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<th>Title</th>
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On the Local Connectivity of the Boundary of Unbounded Periodic Fatou Components of Transcendental Functions

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1 Definitions, Notations and Results

Let $f$ be a transcendental entire function, $F_f \subset \mathbb{C}$ the Fatou set of $f$ and $J_f := \mathbb{C} \setminus F_f$ the Julia set of $f$. We call a connected component of $F_f$ a Fatou component. It is well known that a Fatou component $U$ is either eventually periodic (i.e. there exists a $k_0$ such that $f^{k_0}(U)$ is periodic) or a wandering domain (i.e. $f^m(U) \cap f^n(U) = \emptyset$ for every $m,n \in \mathbb{N}$ ($m \neq n$)) and if it is periodic (i.e. there exists an $n_0 \in \mathbb{N}$ with $f^{n_0}(U) \subseteq U$), there are four possibilities;

1. There exists a point $z_0 \in U$ with $f^{n_0}(z_0) = z_0$ and $|(f^{n_0})'(z_0)| < 1$ and every point $z \in U$ satisfies $f^{n_0 k}(z) \to z_0$ as $k \to \infty$. The point $z_0$ is called an attracting periodic point and the domain $U$ is called an attractive basin.

2. There exists a point $z_0 \in \partial U$ with $f^{n_0}(z_0) = z_0$ (it is possible that $f^{n_1}(z_0) = z_0$ for an $n_1$ with $n_1|n_0$) and $(f^{n_0})'(z_0) = 1$ and every point $z \in U$ satisfies $f^{n_0 k}(z) \to z_0$ as $k \to \infty$. The point $z_0$ is called a parabolic periodic point and the domain $U$ is called a parabolic basin.

3. There exists a point $z_0 \in U$ with $f^{n_0}(z_0) = z_0$ and $(f^{n_0})'(z_0) = e^{2\pi i \theta}$ ($\theta \in \mathbb{R} \setminus \mathbb{Q}$) and $f^{n_0}|U$ is conjugate to an irrational rotation of a unit disk.
The domain $U$ is called a Siegel disk.

4. For every $z \in U$, $f^{n_0 k}(z) \to \infty$ as $k \to \infty$. The domain $U$ is called a Baker domain.

1. an attractive basin

2. a parabolic basin

3. a Siegel disk

4. a Baker domain

Figure 1. Invariant Fatou components

The natural number $n_0$ is called the period of a component $U$. Figure 1 shows these periodic Fatou components schematically in the case that its period $n_0$ is equal to one. In particular in this case, $U$ is called an invariant component. By definition Baker domains are unbounded but attractive basins, parabolic bains and even Siegel disks can be unbounded as follows:

Example 1. Consider the exponential family $E_\lambda(z) := \lambda e^z$.

1. If $E_\lambda$ has an attracting fixed point, then its basin is always unbounded.
(2) If $\lambda = \frac{1}{e}$, then it is easy to see that it has an unbounded parabolic basin.

(3) If there is a Siegel disk on which $E_{\lambda}$ is conjugate to a irrational rotation $z \mapsto e^{2\pi i \theta}z$ and $\theta$ satisfies the Diophantine condition, then it is unbounded ([H]).

So throughout this paper we assume that $f$ has an unbounded periodic Fatou component $U$ with period $n_0$.

Then when is $\partial U \subset \mathbb{C}$ (or $\partial U \cup \{\infty\} \subset \hat{\mathbb{C}}$) locally connected? For this problem we have the following result:

**Theorem A.** If $U$ is either

(i) an attractive basin,  
(ii) a parabolic basin,  
(iii) a Siegel disk, or  
(iv) a Baker domain on which $f^{n_0}|U$ is a $d$ to 1 mapping $(2 \leq d < \infty)$,
then $\partial U \cup \{\infty\} \subset \hat{\mathbb{C}}$ is not locally connected. Also $\partial U \subset \mathbb{C}$ is not locally connected.

The local connectivity of $\partial U$ is intimately related to the local connectivity of $J_f$ by the following proposition:

**Proposition 2.** ([W]) A compact set $K \subset \hat{\mathbb{C}}$ is locally connected if and only if the following two conditions are satisfied:

1. The boundary of each connected component of $K^c$ (:= complement of $K$) is locally connected.

2. For any $\epsilon > 0$ the number of connected components of $K^c$ with diameter (with respect to the spherical metric) greater than $\epsilon$ is finite.

From this proposition and Theorem A we can prove the following result:

**Theorem B.** Assume that a transcendental entire function $f$ has an unbounded periodic Fatou component $U$ with period $n_0$. If $U$ is either

(i) an attractive basin,  
(ii) a parabolic basin,  
(iii) a Siegel disk, or  
(iv) a Baker domain on which $f^{n_0}|U$ is a $d$ to 1 mapping $(1 \leq d < \infty)$,
then $J_f \cup \{\infty\} \subset \hat{\mathbb{C}}$ is not locally connected. Also $J_f \subset \mathbb{C}$ is not locally connected.
2 Outline of the proof of Theorem A

In what follows we shall assume that $n_0 = 1$, that is, $U$ is an invariant component for simplicity. In general cases similar arguments are valid if we consider $f^{n_0}$ instead of $f$.

Since $U$ is an unbounded component, it is simply connected ([EL]). So let $\varphi : \mathbb{D}(:= \{|z| < 1\}) \rightarrow U$ be a Riemann map of $U$. Then the following theorem is well known:

**Theorem 3 (Carathéodory).** Let $U \subset \hat{\mathbb{C}}$ be a simply connected domain.

1. There is one to one correspondence between $\partial \mathbb{D}$ and the set of prime ends: $e^{i\theta} \mapsto$ a prime end $P(e^{i\theta})$ of $U$.
2. $I(P(e^{i\theta}))$ be the impression of a prime end $P(e^{i\theta})$. Then the following three conditions are equivalent:
   1. The Riemann map $\varphi : \mathbb{D} \rightarrow U$ extends to a continuous map $\overline{\varphi} : \mathbb{D} := \{|z| \leq 1\} \rightarrow \overline{U}$.
   2. $\partial U$ is locally connected.
   3. For any prime end $P(e^{i\theta})$ the impression $I(P(e^{i\theta}))$ is reduced to a single point.

**Remark 4.** (1) For the definitions of the prime end, its impression and the proof of Theorem 3, see, for example, [CL].

2. Since $U \subset \mathbb{C}$ is unbounded in our case, we should write $\partial U \cup \{\infty\} \subset \hat{\mathbb{C}}$ in the above theorem.

We also use the following result:

**Theorem 5 ([BaW]).** Let $f$ and $U$ be as above. Suppose that $U$ is not a Baker domain then every impression $I(P(e^{i\theta}))$ of a prime end $P(e^{i\theta})$ of $U$ contains the point $\infty$.

First let us consider the case (i), (ii) and (iii). Suppose that $\partial U \cup \{\infty\} \subset \hat{\mathbb{C}}$ is locally connected. Then by Theorem 3, the Riemann map $\varphi$ extends to a continuous map $\overline{\varphi}$ and moreover by Theorem 5 we have $\overline{\varphi}|\partial \mathbb{D} \equiv \infty$, which contradicts the following fact:
Proposition 6 ([CL]). For almost every point $e^{i\theta} \in \partial \mathbb{D}$ the radial limit
\[
\lim_{r \nearrow 1} \varphi(re^{i\theta})
\]
exists and is nonconstant. Moreover for each $p \in \partial U$ the capacity of the set
\[
\{e^{i\theta} \mid \lim_{r \nearrow 1} \varphi(re^{i\theta}) = p\} \subset \partial \mathbb{D}
\]
is equal to zero.

This completes the proof for the case (i), (ii) and (iii).

In the case (iv), define
\[
I_{\infty} := \{e^{i\theta} \mid I(P(e^{i\theta})) \ni \infty\} \subset \partial \mathbb{D}, \quad V := \partial \mathbb{D} \setminus I_{\infty}.
\]
Then since $U$ is unbounded, we have $I_{\infty} \neq \emptyset$. It is easy to see that $V$ is open in $\partial \mathbb{D}$ and $V \neq \partial \mathbb{D}$. Consider the following commutative diagram:

\[
\begin{array}{ccc}
U & \xrightarrow{f} & U \\
\uparrow \varphi & & \uparrow \varphi \\
\mathbb{D} & \xrightarrow{g := \varphi^{-1} \circ f \circ \varphi} & \mathbb{D}
\end{array}
\]

By the assumption that $f|U$ is a $d$ to 1 mapping ($2 \leq d < \infty$), $g := \varphi^{-1} \circ f \circ \varphi$ is a finite Blaschke product. It can be shown that $g(V) \subseteq V$. On the other hand we can consider the Julia set $J_g$ and it is easy to see that $J_g \subset \partial \mathbb{D}$. Suppose that $V \cap J_g \neq \emptyset$. Then from an elementary property of Julia sets of rational maps, we have $g^n(V) = \partial \mathbb{D}$ for sufficiently large $n \in \mathbb{N}$ and since $g(V) \subseteq V$, it follows that $V = \partial \mathbb{D}$, which contradicts the fact that $V \neq \partial \mathbb{D}$. Consequently we have $V \cap J_g = \emptyset$, that is, $J_g \subset I_{\infty}$. Suppose here that $\partial U \cup \{\infty\} \subset \hat{\mathbb{C}}$ is locally connected. Then from Theorem 3 $\varphi$ has a continuous extension $\overline{\varphi}$ and we must have $\overline{\varphi} \equiv \infty$ on the set $I_{\infty}$. In particular $\overline{\varphi} \equiv \infty$ on $J_g$. But on the contrary since the Hausdorff dimension of the Julia set of a rational map is always positive ([Bea, Theorem 10.3.1]), $J_g$ has positive Hausdorff dimension. In particular its capacity is positive. Then it follows that the set
\[
\{e^{i\theta} \mid \lim_{r \nearrow 1} \varphi(re^{i\theta}) = \infty\}
\]
has positive capacity, which contradicts Proposition 6. This completes the proof for the case (iv).
The non-local connectivity of \( \partial U \subset \mathbb{C} \) follows from the following proposition, since \( U \) is simply connected, \( \partial U \cup \{\infty\} \) is closed and connected.

**Proposition C.** Let \( K \subset \hat{\mathbb{C}} \) be a closed connected subset and \( p \in K \). If \( K \) is not locally connected, then \( K \setminus \{p\} \) is also not locally connected.

We shall omit the proof of this proposition. \( \square \)

**Remark 7.** It is known that the boundary of a Baker domain \( U \) on which \( f \) is 1 to 1 mapping (i.e. univalent) can be a Jordan curve (i.e. \( \partial U \cup \{\infty\} \subset \hat{\mathbb{C}} \) is a Jordan curve and \( \partial U \subset \mathbb{C} \) is a Jordan arc). The function \( f(z) := 2 - \log 2 + 2z - e^{z} \) is such an example ([Ber, Theorem 2]). In particular in this case both \( \partial U \cup \{\infty\} \subset \hat{\mathbb{C}} \) and \( \partial U \subset \mathbb{C} \) are locally connected. So we cannot drop the assumption \( 2 \leq d \) in Theorem A. It is also known that if \( \partial U \cup \{\infty\} \) is a Jordan curve in \( \hat{\mathbb{C}} \), then \( f|U \) is univalent ([BaW, Theorem 4]).

### 3 Proof of Theorem B

By definition \( J_{f} \cup \{\infty\} \) is a compact subset of \( \hat{\mathbb{C}} \) so we can apply Proposition 2. In the case (i), (ii) and (iii), the set \( \partial U \cup \{\infty\} \subset \hat{\mathbb{C}} \) is not locally connected from Theorem A. So by Proposition 2 \( J_{f} \cup \{\infty\} \) is not locally connected.

Next let us consider the case (iv). If \( 2 \leq d \), the proof is completely the same as the previous cases. If \( d = 1 \), take a point \( w_{0} \neq \infty \in \partial U \cup \{\infty\} \) and \( z_{0} \in U \). Then from an elementary property of complex dynamical systems there exist \( n_{k} \in \mathbb{N} \) with \( n_{k} / \infty \) and \( z_{n_{k}} \in f^{-n_{k}}(z_{0}) \) with \( z_{n_{k}} \rightarrow w_{0} \). Since \( f|U \) is univalent we can take \( z_{0}, \{z_{n_{k}}\} \) and \( w_{0} \) satisfying \( z_{n_{k}} \notin U \). Let \( U_{n_{k}} \) be the Fatou component containing \( z_{n_{k}} \). Then it follows that \( U_{n_{k}} \) are mutually disjoint and also we have \( U_{n_{k}} \cap U = \emptyset \). Since \( z_{n_{k}} \rightarrow w_{0}, z_{n_{k}} \in U_{n_{k}} \) and \( U_{n_{k}} \) is unbounded, it follows that the condition 2 in Proposition 2 is not satisfied. Hence again \( J_{f} \cup \{\infty\} \subset \hat{\mathbb{C}} \) is not locally connected.

For the non-local connectivity of \( J_{f} \subset \mathbb{C} \) itself, we can again apply Proposition C, since \( J_{f} \cup \{\infty\} \subset \hat{\mathbb{C}} \) is compact and connected in this case ([K, Corollary 1]). This completes the proof. \( \square \)
References


