<table>
<thead>
<tr>
<th>Title</th>
<th>Moduli spaces of maps with two critical points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>FUJIMURA, Masayo</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1997), 988: 57-66</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/61054">http://hdl.handle.net/2433/61054</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Moduli spaces of maps with two critical points

Masayo FUJIMURA
Dept. of Math., College of Sci. and Tech., Nihon Univ.

Abstract
We give directly a defining equation of the symmetry locus, a singular part of the moduli space of the quadratic rational maps. We show a characterization of this locus. We can expand analogous discussion for the cubic polynomials and give a "chart" making a comparison between properties of these moduli spaces in Appendix A. Moreover, we apply these method to the polynomials of degree $n$, and give some conjectures.

1 Quadratic rational maps
1.1 Moduli space of quadratic rational maps

Let $\overline{\mathbb{C}}$ be the Riemann sphere and $\text{Rat}_2(\mathbb{C})$ the space of all quadratic rational maps from $\overline{\mathbb{C}}$ to itself. The group $\text{PSL}_2(\mathbb{C})$ of Möbius transformations acts on the space $\text{Rat}_2(\mathbb{C})$ by conjugation,

$$g \circ f \circ g^{-1} \in \text{Rat}_2(\mathbb{C}) \quad \text{for} \quad g \in \text{PSL}_2(\mathbb{C}), \ f \in \text{Rat}_2(\mathbb{C}).$$

Two maps $f_1, f_2 \in \text{Rat}_2(\mathbb{C})$ are holomorphically conjugate, denoted by $f_1 \sim f_2$, if and only if there exists $g \in \text{PSL}_2(\mathbb{C})$ with $g \circ f_1 \circ g^{-1} = f_2$. The quotient space of $\text{Rat}_2(\mathbb{C})$ under this action will be denoted by $\mathcal{M}_2(\mathbb{C})$, and called the moduli space of holomorphic conjugacy classes $(f)$ of quadratic rational maps $f$.

Milnor introduced coordinates in $\mathcal{M}_2(\mathbb{C})$ as follows; for each $f \in \text{Rat}_2(\mathbb{C})$, let $z_1, z_2, z_3$ be the fixed points of $f$ and $\mu_i$ the multipliers of $z_i$; $\mu_i = f'(z_i) \ (1 \leq i \leq 3)$. Consider the elementary symmetric functions of the three multipliers,

$$\sigma_1 = \mu_1 + \mu_2 + \mu_3, \quad \sigma_2 = \mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1, \quad \sigma_3 = \mu_1 \mu_2 \mu_3.$$

These three multipliers determine $f$ up to holomorphic conjugacy, and are subject only to the restriction that

$$\sigma_3 = \sigma_1 - 2.$$
Hence the moduli space $\mathcal{M}_2(\mathbb{C})$ is canonically isomorphic to $\mathbb{C}^2$ with coordinates $\sigma_1$ and $\sigma_2$ (Lemma 3.1 in [Mil93]).

For each $\mu \in \mathbb{C}$ let $\text{Per}_n(\mu)$ be the set of all conjugacy classes $\langle f \rangle$ of maps $f$ which having a periodic point of period $n$ and multiplier $\mu$.

Each of $\text{Per}_1(\mu)$ and $\text{Per}_2(\mu)$ forms a straight lines as follows:

$$\text{Per}_1(\mu) = \{ \langle f \rangle \in \mathcal{M}_2(\mathbb{C}) ; \sigma_2 = (\mu + \mu^{-1})\sigma_1 - (\mu^2 + 2\mu^{-1}) \}$$

$$\text{Per}_2(\mu) = \{ \langle f \rangle \in \mathcal{M}_2(\mathbb{C}) ; \sigma_2 = -2\sigma_1 + \mu \},$$

(Lemmas 3.4 and 3.6 in [Mil93]).

Remark $\text{Per}_1(-1) \subseteq \text{Per}_2(1)$ by definition. But, in the case of $\mathcal{M}_2(\mathbb{C})$, it is clear that two families coincide.

### 1.2 Symmetry locus

By an automorphism of a quadratic rational map $f$, we will mean $g \in \text{PSL}_2(\mathbb{C})$ which commutes with $f$. The collection $\text{Aut}(f)$ of all automorphisms of $f$ forms a finite group. It is clear that $\text{Aut}(\tilde{f})$ is isomorphic to $\text{Aut}(f)$ for any $\tilde{f} \in \langle f \rangle$.

The set

$$\mathcal{S} = \{ \langle f \rangle ; \text{Aut}(f) \text{ is non-trivial} \} \subset \mathcal{M}_2(\mathbb{C})$$

is called the symmetry locus.

**Corollary 1** The symmetry locus $\mathcal{S}$ of quadratic rational maps forms an irreducible algebraic curve as follows:

$$S(\sigma_1, \sigma_2) = 2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36 = 0.$$  \hfill (1)

**Proof of Corollary 1.**

$\text{Aut}(f)$ coincides with the group consisting of all permutations of the fixed points which preserve the multipliers. In the case of $f$ has the three distinct fixed points, $\text{Aut}(f)$ has order 1, 2, or 6 according as three multipliers are distinct, two are equal, or all the three are equal, respectively, while, if $f$ has multiple fixed points then $\text{Aut}(f)$ is non-trivial if and only if $f$ has a triple fixed point. The multipliers $\mu_i$ are the roots of the equation:

$$\mu^3 - \sigma_1\mu^2 + \sigma_2\mu - \sigma_1 + 2 = 0.$$  \hfill (2)

The equation (2) has multiple roots if and only if its discriminant is equal to zero. Hence we have

$$(\sigma_2 - 2\sigma_1 + 3)(2\sigma_1^3 + \sigma_1^2\sigma_2 - \sigma_1^2 - 4\sigma_2^2 - 8\sigma_1\sigma_2 + 12\sigma_1 + 12\sigma_2 - 36) = 0.$$
The first factor corresponds with $\text{Per}_1(1)$. Considering the line of the first factor ($\text{Per}_1(1)$) tangent to the curve of the second factor ($S$) with tangency of degree three, the second factor is the required equation.

The following result is obtained immediately by the definition of the envelope of the family of curves.

**Corollary 2** The envelope of $\{\text{Per}_1(\mu)\}_\mu$ coincides with the symmetry locus.

**Remark** (Theorem 5.1 of [Mil93]) A quadratic rational map has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form $f(z) = k(z + \frac{1}{z})$ with $k \in \mathbb{C} \setminus \{0\}$.

### 1.3 Real moduli space

Let $\text{Rat}_2(\mathbb{R})$ be the set of real quadratic rational maps. Then the parameters $\sigma_i$ ($1 \leq i \leq 3$) are all real, because the three fixed points and the corresponding multipliers are either all real or one real and a pair of complex conjugate numbers. According to J. Milnor, we define the real moduli space $\mathcal{M}_2(\mathbb{R})$ for $\text{Rat}_2(\mathbb{R})$ to be simply the real $(\sigma_1, \sigma_2)$-plane. This notation needs some care when used: if we put $S_\mathbb{R} = S \cap \mathcal{M}_2(\mathbb{R})$, and denote by $\langle \_ \rangle_\mathbb{R}$ the real conjugacy class, then $(\text{Rat}_2(\mathbb{R})/\text{PGL}_2(\mathbb{R})) \setminus \{\langle a(x + \frac{1}{x}) \rangle_\mathbb{R}, \langle a(x - \frac{1}{x}) \rangle_\mathbb{R} \}_{a \in \mathbb{R}^\times}$ is canonically isomorphic to $\mathbb{R}^2 \setminus S_\mathbb{R}$, whereas there is a canonical two-to-one correspondence between $\{\langle a(x \pm \frac{1}{x}) \rangle \}_{a \in \mathbb{R}^\times}$ and $S_\mathbb{R}$.
2 Cubic polynomials

2.1 Moduli space of cubic polynomials

Let be \( \text{Poly}_3(\mathbb{C}) \) the space of all cubic polynomials from \( \mathbb{C} \) to itself. The group \( \text{Poly}_3(\mathbb{C}) \) of affine transformations acts on the space \( \text{Poly}_3(\mathbb{C}) \) by conjugation,

\[
g \circ p \circ g^{-1} \in \text{Poly}_3(\mathbb{C}) \quad \text{for} \quad g \in \text{Poly}_1(\mathbb{C}), \ p \in \text{Poly}_3(\mathbb{C}).
\]

Two maps \( p_1, p_2 \in \text{Poly}_3(\mathbb{C}) \) are holomorphically conjugate, denoted by \( p_1 \sim p_2 \), if and only if there exists \( g \in \text{Poly}_1(\mathbb{C}) \) with \( g \circ p_1 \circ g^{-1} = p_2 \). The quotient space of \( \text{Poly}_3(\mathbb{C}) \) under this action will be denoted by \( M_3(\mathbb{C}) \), and called the moduli space of holomorphic conjugacy classes \( \langle p \rangle \) of cubic polynomials \( p \).

Doing the same as the case of quadratic rational maps, we introduce coordinates in \( M_3(\mathbb{C}) \) as follows; for each \( p \in \text{Poly}_3(\mathbb{C}) \), let \( z_1, z_2, z_3, z_4(= \infty) \) be the fixed points of \( p \) and \( \mu_i \) the multipliers of \( z_i; \mu_i = p'(z_i) \) \((1 \leq i \leq 3)\), and \( \mu_4 = 0 \). Consider the elementary symmetric functions of the four multipliers,

\[
\begin{align*}
\sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu_1 + \mu_2 + \mu_3 \\
\sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\
\sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 = \mu_1\mu_2\mu_3 \\
\sigma_4 &= \mu_1\mu_2\mu_3\mu_4 = 0.
\end{align*}
\]

These multipliers determine uniquely \( p \) up to holomorphic conjugacy, and are subject only to the restriction that

\[
3 - 2\sigma_1 + \sigma_2 = 0.
\]

Hence the moduli space \( M_3(\mathbb{C}) \) is canonically isomorphic to \( \mathbb{C}^2 \) with coordinates \( \sigma_1 \) and \( \sigma_3 \).

Proposition 1
The locus \( \text{Per}_1(\mu) \) forms a straight lines as follows:

\[
\text{Per}_1(\mu) = \left\{(f) \in M_3(\mathbb{C}); \sigma_3 = (-\mu^2 + 2\mu)\sigma_1 + \mu^3 - 3\mu \right\}.
\]

The locus \( \text{Per}_2(\mu) \) forms an algebraic curve of degree three as follows:

\[
\text{Per}_2(\mu) = \left\{(f) \in M_2(\mathbb{C}); \sigma_3^2 + (4\sigma_1^2 - (\mu + 57)\sigma_1 + 252)\sigma_3 - (4\mu - 16)\sigma_1^3 + (61\mu - 252)\sigma_1 - (4\mu^2 + 246\mu - 1134)\sigma_1 - \mu^3 + 51\mu^2 - 99\mu - 459 = 0 \right\}.
\]

Note that this curve is irreducible if and only if \( \mu \neq 1 \). In the case of \( \mu = 1 \),

\[
\text{Per}_2(1) = \{\text{Per}_1(-1)\} \cup \left\{(f) \in M_2(\mathbb{C}); \sigma_3 + 4\sigma_1^2 - 61\sigma_1 + 254 = 0 \right\}.
\]
2.2 Symmetry locus

Using conjugation described in above, we can define symmetry locus of this moduli space as one in $\mathcal{M}_2(\mathbb{C})$, and we obtain next results.

**Theorem 1** The symmetry locus $S$ of cubic polynomials forms an irreducible algebraic curve:

$$S(\sigma_1, \sigma_3) = 27\sigma_3 + (\sigma_1 - 6)(2\sigma_1 - 3)^2 = 0. \quad (3)$$

The following result is obtained immediately by the definition of the envelope of the family of curves.

**Corollary 3** The envelope of $\{\text{Per}_1(\mu)\}_\mu$ coincides with the symmetry locus.

![Figure 3: $\mathcal{M}_3(\mathbb{R})$ with the real cut of $S$.](image1)

![Figure 4: Lines $\{\text{Per}_1(\mu)\}$ in the real cut of the moduli space $\mathcal{M}_3(\mathbb{C})$.](image2)

**Remark** A cubic polynomial has non-trivial automorphism if and only if it is conjugate to a map in the unique normal form $p(z) = z^3 + az$.

2.3 Real moduli space

Let $\text{Poly}_3(\mathbb{R})$ be the set of real cubic polynomials. By the same reason for the case of $\mathcal{M}_2$, we define the real moduli space $\mathcal{M}_3(\mathbb{R})$ for $\text{Poly}_3(\mathbb{R})$ to be simply the real $(\sigma_1, \sigma_3)$-plane. This notation needs some care when used: if we put $S_\mathbb{R} = S \cap \mathcal{M}_3(\mathbb{R})$, and denote by $(\ )_\mathbb{R}$ the real conjugacy class, then $(\text{Poly}_3(\mathbb{R})/\text{Poly}_1(\mathbb{R})) \setminus \{(x^3 + ax)_{\mathbb{R}}, (-x^3 + ax)_{\mathbb{R}}\}_{a \in \mathbb{R}^*}$ is canonically isomorphic to $\mathbb{R}^2 \setminus S_\mathbb{R}$, whereas there is a canonical two-to-one correspondence between $\{(\pm x^3 + ax)\}_{a \in \mathbb{R}^*}$ and $S_\mathbb{R}$. 
3 Polynomials of degree \( n \)

3.1 Moduli space of polynomials of degree \( n \)

Now we discuss about the moduli space \( \mathrm{M}_n(\mathbb{C}) \) for the space, \( \text{Poly}_n(\mathbb{C}) \), of polynomials of degree \( n \).

Doing the same as the case of cubic polynomials, we try introducing coordinates in \( \mathrm{M}_n(\mathbb{C}) \) as follows; for each \( p(z) \in \text{Poly}_n(\mathbb{C}) \), let \( z_1, \cdots, z_n, z_{n+1}(=\infty) \) be the fixed points of \( p \) and \( \mu_i \) the multipliers of \( z_i; \mu_i = p'(z_i) \) \((1 \leq i \leq n)\), and \( \mu_{n+1} = 0 \). Consider the elementary symmetric functions of the \( n \) multipliers,

\[
\begin{align*}
\sigma_{n,1} &= \mu_1 + \cdots + \mu_n, \\
\sigma_{n,2} &= \mu_1\mu_2 + \cdots + \mu_{n-1}\mu_n = \sum_{i=1}^{n-1} \mu_i \sum_{j=i+1}^{n} \mu_j, \\
& \quad \vdots \\
\sigma_{n,n} &= \mu_1\mu_2 \cdots \mu_n, \\
\sigma_{n,n+1} &= 0.
\end{align*}
\]

**Example 1** For example, we assume \( p(z) \in \text{Poly}_4(\mathbb{C}) \);

- fixed points: \( z_1, z_2, z_3, z_4, \infty \)
- multiplier: \( \mu_1, \mu_2, \mu_3, \mu_4, 0 \)
- elementary symmetric functions:

\[
\begin{align*}
\sigma_{4,1} &= \mu_1 + \mu_2 + \mu_3 + \mu_4 \\
\sigma_{4,2} &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4 \\
\sigma_{4,3} &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4 \\
\sigma_{4,4} &= \mu_1\mu_2\mu_3\mu_4 \\
\sigma_{4,5} &= 0.
\end{align*}
\]

Applying Fatou-index theorem to these fixed points;

\[
\frac{1}{1-\mu_1} + \frac{1}{1-\mu_2} + \frac{1}{1-\mu_3} + \frac{1}{1-\mu_4} + \frac{1}{1-0} = 1,
\]

(4)

where \( \mu_i \neq 1 \) \((1 < i < n)\). Arranging this equation for the form of elementary symmetric functions;

\[
4 - 3(\mu_1 + \mu_2 + \mu_3 + \mu_4) + 2(\mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4) - (\mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4) = 0.
\]

Hence we have

\[
4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0.
\]

(5)

For the equation (5), the cases \( \mu_i = 1 \) are also allowable.
Now we consider a polynomial $p(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \in \text{Poly}_4(C)$ that has at least two fixed points. After affine conjugation, we can assume they are 0 and 1. Then, we will solve the following question: “Do the four multipliers $
abla_0 = p'(0), \nabla_1 = p'(1), \nabla_2 = p'(z_2), \nabla_3 = p'(z_3),$ where $z_1, z_2$ are fixed points of $p(z)$, determine the five coefficients $a_4, a_3, a_2, a_1, a_0$ of $p(z)$?”

In fact, the following equations hold:

$a_0 = 0$ because of $f(0) = 0,$

$a_1 = \mu_0$ because of $f'(0) = \mu_0,$

$a_2 = a_4 + 3 - 2\mu_0 - \mu_1$ because of $f'(1) = \mu_1,$

$a_3 = 1 - a_4 - a_2 - \mu_0$ because of $f(1) = 1,$

and $a_4$ is a common root of the following two equations:

$A_1 = (\mu_2^2 - 2\mu_3\mu_2 + \mu_3^2 - 2\mu_1\mu_0 - \mu_0^2) a_4^2 + (\mu_4^3 + 4\mu_3 + 8) a_0^2 + (-4\mu_1^2 - 8) \mu_0 + 4\mu_3^2 - 8\mu_1^2 + 8\mu_1) a_2^2 + (-4\mu_0^2 + (-4\mu_1 + 28) \mu_0 + (4\mu_2^2 + 4\mu_1 - 44) \mu_0^2 + (-4\mu_3^2 + 4\mu_1^2 - 8\mu_1 + 32) \mu_0 - 6\mu_1^4 + 28\mu_1^3 - 44\mu_1^2 + 32\mu_1 - 16) a_2^2 + (-4\mu_0^5 + (-12\mu_1 + 32) \mu_0^4 + (-8\mu_1^2 + 64\mu_1 - 96) \mu_0^3 + (8\mu_1^3 - 96\mu_1 + 128) \mu_0^2 + (12\mu_1^4 - 64\mu_1^3 + 96\mu_1^2 - 64) \mu_0 + 4\mu_1^5 - 32\mu_1^4 + 96\mu_1^3 - 128\mu_1^2 + 64\mu_1) a_4 - \mu_0^6 + (-6\mu_1 + 12) \mu_0^5 + (-15\mu_1^2 + 60\mu_1 - 60) \mu_0^4 + (-20\mu_1^3 + 120\mu_1^2 - 240\mu_1 + 160) \mu_0^3 + (-15\mu_1^4 + 120\mu_1^3 - 360\mu_1^2 + 480\mu_1 - 240) \mu_0^2 + (-6\mu_1^5 + 60\mu_1^4 - 240\mu_1^3 + 480\mu_1^2 - 480\mu_1 + 192) \mu_0 - \mu_0^6 + 12\mu_1^5 - 60\mu_1^4 + 160\mu_1^3 - 240\mu_1^2 + 192\mu_1 - 64 = 0,$

$A_2 = (\mu_2 + \mu_3 + \mu_0 + \mu_1 - 4) a_4^2 + (2\mu_0^2 - 4\mu_0 - 2\mu_1^2 + 4\mu_1) a_4 + \mu_3^2 + (3\mu_1 - 6) \mu_0^2 + (3\mu_1^2 - 12\mu_1 + 12) \mu_0 + \mu_3^3 - 6\mu_1^2 + 12\mu_1 - 8 = 0.$

Above two equations have common roots if and only if $\mu_0, \mu_1, \mu_2, \mu_3$ satisfy the equation (5). Since $\mu_0, \mu_1, \mu_2, \mu_3$ are the four multipliers of $p(z)$ and they should satisfy the equation (5), the two equations always have common roots. Hence five coefficients of $p(z)$ are calculated by its four multipliers, however, this calculation is not decisive when they have distinct two common roots.

For the case of $\text{Poly}_n(C)$, it is clear from (4) that the equation corresponds to (5) cannot have the term of $\sigma_{n,n}$. Hence we can put

$c_0 + c_1 \sigma_{n,1} + c_2 \sigma_{n,2} + \cdots + c_{n-1} \sigma_{n,n-1} = 0$

where $c_k$ ($0 \leq k \leq n - 1$) are functions of $n$ variable.

Paying attention to the form of elementary symmetric functions, we obtain the following equation:

$c_k = (-1)^k \binom{n-1}{k} \binom{n}{k} = n - k.$
where \( \binom{n}{k} \) means binomial coefficient. For convenience, put \( \sigma_{n,0} = 1 \). we have

\[
\sum_{k=0}^{n-1}(-1)^k(n-k)\sigma_{n,k} = 0.
\]

\[
(6)
\]

**Question** Is the moduli space \( M_n(\mathbb{C}) \) for polynomials of degree \( n \) canonically isomorphic to \( \mathbb{C}^{n-1} \) with coordinates \( \sigma_1, \sigma_2, \cdots, \sigma_{n-2}, \) and \( \sigma_n \)?

### 3.2 Symmetry locus

**Proposition 2** A polynomial of degree four has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form

\[
\{z^4 + az\}, \quad a \in \mathbb{C}.
\]

For a map \( p(z) \) in this normal form, \( \text{Aut}(p) \) is a cyclic group of order three.

**Outline of proof.** Let \( p(z) \in \text{Poly}_4(\mathbb{C}) \).

1. In the case of a map \( p(z) \) with multiple fixed points.
   
   (a) The case of \( p(z) \) with a fixed point of order four: \( \text{Aut}(p) \) is non-trivial.
   
   (b) The case of \( p(z) \) with a fixed point of order three: \( \text{Aut}(p) \) is trivial.
   
   (c) The case of \( p(z) \) with two fixed points of order two: there is not such \( p(z) \).
   
   (d) The case of \( p(z) \) with a fixed point of order two: \( \text{Aut}(p) \) is trivial.

2. In the case of a map \( p(z) \) with four distinct fixed points.
   
   (a) The case of four distinct multipliers: \( \text{Aut}(p) \) is trivial.
   
   (b) The case that only two of multipliers are coincide: \( \text{Aut}(p) \) is trivial.
   
   (c) The case of two pair of same multipliers: there is not such \( p(z) \).
   
   (d) The case of three same multipliers: By an affine conjugation, if three fixed points (whose multipliers are same) are mapped on the vertices of a regular triangle whose barycenter is the origin and the other fixed point on the origin, then \( \text{Aut}(p) \) is non-trivial. Otherwise \( \text{Aut}(p) \) is trivial.
   
   (e) The case of four same multipliers: there is not such \( p(z) \).
Therefore a map \( p(z) \) has non-trivial automorphisms if and only if \( p(z) \) is in the case 1-(a) and the first part of 2-(d). We can check easily that these maps coincide with the normal form \( \{z^4 + az\} \).

**Conjecture**  A polynomial of degree \( n \) has a non-trivial automorphism if and only if it is conjugate to a map in the unique normal form

\[
\left\{ z^n + \sum_{k|\,(n-1),k\neq n-1} A(k)z^k \right\}
\]

where \( A(k) \) are parameters in \( \mathbb{C} \).

**References**


A Comparison between the quadratic rational maps and cubic polynomials

<table>
<thead>
<tr>
<th></th>
<th>Quadratic rational maps</th>
<th>Cubic polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Moduli Space</strong></td>
<td>$\mathcal{M}_2(\mathbb{C}) \simeq \mathbb{C}^2$</td>
<td>$\mathcal{M}_3(\mathbb{C}) \simeq \mathbb{C}^2$</td>
</tr>
<tr>
<td><strong>Real Moduli Space</strong></td>
<td>$\mathcal{M}_2(\mathbb{R}) \simeq \mathbb{R}^2$</td>
<td>$\mathcal{M}_3(\mathbb{R}) \simeq \mathbb{R}^2$</td>
</tr>
<tr>
<td></td>
<td><em>excepts on the symm. locus</em></td>
<td><em>excepts on the symm. locus</em></td>
</tr>
<tr>
<td><strong>Coordinates</strong></td>
<td>$(\sigma_1, \sigma_2), \quad \sigma_3 = \sigma_1 - 2$</td>
<td>$(\sigma_1, \sigma_3), \quad 3 - 2\sigma_1 + \sigma_2 = 0$</td>
</tr>
<tr>
<td><strong>Normal Forms</strong></td>
<td>Fixed Pint Normal Form, etc.</td>
<td>${ f(z) = z^3 + az + b }_{(a,b)}$</td>
</tr>
<tr>
<td><strong>Periodic Orbits</strong></td>
<td>$\text{Per}_1(\mu)$: $\sigma_2 = (\mu + \frac{1}{\mu})\sigma_1 - (\mu^2 + 2\mu)$</td>
<td>$\text{Per}_1(\mu)$: $\sigma_3 = (-\mu^2 + 2\mu)\sigma_1 + \mu^3 - 3\mu$</td>
</tr>
<tr>
<td></td>
<td>$\text{Per}_2(\mu)$: $2\sigma_1 + \sigma_2 = \mu$</td>
<td>$\text{Per}_2(\mu)$:</td>
</tr>
<tr>
<td></td>
<td>$\text{Per}_1(-1) = \text{Per}_2(1)$</td>
<td><em>cubic algebraic curve</em></td>
</tr>
<tr>
<td><strong>Symmetry Locus</strong></td>
<td><em>the envelope of ${\text{Per}_1(\mu)}$</em></td>
<td><em>the envelope of ${\text{Per}_1(\mu)}$</em></td>
</tr>
<tr>
<td></td>
<td><em>normal form: ${k(z + \frac{1}{z})}$</em></td>
<td><em>normal form: ${z^3 + az}$</em></td>
</tr>
<tr>
<td><strong>Topological Partition</strong></td>
<td>$\text{degree}\pm 2$, monotone, unimodal, bimodal</td>
<td>$\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$</td>
</tr>
<tr>
<td><strong>Hyp. Components</strong></td>
<td>B, C, D, E</td>
<td>A, B, C, D</td>
</tr>
</tbody>
</table>

Masayo FUJIMURA  
Dept. of Math., College of Sci. and Tech., Nihon Univ.  
1-8, Kanda-Surugadai, Chiyoda-ku, Tokyo, 101 JAPAN  
e-mail: masayo@math.cst.nihon-u.ac.jp