数理解析研究所講究録
989 巻 1997 年 158-168

The application of PVM to the computation of vortex sheet motion

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1 Vortex sheet and its governing equation

We consider a motion of incompressible, inviscid fluid. Attention is restricted to a vortex sheet motion. A vortex sheet is a surface, along which the velocity changes discontinuously. The vorticity concentrates on the surface, outside which the flow is irrotational. We assume a further simplification that the flow is two-dimensional. Mathematically, two-dimensional vortex sheet is represented by a curve. When we identify the two-dimensional space with complex plane, a vortex sheet in two-dimensional space is expressed by a complex valued function $z(\Gamma, t)$, where $\Gamma \in \mathbf{R}$ is a Lagrangian parameter along the curve, which represents the circulation of the flow. t represents time.

We consider the dynamics of a two-dimensional vortex sheet with a periodic boundary condition;

$$z(\Gamma+1, t) = z(\Gamma, t) + 1.$$

The equation which describes the motion of vortex sheet is known as the Birkhoff-Rott equation([16]):

$$\frac{\partial z^*(\Gamma, t)}{\partial t} = \frac{1}{2\pi i}\mathrm{p.v.}\int \frac{d\Gamma'}{z(\Gamma, t) - z(\Gamma', t)}$$

The integral on the right hand side is Cauchy's principal value. i is the imaginary unit. $*$ denotes complex conjugate. Taking the periodic boundary condition into account, we rewrite the equation and obtain([8]):

$$\frac{\partial z^*(\Gamma, t)}{\partial t} = \frac{1}{2i}\mathrm{p.v.}\int_0^1 \cot\pi(z(\Gamma, t) - z(\Gamma', t))d\Gamma'. \quad (1)$$

This is the equation which we are going to study numerically.

A flat vortex sheet of constant strength, $z(\Gamma, t) \equiv \Gamma$, is an equilibrium solution of the equation (1). This equilibrium is known to be unstable: even a small perturbation can grows very rapidly and the vortex sheet shows extreme complexity for large t ([8, 9, 17]). The following properties are known:

- Linearized stability analysis shows that perturbations of short wave length grow exponentially. The shorter the wavelength is, the faster the perturbation grows([16]). (Kelvin-Helmholtz instability.)
- If the initial perturbation is an analytic function of Γ, then the sheet remains analytic for a positive time interval ([2, 20]).
- The vortex sheet loses analyticity in finite time([14]).
- The initial value problem for (1) is ill-posed in the sense of Hadamard([3]).
- A vortex sheet evolves into a complex form having spirals (see, for instance, [8, 9, 17]).

Because of the ill-posedness of the equation, it is difficult to apply naive numerical methods to the computation of a vortex sheet. We apply Chorin's vortex blob method, which we are going to explain in the next section.

2 Numerical method

Instead of the original equation, we consider the following smoothed equation. (This equation is given by Krasny([8]).)

$$\frac{\partial z^*(\Gamma, t)}{\partial t} = \int_0^1 K_\delta(z(\Gamma, t) - z(\Gamma', t))d\Gamma', \tag{2}$$

where

$$K_\delta(x + iy) = -\frac{1}{2}\frac{\sinh(2\pi y) + i\sin(2\pi x)}{\cosh(2\pi y) - \cos(2\pi x) + \delta^2}.$$

This equation is well-posed for any time interval if $\delta > 0$. δ is an artificial parameter that makes the equation well-posed. When $\delta = 0$, the equation reduces to the original equation. The convergence of the solution of the smoothed equation to that of the Birkhoff-Rott equation is proven as far as the solution of (1) is smooth. However, after the appearance of the singularity, only convergence in some weak sense is proven ([12, 13]).

2.1 Discretization

In order to compute (2), we approximate the vortex sheet by N points.

$$\Gamma_i = \frac{i}{N}, \qquad z(\Gamma_i, t) = z_i(t), \quad i = 0, \cdots, N-1.$$

Then, we discretize the integral by trapezoidal rule and obtain the following system of ordinary differential equations:

$$\frac{\partial z_n^*(t)}{\partial t} = \frac{1}{N} \sum_{m=0}^{N-1} K_\delta(z_n(t) - z_m(t)), \quad i = 0, \cdots, N-1. \tag{3}$$

In order to integrate the system of O.D.E, we use the fourth-order Runge-Kutta method. The parameters we can change are as follows:

- N $\cdots$ the number of vortices
- Δt $\cdots$ time step size for Runge-Kutta method
- δ $\cdots$ smoothing parameter of vortex blob method

This is a rough description of the vortex method. For more details, see, e.g., Puckett [15].

Numerical computation of the right hand side of (3) requires $O(N)$ multiplications for each vortex point. Since there are N vortices, $O(N^2)$ operations are necessary to compute the velocity fields at all the positions of particles. In order to obtain an accurate numerical solution, we need a large number of vortices to discretize the vortex sheet. Thus the computation becomes too slow when N is large. This is the most serious disadvantage of the vortex method. There are some algorithms which evaluate the velocity field in $O(N \log N)$ operations within some errors([4, 6, 7, 18]). Although they are promising, they have some defects, too([7, 18]). In the present paper, we would like to study another method, Parallel Virtual Machine software, to evaluate the velocity field at the vortex points. In the following section, we will explain how to implement this tool for numerical computations and the efficiency will be examined.

3 Parallel Virtual Machine

The Parallel Virtual Machine (shortly PVM) is a software framework for heterogeneous parallel computing in networked environments ([5]). PVM supports complete message passing model and it emulates a distributed memory model in heterogeneous network.

Our virtual parallel machine consists of four computers. They have the following CPU's, memory, and operating systems, respectively

- A $\cdots$ PA-RISC 7000, 32MB, HP-UX 9.0.5

- B ⋯ Pentium 120MHz, 64MB, FREEBSD
- C ⋯ Pentium 100MHz, 64MB, FREEBSD
- D ⋯ Pentium 60MHz, 24MB, FREEBSD

These computers are connected through 10Base-T Ethernet.

We use the following master-slave type algorithm to implement the computation of velocity field:

1. Divide N points by k group (k is the number of computers). n_j is the number of vortices, the velocities of which are computed in the j-th computer, whence $\sum_{j=1}^{k} n_j = N$.
2. Send the position of n_j vortices to each slave computers
3. Each slave computer evaluates the velocities at the positions of n_j points
4. Send back the results to the master computer
5. loop to 2

n_j is determined by CPU speed to make the evaluation time of each computers as even as possible.

3.1 Test problem : Two Vortex Sheets

Using PVM and vortex blob method for a vortex sheet, we compute the motion of two vortex sheets. We consider two, nearly parallel, vortex sheets. We denotes upper vortex sheet and lower vortex sheet by $z(\Gamma, t)$ and $w(\Gamma, t)$, respectively. Then the equation of motion of two vortex sheets is written as follows:

$$\begin{aligned}
\frac{\partial z^*(\Gamma,t)}{\partial t} &= \frac{\sigma_1}{2i}\text{p.v.}\int_0^1 \cot\pi(z(\Gamma,t)-z(\Gamma',t))d\Gamma' \\
&+ \frac{\sigma_2}{2i}\int_0^1 \cot\pi(z(\Gamma,t)-w(\Gamma',t))d\Gamma', \\
\frac{\partial w^*(\Gamma,t)}{\partial t} &= \frac{\sigma_2}{2i}\text{p.v.}\int_0^1 \cot\pi(w(\Gamma,t)-w(\Gamma',t))d\Gamma' \\
&+ \frac{\sigma_1}{2i}\int_0^1 \cot\pi(w(\Gamma,t)-z(\Gamma',t))d\Gamma',
\end{aligned}$$

where σ_1 is the vorticity density of upper vortex sheet and σ_2 is that of lower vortex sheet.

The initial value of z and w is taken as follows:

$$\begin{aligned} z(\Gamma,0) &= \Gamma + \epsilon\sin 2\pi\Gamma - i\epsilon\sin 2\pi\Gamma + i\frac{H}{2}, \\ w(\Gamma,0) &= \Gamma + \epsilon\sin 2\pi\Gamma - i\epsilon\sin 2\pi(\Gamma+\alpha) - i\frac{H}{2} \\ & \quad (0 \le \alpha < 1, H \ne 0, 0 \le \Gamma < 1), \end{aligned}$$

where H is average distance between two vortex sheets and α is the phase difference of two vortex sheets.

The numerical parameters of the computations are

- N $\cdots$ the number of vortices
- Δt $\cdots$ time step size for Runge-Kutta method
- δ $\cdots$ smoothing parameter of vortex blob method
- ϵ $\cdots$ the amplitude of the disturbance
- H $\cdots$ the average distance between two vortex sheets
- α $\cdots$ initial phase difference of two vortex sheets
- σ_1,σ_2 $\cdots$ the vorticity of two vortex sheets.

3.2 Reduction of execution time

We show the execution time when we use PVM. The execution time (in second) is measured by one time step of Runge-Kutta method. (The time to evaluate velocity field four times.) Among four computers, machine B is the fastest of the four computers. The ratio of CPU speeds is approximately equal to A:B:C:D=9:10:8:5. We divide N vortex points according as this ratio. The following are the list of of performances.

3.2.1 The result for $N = 2048$

1. Single processor $k = 1$
   - A $\cdots$ 40.68 seconds
   - B $\cdots$ 33.79 seconds
   - C $\cdots$ 41.19 seconds
   - D $\cdots$ 67.22 seconds
2. Two processors $k = 2$

- A+B $\cdots$ 20.87 seconds ($\times$ 1.62)

3. Three processors $k = 3$
   - A+B+C $\cdots$ 14.81 seconds ($\times$ 2.28)
4. Four processors $k = 4$
   - A+B+C+D $\cdots$ 13.26 seconds ($\times$ 2.54)

Here and hereafter, $\times 1.77$, for instance, implies that the computation is 1.77 times faster than the computation of single B processor.

3.2.2 The result for $N = 4096$

1. Single processor
   - A $\cdots$ 153.09 seconds
   - B $\cdots$ 134.98 seconds
   - C $\cdots$ 263.29 seconds
   - D $\cdots$ 159.49 seconds
2. Two processors
   - A+B $\cdots$ 75.90 seconds ($\times$ 1.77)
3. Three processors
   - A+B+C $\cdots$ 55.29 seconds ($\times$ 2.44)
4. Four processors
   - A+B+C+D $\cdots$ 49.19 seconds ($\times$ 2.74)

3.2.3 The result for $N = 8192$

1. Single processor
   - A $\cdots$ 635.06 seconds
   - B $\cdots$ 521.05 seconds
   - C $\cdots$ 640.54 seconds
   - D $\cdots$ 1050.41 seconds
2. Two processors

- A+B $\cdots$ 308.12 seconds ($\times$ 1.69)

3. Three processors

- A+B+C $\cdots$ 219.82 seconds ($\times$ 2.37)

4. Four processors

- A+B+C+D $\cdots$ 188.56 seconds ($\times$ 2.76)

The above results show that the usefulness of PVM at least when a small number of computers are combined.

3.3 Numerical results

We choose $N = 4096$. Figure 1 is the long time evolution of two vortex sheets. Initial average distance H is 0.2, initial phase difference α is 0,and $(\sigma_1, \sigma_2) = (1, -1)$. Figure 2 is the long time evolution of two vortex sheets. Initial average distance H is 0.2, initial phase difference α is 0.5,and $(\sigma_1, \sigma_2) = (1, -1)$. Both figures show complicated spiral structures. Since the vorticity is not of distinguished sign $(\sigma_1, \sigma_2) = (1, -1)$, such complexities seem to comply with what are predicted in [13].

4 Summary and acknowledgment

When the number of vortices exceeds a few thousands, the efficiency of PVM is satisfactory. PVM is a useful tool for particle simulations. However, if PVM environment consists of a lot of node computers, the rapid increase of data transfer may well make it impossible for us to execute fast computation. To make more efficient computation, we must choose algorithms with more effective data transfer. There is a possibility to combine the fast algorithms [4, 6] and PVM effectively.

Professor O. Pironneau advised us to try computations with adaptive increase of the number of vortices. This attempt is in progress and will be reported elsewhere. We thank him for the advice.

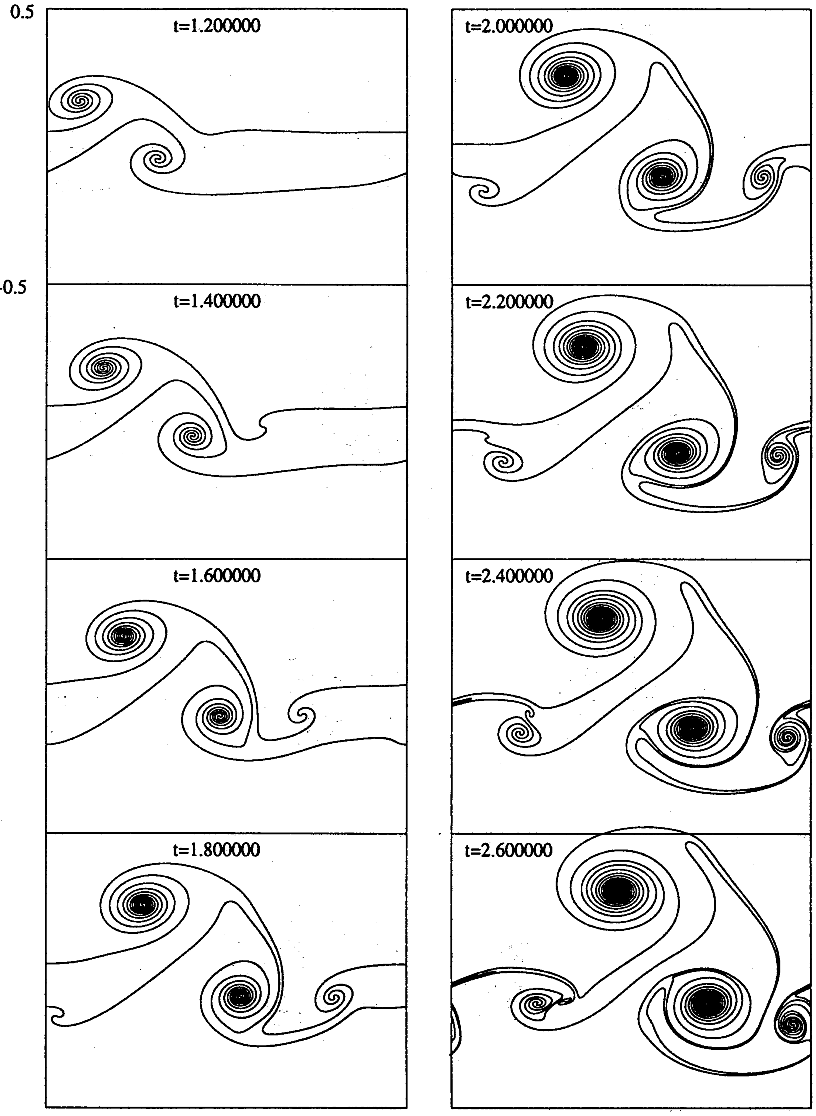

Figure 1: Long time evolution of two vortex sheets. The initial parameters are $H = 0.2$, $\alpha = 0.0$,and $(\sigma_1, \sigma_2) = (1, -1)$.

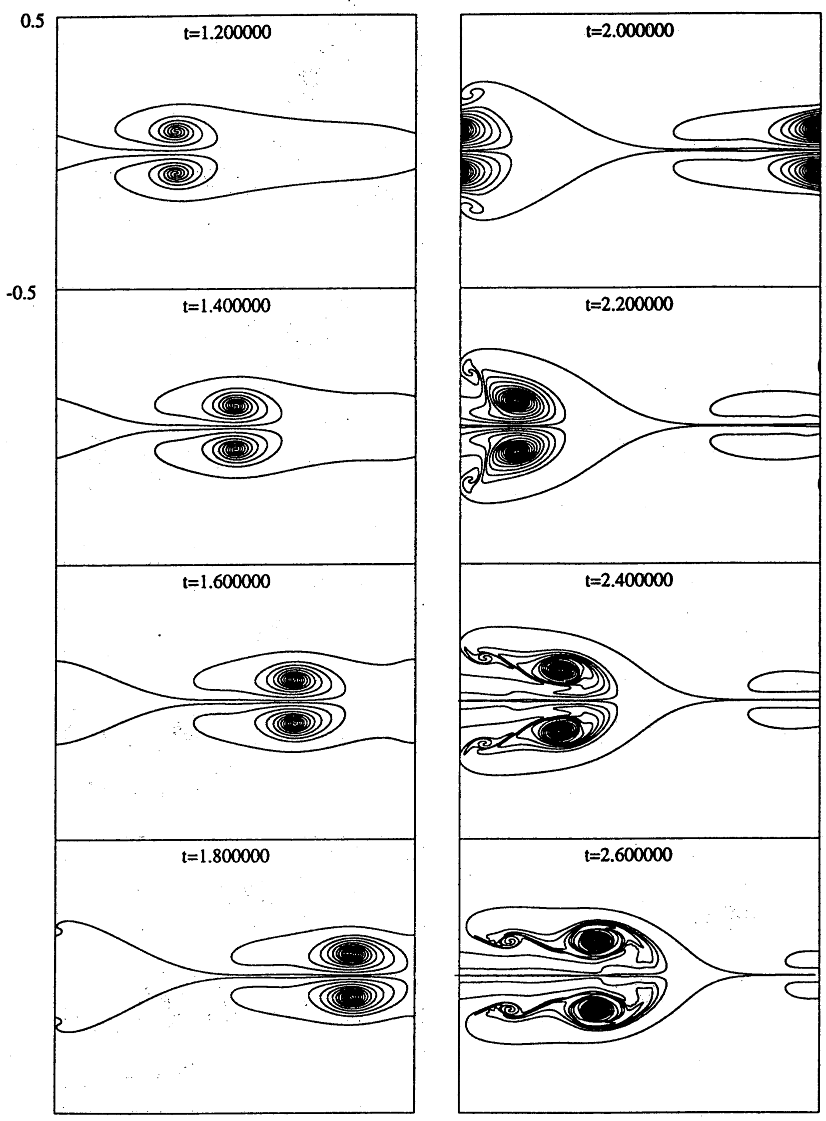

Figure 2: Long time evolution of two vortex sheets. The initial parameters are $H = 0.2$, $\alpha = 1.0$,and $(\sigma_1, \sigma_2) = (1, -1)$.

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