that $\varphi(\Omega) \subset \Omega$. Then either

(1) Ω is a rotation domain, or

(2) there exist a complex submanifold V in Ω of dimension 1 and a holomorphic retraction $R : \Omega \to V$ such that $\varphi(\Omega) = V$ and $\varphi = \phi \circ R$ with $\phi \in Aut(V)$.

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