<table>
<thead>
<tr>
<th>Title</th>
<th>Heat convection of compressible fluid (Domain Decomposition Methods and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>AYE, PYI; NISHIDA, Takaaki</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 989: 169-179</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/61063">http://hdl.handle.net/2433/61063</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
</tbody>
</table>

Kyoto University
Heat convection of compressible fluid

Dedicated to the sixtieth birthday of Professor Hideo Kawarada

PYI AYE and Takaaki NISHIDA
Department of Mathematics, Faculty of Science
Kyoto University

1. Mathematical Formulation

We consider the heat convection problem for the compressible viscous and heat conducting fluid in the layer between the upper plane $X_n = 0$ and the lower plane $X_n = d$, where we use the orthonormal basis $\{e_1, e_2, ..., e_n\}$ in $\mathbb{R}^n$, $n = 2$ or $3$, and $e_n$ is considered in the vertically downward direction. Using the velocity of fluid: $u' = (u'_1, u'_2, ..., u'_n)$, temperature: $\theta'$, density: $\rho'$, the governing equations of motion are

$$\frac{\partial \rho'}{\partial t'} + \nabla' \cdot (\rho' u') = 0, \quad (1)$$

$$\rho'\left(\frac{\partial u'}{\partial t'} + u' \cdot \nabla' u'\right) = -\nabla' p' + g \rho'e_n + \mu\Delta' u' + \frac{1}{3} \mu \nabla'(\nabla' \cdot u'), \quad (2)$$

$$\rho' c_v \left(\frac{\partial \theta'}{\partial t'} + u' \cdot \nabla' \theta'\right) = \kappa \Delta' \theta' + 2 \mu \mathbf{D}' : \mathbf{D}' - \frac{2}{3} \mu (\nabla' \cdot u')^2. \quad (3)$$

Here $p'$ is the pressure, $g e_n$ the acceleration of gravity, $\mu$ viscosity, $\kappa$ heat conduction coefficient, $c_v$ specific heat at constant volume, $\mathbf{D}'$ the deformation tensor. We assume the equation of state for the ideal gas:

$$p = R_* \rho \theta, \quad (4)$$
where \( R_* = c_p - c_v \) is the gas constant and \( c_p \) specific heat at constant pressure. Let the temperatures on the boundaries be given as

\[
\theta' = T_u \text{ at } X_n = 0 \text{ and } \theta' = T_l \text{ at } X_n = d ,
\]

(5)

where \( 0 < T_u < T_l \), and \( 0 < \beta_0 = (T_l - T_u)/d \) is the constant gradient of the temperature across the layer. Then the equilibrium solution \( s_0 = (\rho_0, u_0, \theta_0) \) is the purely heat conducting one and is given by

\[
\begin{align*}
    u_0 &= 0 , \quad \theta_0 = \beta_0 x_n' , \quad \rho_0 = \frac{P}{R_* \beta_0} x_n^m , \\
    x_i' &= X_i , \quad 1 \leq i \leq n - 1 , \quad x_n' = \frac{T_u}{\beta_0} + X_n ,
\end{align*}
\]

(6)

where

\[
x_i' = X_i , \quad 1 \leq i \leq n - 1 , \quad x_n' = \frac{T_u}{\beta_0} + X_n ,
\]

\( P \) is an integration constant and \( m \) is the polytropic index:

\[
m = \frac{g}{R_* \beta_0} - 1 .
\]

(7)

We consider the perturbation to the equilibrium solution in the following dimensionless form:

\[
\begin{align*}
    u' &= \tilde{u} , \quad \rho' = \tilde{\rho} + \rho_0 , \quad T' = \tilde{\theta} + \theta_0 .
\end{align*}
\]

Defining Prandtl number and other constants as follows:

\[
\begin{align*}
    \mathcal{Pr} &= \frac{c_v \mu}{\kappa} , \quad \beta = \beta_0 - \frac{g}{c_p} , \quad b = \frac{\beta}{\beta_0} \quad \text{and} \quad \gamma = \frac{c_p}{c_v} ,
\end{align*}
\]

we introduce dimensionless variables

\[
\begin{align*}
    t &= At' , \quad u = B \tilde{u} , \quad \theta = C \tilde{\theta} , \quad \rho = D \tilde{\rho} , \quad x_i = \frac{x_i'}{d} , \quad 1 \leq i \leq n ,
\end{align*}
\]
where
\[ A = \frac{R_\ast \beta_0 \mu}{P_r P d^{m+2}}, \quad B = \left( \frac{P_r}{g d b \gamma} \right)^{\frac{1}{2}}, \quad C = \frac{1}{\beta_0 d}, \quad D = \frac{R_\ast \beta_0}{P d^m}. \]

Then the layer becomes \( \Omega = \{ z_0 \leq x_n = z \leq z_0 + 1 \} \), where \( z_0 = \frac{T}{\beta_0 d} \).

We also define the Rayleigh number for the upper plane \( z = z_0 \)
\[ \mathcal{R}_a(z_0) = \mathcal{R}^2 z_0^{2m-1} = \frac{P^2 \beta R_\ast c_p(m + 1) d^{2m+3}}{g^{2} \mu \kappa} z_0^{2m-1}. \]

Thus we obtain the dimensionless system for the perturbation:
\[ \frac{\partial \rho}{\partial t} + \mathcal{R} \nabla \cdot (z^m \textbf{u}) = N_1, \tag{8} \]
\[ \frac{1}{P_r} z^m \frac{\partial \textbf{u}}{\partial t} - \Delta \textbf{u} - \frac{1}{3} \nabla (\nabla \cdot \textbf{u}) = -\frac{\mathcal{R}}{b \gamma (m + 1)} \{ z^{m+1} \nabla (z^{-m} \rho) + \nabla (z^m \theta) \} + N_2, \tag{9} \]
\[ z^m \frac{\partial \theta}{\partial t} - \Delta \theta + \mathcal{R} z^m \textbf{u} \cdot \nabla \textbf{u} = -\mathcal{R} (\gamma - 1) z^{m+1} \nabla \cdot \textbf{u} + N_3, \tag{10} \]

where the nonlinear terms are the followings
\[ N_1 = -\mathcal{R} \nabla \cdot (\rho \textbf{u}), \tag{11} \]
\[ N_2 = -\frac{1}{P_r} \rho \frac{\partial \textbf{u}}{\partial t} - \frac{\mathcal{R}}{P_r} \rho \textbf{u} \cdot \nabla \textbf{u} - \frac{\mathcal{R}}{b \gamma (m + 1)} \nabla (\rho \theta), \tag{12} \]
\[ N_3 = -\rho \frac{\partial \theta}{\partial t} - \mathcal{R} \rho \textbf{u} \cdot \nabla \theta - \mathcal{R} z^m \textbf{u} \cdot \nabla \theta - \mathcal{R} \rho u_n - \mathcal{R} (\gamma - 1)(\rho + z^m) \theta \nabla \cdot \textbf{u} \]
\[ -\mathcal{R} (\gamma - 1) z \rho \nabla \cdot \textbf{u} + \frac{2 \alpha_b r}{\beta_0 c_v} D : D - \frac{2 \alpha_b r}{3 \beta_0 c_v} (\nabla \cdot \textbf{u})^2. \tag{13} \]

The system is formulated and treated by [7], [3], [4], [6], [1] and many others. Here we consider the instability and introduce a method to obtain the critical Rayleigh number as a computer assisted proof.
2. The linearized problem and the stability

We consider the linearized problem

\begin{align}
\frac{\partial \rho}{\partial t} &= -\mathcal{R} \nabla \cdot (z^m u), \\
\frac{1}{\mathcal{P}_r} z^m \frac{\partial u}{\partial t} - \Delta u + \frac{1}{3} \nabla (\nabla \cdot u) &= -\frac{\mathcal{R}}{b\gamma(m + 1)} \{z^{m+1} \nabla (z^{-m} \rho) + \nabla (z^m \theta)\}, \\
z^m \frac{\partial \theta}{\partial t} - \Delta \theta &= -\mathcal{R} z^m u_n - \mathcal{R} (\gamma - 1) z^{m+1} \nabla \cdot u,
\end{align}

in the strip \( \Omega = \{z_0 \leq z = x_n \leq z_0 + 1\} \), where the slip boundary condition for the velocity and Dirichlet boundary condition for the temperature are assumed

\begin{align}
\frac{\partial u_i}{\partial z} &= 0, \ i = 1, \cdots, n - 1, \\
u_n &= \theta = 0 \\
&\text{at } z = z_0 \text{ and } z = z_0 + 1. \tag{17}
\end{align}

We can treat also the Dirichlet boundary condition for the velocity. We consider the solution \((\rho, u, \theta)\) in the strip \( \Omega \), which is periodic in \( x_i \) with period \( \ell_i \), \( 1 \leq i \leq n - 1 \) and satisfies the boundary condition.

**Theorem 1** If Rayleigh number is small, then the solution \( s_0 = (\rho_0, u_0, \theta_0) \) is asymptotically energy stable.


3. Eigenvalue problems for the linearized system

We notice that main dimensionless parameters are Rayleigh number, Prandtl number, polytropic index \( m \), periodicity \( \ell_i \) and the shallowness of
the layer which is proportional to $1/z_0$. We want to consider the instability of the purely heat conducting state, which is given by the critical Rayleigh number when we change Rayleigh number, $\ell_i$ and $z_0$.

The eigenvalue problem for the linearized system is the following:

$$\lambda \rho = -\mathcal{R} \nabla \cdot (z^m u) + f,$$

$$\frac{\lambda}{P_r} z^m u - \Delta u + \frac{1}{3} \nabla (\nabla \cdot u) = -\frac{\mathcal{R}}{b\gamma(m+1)} \{ z^{m+1} \nabla (z^{-m}\rho) + \nabla (z^m \theta) \} + g,$$

$$\lambda z^m \theta - \Delta \theta = -\mathcal{R} z^m u_n - \mathcal{R} (\gamma - 1) z^{m+1} \nabla \cdot u + h,$$

in the strip $\Omega = \{z_0 \leq z = x_n \leq z_0 + 1\}$, with the boundary condition (17).

**Theorem 2** The linearized system (18-20) forms a sectorial operator for any Rayleigh number for $f \in H^1$, $g \in L^2$, $h \in L^2$.

See Pyi Aye [6].

Hereafter we consider the two-dimensional problem and since we assume the periodicity with respect to the horizontal direction $x$, we may consider the eigenfunctions in the form:

$$\rho = \rho(\lambda, z) \cos(nx), \quad u_1 = u(\lambda, z) \sin(nx),$$

$$u_2 = w(\lambda, z) \cos(nx), \quad \theta = \theta(\lambda, z) \cos(nx) \quad \text{for } z_0 \leq z \leq z_0 + 1.$$

Then the system for the eigenvalue problem becomes the following system of ordinary differential equations.

$$\left(\frac{\lambda}{R z^m} + \frac{3 R}{4 b \gamma(m+1)}\right) \frac{d\rho}{dz} = \left(\frac{2 \lambda m}{R z^{m+1}} + \frac{3 R m}{4 b \gamma(m+1)}\right) \rho + \frac{m n u}{z} - \frac{3}{4} n \frac{du}{dz}$$
\[- \left( \frac{3\lambda}{4P_r} z^m + \frac{3}{4} n^2 - \frac{m(m+1)}{z^2} \right) w \]
\[- \frac{3 \mathcal{R}}{4 b \gamma (m+1)} \left( m z^{m-1} \theta + z^m \frac{d\theta}{dz} \right) \]
\[
\begin{align*}
\frac{dw}{dz} &= - n u - m \frac{w}{z} - \frac{\lambda}{\mathcal{R} z^m} \rho \\
\frac{d^2 u}{dz^2} &= - \frac{\mathcal{R} n z}{b \gamma (m+1)} \rho + \left( \frac{\lambda}{P_r} z^m + \frac{4}{3} n^2 \right) u \\
&\quad + \frac{1}{3} n \frac{dw}{dz} - \frac{\mathcal{R} n z}{b \gamma (m+1)} \theta \\
\frac{d^2 \theta}{dz^2} &= \mathcal{R} (\gamma - 1) n z^{m+1} u + \mathcal{R} (\gamma - 1) z^{m+1} \frac{dw}{dz} \\
&\quad + \mathcal{R} z^m w + \left( \lambda z^m + n^2 \right) \theta,
\end{align*}
\]

in the interval \( z_0 \leq z = x_n \leq z_0 + 1 \), with the boundary condition
\[
\frac{du}{dz} = 0, \quad w = \theta = 0 \text{ at } z = z_0 \text{ and } z = z_0 + 1. \quad (25)
\]

By this formulation, the original problem of instability is reduced to investigate the behavior of the real part of the eigenvalue \( \lambda \) when the parameters \( \mathcal{R}, z_0 \) and \( n \) vary. In order to see the instability we use the method given in [5] to prove the existence of the purely imaginary eigenvalue and the critical Rayleigh number in a small neighbourhood of the computed purely imaginary eigenvalue and critical Rayleigh number based on the Newton method. To obtain the eigenvalue and the eigenfunction for (21-25), we use the shooting method, i.e., we consider the fundamental solutions of the initial value problem for (21-24) in \( z \geq z_0 \) and express the eigenfunction by them as
\[
\rho = a \rho_1(z) + b \rho_2(z) + c \rho_3(z), \quad u = a u_1(z) + b u_2(z) + c u_3(z),
\]
\[
w = a w_1(z) + b w_2(z) + c w_3(z), \quad \theta = a \theta_1(z) + b \theta_2(z) + c \theta_3(z) \quad (26)
\]
where $\rho_j(z), u_j(z), w_j(z), \theta_j(z), j = 1,2,3$ satisfy (21-24) in $z > z_0$ and the initial conditions at $z = z_0$

\[
\begin{align*}
\rho_1(z_0) &= 1, \ u_1(z_0) = 0, \ \theta'_1(z_0) = 0, \\
\rho_2(z_0) &= 0, \ u_2(z_0) = 1, \ \theta'_2(z_0) = 0, \\
\rho_3(z_0) &= 0, \ u_3(z_0) = 0, \ \theta'_3(z_0) = 1,
\end{align*}
\]

(27)

$a$, $b$ and $c$ are constants to be determined. In order that the function (26) is the eigenfunction, it must satisfy the boundary condition (25). This condition is written as follows

\[
\begin{pmatrix}
\frac{du_1}{dz}(z_0 + 1) & \frac{du_2}{dz}(z_0 + 1) & \frac{du_3}{dz}(z_0 + 1) \\
w_1(z_0 + 1) & w_2(z_0 + 1) & w_3(z_0 + 1) \\
\theta_1(z_0 + 1) & \theta_2(z_0 + 1) & \theta_3(z_0 + 1)
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = 0.
\]

(28)

Then for the eigenfunction, we have to have

\[
\mathcal{F}(\mathcal{R}, \lambda; \mathcal{P}_r, z_0, n) \equiv \det A = 0,
\]

(29)

where the matrix in (28) is denoted by $A = (a_{ij})$. Thus, we come to the position to search the values of $\mathcal{R} = \mathcal{R}_c, \ \lambda = \mathrm{i}\omega_c$ satisfying (29) for the fixed parameters $\mathcal{P}_r, z_0$ and $n$. Noting that (29) can be rewritten as

\[
\mathcal{F}(\mathcal{R}, \lambda) = \mathcal{F}(\mathcal{R}_0, \lambda_0) + \frac{\partial \mathcal{F}}{\partial \mathcal{R}}(\mathcal{R} - \mathcal{R}_0) + \frac{\partial \mathcal{F}}{\partial \lambda}(\lambda - \lambda_0) = 0,
\]

we can state our criterion for existence of the critical eigenvalue based on the simplified Newton method as follows:
Theorem 3 Suppose, for a small $\epsilon > 0$, there exist $R_0$ and $\lambda_0$ such that

$$\| \mathcal{F}(R_0, \lambda_0) \| < \epsilon .$$  \hspace{1cm} (30)

Put

$$L_0 \equiv \left( \frac{\partial \mathcal{F}}{\partial R}(R_0, \lambda_0), \frac{\partial \mathcal{F}}{\partial \lambda}(R_0, \lambda_0) \right),$$  \hspace{1cm} (31)

where the bar means an appropriate approximation of the quantity. Suppose further that, for a small $\delta$, there is a $\rho_1$ such that the estimate

$$\| D\mathcal{F}(R, \lambda) - L_0 \| < \delta$$  \hspace{1cm} (32)

holds for any $(R, \lambda)$ such that

$$(R - R_0)^2 + |\lambda - \lambda_0|^2 < \rho_1^2 .$$

For $\epsilon$, $\rho_1$, $\delta$ and $L_0$ as above, if it holds that

$$\| L_0^{-1} \| \left( \frac{\epsilon}{\rho_1} + \delta \right) \leq 1 ,$$  \hspace{1cm} (33)

then there exist some $R_c$ and $\lambda_c$ in the $\rho_1$-neighborhood of $R_0$ and $\lambda_0$ satisfying

$$\mathcal{F}(R_c, \lambda_c) = 0 .$$  \hspace{1cm} (34)

To utilize this criterion to our problem, we need to justify the following steps:

(i) To find appropriate values $R_0$ and $\lambda_0$, we use the numerical computation by the shooting method and Newton method. The fundamental solutions are obtained by the fourth order Taylor finite difference scheme .

(ii) To estimate $\epsilon$ we need the interval analysis by a computer software for the
bound of round-off errors in the computation of the fundamental solutions and the theory of pseudo trajectory to estimate the difference between the genuine fundamental solutions and the numerically computed ones.

(iii) At this pair of \( R_0, \lambda_0 \), find an approximate derivative \( L_0 \) and estimate the norm \( \| L_0^{-1} \| \);

(iv) Estimate \( \delta \) for which the estimate (32) holds in the \( \rho_1 \)-neighborhood of \( R_0 \) and \( \lambda_0 \);

(v) For these values in (i, ii, iii, iv), prove that the criterion (33) holds.

Following these steps we see that there exist the exact eigenvalue \( \lambda = i\omega_c \) and the critical Rayleigh number \( \mathcal{R} = \mathcal{R}_c \) for (21-25) in the \( \rho_1 \)-neighborhood of numerically computed values \((\mathcal{R}_0, \lambda_0)\) in (i).

In order to see the motion of the eigenvalue crossing the imaginary axis when \( \mathcal{R} \) increases, we can apply such arguments as in [5] which uses the adjoint system of the equations to (21-24). For notational convenience we write the eigenvalue \( \lambda_c \) and the eigenfunction \( \Phi = (\rho, u, w, \theta) \) with the critical Rayleigh number \( \mathcal{R}_c \) for the system of equations (21-24) and the boundary conditions (25) as

\[
L \Phi = 0 \quad \text{and} \quad B \Phi = 0 .
\]

Let us denote the eigenvalue \( \lambda_c \) and the eigenfunction \( \Psi = (\rho^*, u^*, w^*, \theta^*) \) which satisfy the the adjoint problem

\[
L^* \Psi = 0 \quad \text{and} \quad B^* \Psi = 0 .
\]

Taking the derivative of (35) with respect to the Rayleigh number and the
\[ L^2(0,1)-\text{inner product with } \Psi, \text{ we obtain} \]
\[
\left. \frac{\partial \lambda}{\partial R} \right|_{R=R_c} = -\frac{(\frac{\partial L}{\partial \mathcal{R}} \Phi, \Psi)_{L^2}}{(\frac{\partial L}{\partial \lambda} \Phi, \Psi)_{L^2}}.
\]

Example 1. We take \( \gamma = 5/3 \), \( c_p = 1 \), \( c_v = 0.6 \), \( m = 1.4 \), \( \mathcal{P}_r = 1 \), and \( b = 0.04 \).

<table>
<thead>
<tr>
<th>( z_0 )</th>
<th>( n )</th>
<th>( \lambda )</th>
<th>( \mathcal{R}_0 )</th>
<th>( \mathcal{R}_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.125</td>
<td>2.4797</td>
<td>0.0</td>
<td>54.8836</td>
<td>1292.61</td>
</tr>
<tr>
<td>0.25</td>
<td>2.3925</td>
<td>0.0</td>
<td>42.6609</td>
<td>1084.35</td>
</tr>
<tr>
<td>0.5</td>
<td>2.3136</td>
<td>0.0</td>
<td>29.7561</td>
<td>885.428</td>
</tr>
<tr>
<td>1.0</td>
<td>2.2614</td>
<td>0.0</td>
<td>19.0653</td>
<td>754.139</td>
</tr>
<tr>
<td>2.0</td>
<td>2.2357</td>
<td>0.0</td>
<td>11.5268</td>
<td>691.377</td>
</tr>
<tr>
<td>4.0</td>
<td>2.2258</td>
<td>0.0</td>
<td>6.6749</td>
<td>667.851</td>
</tr>
<tr>
<td>8.0</td>
<td>2.2226</td>
<td>0.0</td>
<td>3.7447</td>
<td>660.399</td>
</tr>
<tr>
<td>16.0</td>
<td>2.2217</td>
<td>0.0</td>
<td>2.0581</td>
<td>658.276</td>
</tr>
<tr>
<td>32.0</td>
<td>2.2215</td>
<td>0.0</td>
<td>1.1176</td>
<td>657.708</td>
</tr>
<tr>
<td>64.0</td>
<td>2.22146</td>
<td>0.0</td>
<td>0.60316</td>
<td>657.561</td>
</tr>
<tr>
<td>128.0</td>
<td>2.22145</td>
<td>0.0</td>
<td>0.32429</td>
<td>657.523</td>
</tr>
<tr>
<td>\text{Bouss.}</td>
<td>2.22144</td>
<td>0.0</td>
<td></td>
<td>657.511</td>
</tr>
</tbody>
</table>

Here \( \mathcal{R}_m \) is the Rayleigh number on the middle plane \( z = z_0 + 0.5 \)

\[
\mathcal{R}_m = \mathcal{R}_a(z_0 + 0.5).
\]

It approaches to that of Boussinesq approximation for heat convection as \( z_0 \) gets large.

This example suggests the occurrence of the stationary bifurcation at the critical Rayleigh number. However the usual bifurcation theory does not apply to the original system (8-13), because the mass conservation law has a high nonlinearity and the sectorial properties of the theorem 2 is not sufficient to guarantee the bifurcation. Further investigations are required.
References


