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YAMAGISHI, YOSHIKAZU

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SPIRAL CONVERGENCE OF SOR DURAND-KERNER'S METHOD

YAMAGISHI YOSHIKAZU

(Department of Applied Mathematics and Informatics, Ryukoku University)

ABSTRACT. It is proved that SOR Durand-Kerner's method has spiral trajectories of approximants toward multiple roots.

1. INTRODUCTION

Durand-Kerner’s method is an iterative algorithm for finding zeros of a monic complex polynomial \( p(x) \) of degree \( d > 1 \). It was proposed by Weierstrass[8], Durand[2], Dochev[1], Kerner[5] and Prešić[6]. Let \( \vec{x} = (x_0, \ldots, x_{d-1}) \) be a point in the complex Euclidean space \( \mathbb{C}^d \). Let \( I_k = \{0,1,\ldots,k-1\} \), \( I_k' = \{1,\ldots,k-1\} \) be finite sets of indices. Let \( \pi_i : \mathbb{C}_x^d \to \mathbb{C}, \pi_i \vec{x} = x_i, i \in I_d \), be the projection to the \( i \)-th coordinate. If \( f \) is a self-map of \( \mathbb{C}_x^d \), we denote the iteration of \( f \) by \( f^k : f^0(\vec{x}) = \vec{x}, f^{k+1}(\vec{x}) = f(f^k(\vec{x})) \).

In this paper we consider Durand-Kerner’s method with ‘successive-over-relaxation’. It can be defined in several ways. SOR Durand-Kerner’s method is:

- the iteration of the rational mapping

\[
\sigma f : \mathbb{C}_x^d \to \mathbb{C}_x^d
\]

where \( f : \mathbb{C}_x^d \to \mathbb{C}_x^d \) is the rational function defined by

\[
\pi_if = \begin{cases} 
  x_0 - \lambda \frac{p(x_0)}{(x_0-x_1)\cdots(x_0-x_{d-1})}, & i = 0, \\
  x_i, & i \in I_d'
\end{cases}
\]

and \( \sigma : \mathbb{C}_x^d \to \mathbb{C}_x^d \) is the linear automorphism

\[
\sigma(x_0,\ldots,x_{d-1}) = (x_1,\ldots,x_{d-1},x_0).
\]

- the iteration of the mapping

\[
F = f_{d-1} \cdots f_0 : \mathbb{C}_x^d \to \mathbb{C}_x^d
\]

where \( f_i = \sigma^{-i}f\sigma^i, i \in I_d \).

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the recursive formula
\[ x_{i+d} = x_i - \frac{p(x_i)}{(x_i - x_{i+1})\cdots(x_i - x_{i+d-1})} \]
with initial values \(x_0, \ldots, x_{d-1}\) that generates the sequence of complex numbers \(\{x_i\}_{i=0,1,\ldots}\).

**Remark 1.** The constant \(\lambda \in \mathbb{R}\) is called the relaxation parameter. The case with \(\lambda = 1\) is especially called Gauss-Seidel Durand-Kerner's method.

**Remark 2.** We have \(F = (\sigma f)^d\) because \(\sigma^{-d+1} = \sigma\). Each \(f_i\) leaves \(x_j\) invariant if \(j \neq i\):
\[ \pi_j f_i = \pi_j. \]
Let \(r_i, i \in I_\nu\), be the roots of \(p(x)\) with multiplicities \(m_i\), so that \(\sum_{i=0}^{\nu-1} m_i = d\). Let \(R\) the set of mappings \(\rho : I_d \rightarrow I_\nu\) such that \(\#\rho^{-1}(i) = m_i\) for \(i \in I_\nu\). If \(\rho \in R\) is given, let \(\theta_i : I_{m_i} \rightarrow I_d, i \in I_\nu\), be the injective mapping such that image(\(\theta_i\)) = \(\rho^{-1}(i)\), and \(\theta_i(j) < \theta_i(k)\) for \(j < k\).

Let
\[ \ell_d(\gamma) = (1 - \gamma)(1 - \gamma^2)\cdots(1 - \gamma^d), \quad \gamma \in \mathbb{C}. \]
Then for each primitive \(d\)-th root of unity \(\zeta\), there exists a function \(\lambda \mapsto \gamma(\lambda)\) defined for \(0 < \lambda < \epsilon\) with \(\epsilon\) small such that \(\ell_d(\gamma \zeta) = \ell_d(\gamma)\), \(\lim_{\lambda \rightarrow 0} \gamma(\lambda) = \zeta\) and
\[ \gamma(\lambda) = \zeta - \frac{\zeta}{d^2} \lambda + O(|\lambda|^2) \quad \text{as} \quad \lambda \rightarrow 0. \]

We will prove the following theorems.

**Theorem 1.** Let \(d \geq 2, 0 < \lambda < \epsilon\) with \(\epsilon\) small, \(\zeta\) a primitive \(d\)-th root of unity, and \(\gamma(\lambda)\) the function defined as above. There exists a complex manifold \(W \subset \mathbb{C}^d\) holomorphically isomorphic to the punctured disk \(\mathbb{D}^* = \{z \in \mathbb{C} | 0 < |z| < 1\}\) such that each \(\tilde{x}_0 \in W\) has a backward orbit \(\tilde{x}_{-n} \in W, -n \leq 0, \) with \(\sigma f(\tilde{x}_{-(n+1)}) = \tilde{x}_{-n}\), \(\lim_{n \rightarrow -\infty} \pi_0 \tilde{x}_{-n} = \infty,\) and
\[ \lim_{n \rightarrow -\infty} \frac{\pi_0 \tilde{x}_{-n}}{\pi_0 \tilde{x}_{-(n+1)}} = \gamma(\lambda). \]

**Remark.** Existence of the spiral trajectory \(\{\pi_0 \tilde{x}_{-nd}\}_{n=0,-1,\ldots}\) of 'period' \(d\) was observed by Kanno et al. [4].

**Theorem 2.** Let \(d \geq 2, 0 < \lambda < \epsilon\) with \(\epsilon\) small, \(\rho \in R\), \(\zeta_i\) a primitive \(m_i\)-th root of unity, \(\theta_i\) and \(\gamma(\lambda)\) the functions defined as above. Denote by \(\gamma(\lambda) = \gamma(\lambda)\). There is an open set \(U \subset \mathbb{C}^d\) containing the point \(\tilde{x}_\rho = (r_\rho(0), \ldots, r_\rho(d-1))\) on its boundary, such that for each initial value \(\tilde{x} \in U\) we have
\[ \lim_{n \rightarrow +\infty} F^n(\tilde{x}) = \tilde{x}_\rho. \]
and, for each \( i \in I_{\nu} \),

\[
\lim_{n \to \infty} \frac{\pi_{\theta_i(j)} F^n(\vec{x}) - r_i}{\pi_{\theta_i(j-1)} F^n(\vec{x}) - r_i} = \gamma_i(\lambda), \quad j \in I'_{m_i},
\]

\[
\lim_{n \to \infty} \frac{\pi_{\theta_i(0)} F^n(\vec{x}) - r_i}{\pi_{\theta_i(m_i-1)} F^n(\vec{x}) - r_i} = \gamma_i(\lambda).
\]

Our argument is based on the Unstable Manifold Theorem and the deformation of the phase space \( \mathbb{C}^d_{\sigma} \). In section 2 we recall the dynamics of \( \sigma f \) in the simplest but important case \( p(x) = x^d \) that was studied in [9]. In section 3 we study the dynamics at infinity and prove Theorem 1. In section 4 we study the dynamics close to the root \( r_{\rho} \) and prove Theorem 2.

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2. THE CASE \( p(x) = x^d \)

In [9], we proved the Theorems above in the case \( p(x) = x^d \). We take the coordinate change \( \vec{y} = \chi(\vec{x}) \) defined by

\[
y_0 = x_0
\]

\[
y_i = x_i / x_{i-1}, \quad i \in I'_{d}.
\]

The rational map \( g = \chi \sigma f \chi^{-1} : \mathbb{C}^d_{\sigma} \rightarrow \mathbb{C}^d_{\sigma} \) is written by

\[
\pi_i g(\vec{y}) = \begin{cases} y_0 y_1, & i = 0, \\ y_{i+1}, & 1 \leq i \leq d - 2, \\ \frac{1}{y_1 \cdots y_{d-1}} \left(1 - \lambda \frac{1}{(1-y_1) \cdots (1-y_{d-1})}\right), & i = d - 1. \end{cases}
\]

The origin of \( \mathbb{C}^d_{\sigma} \) is blown-up to the hyperplane

\[
\alpha = \{ \vec{y} \in \mathbb{C}^d_{\sigma} \mid y_0 = 0 \}
\]

which is forward invariant under \( g \). A point \( \vec{y} \in \mathbb{C}^d_{\sigma} \) is fixed under \( g \) if and only if \( \vec{y} = \vec{\gamma} \) where \( \vec{\gamma} = (0, \gamma, \ldots, \gamma) \in \alpha \) and \( \gamma \) is a root of the equation \( \lambda = \ell_d(\gamma) \).

**Lemma.** Let \( d \geq 2, p(x) = x^d, \) and \( 0 < \lambda < \epsilon \) with \( \epsilon \) small. A point \( \vec{y} \in \mathbb{C}^d_{\sigma} \) is a stable fixed point under \( g \) if and only if \( \vec{y} = \vec{\gamma}_{\zeta}(\lambda) \in \alpha \) where \( \zeta \) is a primitive \( d \)-th root of unity and \( \vec{\gamma}_{\zeta}(\lambda) = (0, \gamma_{\zeta}(\lambda), \ldots, \gamma_{\zeta}(\lambda)) \).

The multipliers of \( \vec{\gamma}_{\zeta}(\lambda) \) under \( g|\alpha \) are written by \( t_k, k \in I'_{d} \), where

\[
t_k = \zeta^k - \frac{k \zeta^k}{d^2} \lambda + O(|\lambda|^2) \quad \text{as} \quad \lambda \to 0.
\]

**Proof.** [9].
3. DYNAMICS AT INFINITY

Here we prove Theorem 1. Let \( p(x, x') = x'^d p(x/x') \) be the homogeneous polynomial of degree \( d \) of two variables \( x, x' \). We take the coordinate change \( \tilde{z} = \chi(\vec{x}) \) defined by

\[
\begin{align*}
  z_0 &= 1/x_0 \\
  z_i &= x_i/x_{i-1}, & i &\in I_d'.
\end{align*}
\]

The rational map \( h = \chi \sigma f \chi^{-1} : \mathbb{C}_z^d \to \mathbb{C}_z^d \) is written by

\[
\pi_i h(\tilde{z}) = \begin{cases} 
  z_0/z_1, & i = 0, \\
  z_{i+1}, & 1 \leq i \leq d-2, \\
  \frac{1}{z_1 \cdots z_{d-1}} \left( 1 - \lambda \frac{p(1,z_0)}{(1-z_1) \cdots (1-z_1 \cdots z_{d-1})} \right), & i = d-1.
\end{cases}
\]

The hyperplane \( \beta \subset \mathbb{C}_z^d \) defined by \( z_0 = 0 \) corresponds to the set of 'points at infinity' of \( \mathbb{C}_z^d \), and is forward invariant under \( h \).

Since \( p(1,0) = 1 \), we have \( h|\beta = g|\alpha \) if we identify \( \beta \subset \mathbb{C}_z^d \) with \( \alpha \subset \mathbb{C}_z^d \). For each primitive \( d \)-th root of unity \( \zeta \), the point \( \gamma_\zeta(\lambda) \in \beta \) is a stable fixed point of \( h|\beta \), but is a saddle of \( h \) with a multiplier \( 1/\gamma_\zeta(\lambda) \) and the eigenvector tangent to the complex line

\[
L_\zeta = \{ (y, \gamma_\zeta(\lambda), \ldots, \gamma_\zeta(\lambda)) \mid y \in \mathbb{C} \}.
\]

Thus it has a holomorphic unstable manifold \( V \) of complex dimension 1 tangent to \( L_\zeta \) (by an argument of Hirsch-Pugh-Shub [3] adapted to the holomorphic category). We take \( W = \chi^{-1}(V - \{ \gamma_\zeta(\lambda) \}) \) and all the assertions in Theorem 1 follows.

4. DYNAMICS AROUND THE ROOT

Here we prove Theorem 2. We denote the rational map \( g \) defined in (1) by \( g_d \), and the hyperplane \( \alpha \) defined in (2) by \( \alpha_d \).

Let \( \sigma_i : \mathbb{C}^d \to \mathbb{C}^d, i \in I_\nu \), be the linear automorphism defined by

\[
\pi_k \sigma_i(\vec{x}) = x_k, \quad k \in I_d \text{ with } \rho(k) \neq i,
\]

and

\[
\pi_{\theta_i(j)} \sigma_i(\vec{x}) = \begin{cases} 
  x_{\theta_i(j+1)}, & 0 \leq j \leq m_i - 2 \\
  x_{\theta_i(0)}, & j = m_i - 1.
\end{cases}
\]

Let \( \hat{f}_i = f_{\theta_i(0)} \) for \( i \in I_\nu \). It is easily seen that

\[
\begin{align*}
  \sigma_i^{m_i} &= id, \\
  f_{\theta_i(j)} &= \sigma_i^{-j} \hat{f}_i \sigma_i^j \text{ for } i \in I_\nu, j \in I_{m_i}, \\
  \sigma_i f_k &= f_k \sigma_i \text{ if } i \neq \rho(k), \\
  \sigma_i \hat{f}_j &= \hat{f}_j \sigma_i \text{ if } i \neq j.
\end{align*}
\]
Thus we can re-factor $F = f_{d-1} \cdots f_{0}$ by the composite of $\sigma_i \hat{f}_i$, $i \in I_{\nu}$, as

\begin{equation}
F = \sigma_{\rho(d-1)} \hat{f}_{\rho(d-1)} \cdots \sigma_{\rho(0)} \hat{f}_{\rho(0)}.
\end{equation}

Denote by $(z_{i,0}, \ldots, z_{i,m_i-1})$ a point in $\mathbb{C}_{z_i}^{m_i}$, $i \in I_{\nu}$, and let $M = \mathbb{C}_{z_0}^{m_0} \times \cdots \times \mathbb{C}_{z_{\nu-1}}^{m_{\nu-1}}$. Let $\pi_{i,j} : M \to \mathbb{C}$, $\pi_{i,j}(\mathbf{z}) = z_{i,j}$, $i \in I_{\nu}$, $j \in I_{m_i}$, be the projection to the $(i,j)$-th component. Let $\chi_i : \mathbb{C}^d \to \mathbb{C}_{z_i}^{m_i}$, $i \in I_{\nu}$, be the rational map

$\chi_i = \left\{ \begin{array}{ll}
x_{\theta_i(0)} - r_i, & j = 0, \\
(\x_{\theta_i(j)} - r_i)/(\x_{\theta_i(j-1)} - r_i), & j \in I_{m_i}.
\end{array} \right.$

We take the coordinate change

$\chi = \chi_0 \times \cdots \times \chi_{\nu-1} : \mathbb{C}^d \to M.$

The rational mapping $h_i = \chi \sigma_i \hat{f}_i \chi^{-1} : M \to M$ is written by

\begin{equation}
\pi_{k,j} h_i(\mathbf{z}) = z_{k,j}, \quad k \in I_{\nu}, j \in I_{m_k}, \text{ with } k \neq i
\end{equation}

and

\begin{equation}
\pi_{i,j} h_i(\mathbf{z}) = \left\{ \begin{array}{ll}
z_{i,0} z_{i,1}, & j = 0, \\
z_{i,j+1}, & 1 \leq j \leq m_i - 2, \\
\frac{1}{z_{i,1} \cdots z_{i,m_i-1}} \left( 1 - \lambda H_i(\mathbf{z}) / \prod_{k=1}^{m_i-1} (1 - z_{i,1} \cdots z_{i,k}) \right), & j = m_i - 1
\end{array} \right.
\end{equation}

where

\begin{equation}
H_i(\mathbf{z}) = \frac{\prod_{k \in I_{\nu}, k \neq i} (r_i - r_k + z_{i,0})^{m_k}}{\prod_{k \in I_{\nu}, k \neq i} (r_i - r_k + z_{i,0} - z_{k,0} \cdots z_{k,1})^{m_k}}.
\end{equation}

By (3) we have

$\chi F \chi^{-1} = h_{\rho(d-1)} \cdots h_{\rho(0)}.$

Let $\beta_i \subset \mathbb{C}_{z_i}^{m_i}$ be the hyperplane defined by $z_{i,0} = 0$. The product $B = \beta_0 \times \cdots \times \beta_{\nu-1} \subset M$ corresponds under $\chi$ to the point $\vec{\gamma}_0 \in \mathbb{C}_x^d$, and is forward invariant under every $h_i$, $i \in I_{\nu}$. Since $H_i(\mathbf{z}) = 1$ on $B$, $i \in I_{\nu}$, we have

$h_i|B = id \times \cdots \times (g_{m_i} | \alpha_{m_i}) \times \cdots \times id, \quad i \in I_{\nu},$

if we identify $\beta_i \subset \mathbb{C}_{z_i}^{m_i}$ with $\alpha_{m_i} \subset \mathbb{C}_{y}^{m_i}$. Note that $h_i$'s are commutative on $B$: $h_i h_j|B = h_j h_i|B$, $i, j \in I_{\nu}$.

A point $\vec{z} \in B$ is fixed under every $h_i|B$ if and only if $\vec{z} = \vec{\gamma}_0 \times \cdots \times \vec{\gamma}_{\nu-1}$ where $\vec{\gamma}_i = (0, \gamma_i, \ldots, \gamma_i) \in \beta_i$ and $\gamma_i$ is a root of the equation $\lambda = \ell_{m_i} (\gamma)$. A point $\vec{z} \in B$ is a stable fixed point of every $h_i|B$, $i \in I_{\nu}$, if and only if $\vec{z} = \vec{\gamma}_0 \times \cdots \times \vec{\gamma}_{\nu-1}$ where $\zeta_i$, $i \in I_{\nu}$, is a primitive $m_i$-th root of unity. Such fixed point $\vec{\gamma}_0 \times \cdots \times \vec{\gamma}_{\nu-1}$ is also a stable fixed point of every $h_i$, $i \in I_{\nu}$, with a multiplier $\gamma_{\zeta_i}(\lambda)$ and the eigenvector tangent to the complex line $\vec{\gamma}_0 \times \cdots \times L_{\zeta_i} \times \cdots \times \vec{\gamma}_{\nu-1}$. Thus it has an attracting region $V$. We take $U = \chi^{-1}(V - \{ \vec{\gamma}_0 \times \cdots \times \vec{\gamma}_{\nu-1} \})$ and all the assertions in Theorem 2 follows.
5. DISCUSSION

In section 4, we only studied the points $z \in B$ that is fixed under 'every' $h_i|B$, $i \in I_\nu$. It is desirable that our argument be extended to the stable fixed points of the mapping $h_{\nu-1}^{m_{\nu-1}} \cdots h_0^{m_0}|B$ which will also have the spiral trajectories in the space $C_d$.

REFERENCES