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<thead>
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<th>Title</th>
<th>An introduction to the piecewise algebraic curve</th>
</tr>
</thead>
<tbody>
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An introduction to the piecewise algebraic curve

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Abstract

In this paper, we review the recent development of our research on piecewise algebraic curves.

Keywords  piecewise algebraic curve, spline function, conformality condition.

1 Introduction

Let us recall the formulation of splines at first. Let $D$ be a bounded polygonal domain of $R^2$ and we partition $D$ with irreducible algebraic curves into cells $\Delta_i$, $i = 1, \ldots, N$. The partition is denoted by $\Delta$. A function $f(x)$ defined on $D$ is a spline function if $f(x) \in C^r(D)$ and $f(x)|_{\Delta_i} = p_i \in P_k$, which is expressed for short as follows:

$$f(x) \in s_n^r(D, \Delta).$$

In [1] R. H. Wang got the following basic results:

Let $\Delta_i$ and $\Delta_j$ be two adjacent cells with partitioning curve $l_{ij} = 0$. $f(x) \in c^r(\Delta_i \cup \Delta_j)$ if and only if

$$p_i - p_j = l_{ij}^{r+1} * q_{ij},$$

where $q_{ij} \in P_{k-(\mu+1)d_{ij}}$ is called a smooth cofactor of the partitioning curve $l_{ij}$ and $d_{ij}$ is the degree of $l_{ij}$.

Further, $f(x) \in s_n^r(D, \Delta)$, if and only if there exists a smooth cofactor on each interior partitioning curve and

$$\sum_{l_{ij} \in L_k} l_{ij}^{r+1} * q_{ij} \equiv 0.$$

where $L_k$ is the set of partitioning curves sharing the same interior vertex.

Algebraic curve $\Gamma$ is defined as follows

$$\Gamma = \{(x,y)|p(x,y) = 0, p \in P\}.$$

The so-called piecewise algebraic curve is defined by using the piecewise polynomial or polynomial spline function $s(x,y)$ to replace the polynomial $p(x,y)$ in $(\ast)$, we have

$$\Gamma = \{(x,y) s(x,y) = 0\}.$$
Let $\Gamma : s(x, y) = 0$ and $\gamma : t(x, y) = 0$ be two piecewise algebraic curves. $\gamma$ is called a local branch of $\Gamma$, if there exists a union of cells in $\Delta$

$$\Omega = \bigcup \delta_i$$

such that $\gamma$ is a branch of $\Gamma$ on $\Omega$.

Why do we have to study piecewise algebraic curves? Let us consider the following interpolation problem: Let $d = \text{dim} S_k^\mu(\Delta)$. How can we choose a set of knots $K = \{(x_i, y_i)\}_{i=1}^d$ such that for any given values $z_1, \ldots, z_d$, there exists a unique $s \in S_k^\mu(\Delta)$ satisfying

$$s(x_i, y_i) = z_i, i = 1, \ldots, d$$

According to the theory on bivariate spline mentioned above, the interpolation problem is a linear algebraic problem. Therefore there is a unique solution if and only if the linear homogeneous equations

$$s(x_i, y_i) = 0, i = 1, \ldots, d$$

has only a trivial solution, that is, if and only if $K$ does not lie on any piecewise algebraic curve $\Gamma : s(x, y) = 0, s \in S_k^\mu(\Delta)$. Denote by $p_i(x, y) \in P_k$ the polynomial defined by $s(x, y) \in S_k^\mu(\Delta)$ on $\Delta_i$. Because there is the possibility that

$$\{(x, y) | p_i(x, y) = s|_{\Delta_i} = 0\} \bigcap \overline{\Delta_i} = \emptyset$$

it is difficult to derive the piecewise algebraic curve.

2 Some Examples

Example 1 $D = R^2, \Delta : x = 0$, 2 cells

\[
\begin{array}{c|cc}
 & R^2_+ & R^2_- \\
\hline
x = 0 & \{ & \}
\end{array}
\]

\[
R^2_- = \{(x, y) \in R^2 : x < 0\} \\
R^2_+ = \{(x, y) \in R^2 : x \geq 0\}
\]

Define $s \in S^0_1(\Delta)$ as follows

$$s(x, y) = \begin{cases} 
  x - 1 & (x, y) \in R^2_- \\
  -x - 1 & (x, y) \in R^2_+
\end{cases}$$
The piecewise algebraic curve $\Gamma : s(x,y) = 0$ is empty.

Example 2  \[ s \in S^0_0(\Delta) \] is defined by
\[
s(x,y) = \begin{cases} 
  x - 1 & (x,y) \in R^2_-, \\
  3x - 1 & (x,y) \in R^2_+.
\end{cases}
\]
The piecewise algebraic curve $\Gamma : s(x,y) = 0$ is $s - \frac{1}{3} = 0$.

Example 3  \[ s \in S^0_1(\Delta) \] is defined as follows
\[
s(x,y) = \begin{cases} 
  x - y & (x,y) \in R^2_-, \\
  2x - y & (x,y) \in R^2_+.
\end{cases}
\]

Example 4  \[ D = R^2, \Delta : x = 0, y = 0 \] \[ s \in S^1_2(\Delta) \] is defined as follows
\[
s(x,y) = \begin{cases} 
  3x^2 + 3y^2 - 1 & (x,y) \in D_1, \\
  x^2 + 3y^2 - 1 & (x,y) \in D_2, \\
  x^2 - 1 & (x,y) \in D_3, \\
  3x^2 - 1 & (x,y) \in D_4.
\end{cases}
\]
Example 5 \[D = R^2, \Delta : x = 0\]

\[\Gamma : s(x, y) = xy - y^2 - yx_+ = 0, s \in S_2^0(\Delta)\]

\[\gamma : t(x, y) = x - y = 0, t \in S_1(\Delta)\]

\(\gamma\) is a local branch of \(\Gamma\) on \(R^2\).

\[\begin{array}{c}
\begin{array}{c}
\gamma \\
\Gamma \\
x = 0
\end{array}
\end{array}\]

3 Intersection of piecewise algebraic curves

Denote by \(\text{Inter}(\Gamma_1, \Gamma_2)\) the intersection set of the two piecewise algebraic curves \(\Gamma_1 : s_1(x, y) = 0\) and \(\Gamma_2 : s_2(x, y) = 0\). The number

\[BN(m_1, r_1; m_2, r_2) = \max\{\text{Card} \, \text{Inter}(\Gamma_1, \Gamma_2) < \infty; \Gamma_i : s_i(x, y) = 0, s_i \in S_{m_i}^r(\Delta), i = 1, 2\}\]

is called the Bezout number of \(S_{m_1}^{r_1}\) and \(S_{m_2}^{r_2}\). It is obvious that

\[BN(m_1, r_1; m_2, r_2) \leq N m_1 m_2,\]

where \(N\) is the number of cells in \(\Delta\).
X.Q. Shi and R.H. Wang discussed the Bezout number of $S_m^0(\Delta)$ and $S_n^0(\Delta)$. We find that the Bezout number $BN(m,0;n,0)$ depends on some property of the triangulation $\Delta$.

A triangulation $\Delta$ is called to be 2-signs, if one can mark $-1$ or $1$ on each vertex of $\Delta$ such that the numbers marked on 3 vertices of any cell in $\Delta$ are not the same one. A triangulation $\Delta$ is called to be 3-signs, if one can mark $-1, 0$ or $1$ on each vertex of $\Delta$ such that the numbers marked on 3 vertices of any cell in $\Delta$ are totally different.

A triangulation $\Delta$ is called to be even, if all of its interior vertices are even.

Let $v$ be an interior vertex of $\Delta$. Denote by $d(v)$ the number of boundary vertices of the star $st(v)$. $d(v)$ is called the degree of $v$. An interior vertex is called to be even(odd) if $d(v)$ is even(odd). A triangulation $\Delta$ is called to be even, if all of its interior vertices are even.

Proposition  The even triangulation of a simple connected domain is of 3-signs.


Theorem 1  If $\Delta$ is a triangulation of a simple connected domain, then

1°  $BN(1,0;1,0) = t$,  if $\Delta$ is even;
2°  $BN(1,0;1,0) \leq T - [(V_{odd} + 2)/3]$, otherwise,

where $T$ is the number of cells in $\Delta$, $V_{odd}$ is the number of odd vertices of $\Delta$, and $[x]$ denotes the maximum integer less than or equal to $x$.

Denote by $\delta = [v_1, v_2, v_3]$ the triangle with vertices $v_1$, $v_2$ and $v_3$. Let $f, g \in S_1^0(\Delta)$, and

$$f_i = f(v_i), g_i = g(v_i), i = 1, 2, 3.$$
Then the piecewise algebraic curves $f = 0$ and $g = 0$ can be represented on $\delta$ as follows
\[ f_{1}u_{1} + f_{2}u_{2} + f_{3}u_{3} = 0, \]  
(1)

and
\[ g_{1}u_{1} + g_{2}u_{2} + g_{3}u_{3} = 0 \]  
(2)
respectively, where \((u_{1}, u_{2}, u_{3})\) are the barycentric coordinates of a point \(v\) with respect to the triangle \(\delta\). Suppose that \((u_{1}^{*}, u_{2}^{*}, u_{3}^{*})\) are the barycentric coordinates of the intersection point of \(f_{1}u_{1} + f_{2}u_{2} + f_{3}u_{3} = 0\) and \(g_{1}u_{1} + g_{2}u_{2} + g_{3}u_{3} = 0\), then
\[ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = u_{1}^{*} \begin{pmatrix} f_{1} \\ g_{1} \end{pmatrix} + u_{2}^{*} \begin{pmatrix} f_{2} \\ g_{2} \end{pmatrix} + u_{3}^{*} \begin{pmatrix} f_{3} \\ g_{3} \end{pmatrix} \]  
(3)

\[ u_{i}^{*} \geq 0, \quad u_{1}^{*} + u_{2}^{*} + u_{3}^{*} = 1 \]

Lemma 1 Suppose (1) and (2) have only one intersection point \(p\). Then the point \(p\) is an interior point of the triangle \(\delta = [v_{1}, v_{2}, v_{3}]\), if and only if the origin \((0,0)\) is an interior point of triangle \(\delta^{*} = [\omega_{1}, \omega_{2}, \omega_{3}]\), where \(\omega_{i}, i = 1, 2, 3\) are defined by
\[ \omega_{i} = (f_{i}, g_{i}), \]

Note: \(u_{i}^{*} > 0, i = 1, 2, 3\).

Lemma 2 Let \(v\) be an interior vertex of the triangulation \(\Delta\), and \(f = 0, g = 0\) have only finite intersection points on \(st(v)\), where \(f, g \in S_{1}^{0}(\Delta)\). Then \(f = 0\) and \(g = 0\) have at most \(N\) intersection points:
\[ N = \begin{cases} \quad d(v), & \text{if } d(v) \text{ is even,} \\ d(v) - 1, & \text{if } d(v) \text{ is odd.} \end{cases} \]

Proof. Assume \(d(v) = 2m\), and \(v_{0}, v_{1}, \ldots, v_{2m}\) are the vertices of \(st(v)\), where \(v_{0} = v\). Let \(\omega_{0}, \omega_{1}\) and \(\omega_{2}\) be some points on \(R^{2}\) such that the origin is an interior point of the triangle \([\omega_{0}, \omega_{1}, \omega_{2}]\), for example,
\[ \omega_{0} = (-1, -1), \omega_{1} = (1, 0), \omega_{2} = (0, 1). \]

Now we define two piecewise linear curves \(f = 0, g = 0\) on \(st(v)\) by using the following values
\[ (f(v_{0}), g(v_{0})) = \omega_{0} = (-1, -1), \]
\[ (f(v_{2i-1}), g(v_{2i-1})) = \omega_{1} = (1, 0), (i = 1, \ldots, m) \]
\[ (f(v_{2i}), g(v_{2i})) = \omega_{2} = (0, 1). \]
Because the origin is inside the triangle $[\omega_0, \omega_1, \omega_2]$, By Lemma 1, $N = d(v)$.

Now assume $d(v) = 2m + 1$. For two piecewise linear curves $f = 0$ and $g = 0$, suppose

$$\omega'_0 = (f(v_0), f(v_0)), \omega'_i = (f(v_i), g(v_i)), \quad i = 1, \ldots, 2m + 1, v_0 = v$$

Lemma 1 shows that $f = 0$ and $g = 0$ have an intersection point inside the triangle $[v_0, v_i, v_{i+1}]$ if and only if the origin is inside the triangle $[\omega'_0, \omega'_i, \omega'_{i+1}] (i = 1, \ldots, 2m + 1, \omega'_1 = \omega'_{2m+2})$.

By joining the origin and $\omega'_0$, we obtain a straight line $L$. According to Lemma 1, two piecewise algebraic curves have a unique intersection point inside the triangle $[\omega'_0, \omega'_i, \omega'_{i+1}]$, if and only if the vertices $\omega'_i$ and $\omega'_{i+1}$ are located at two different sides of the straight line $L$. So it is obvious that

$$N \leq d(v) - 1,$$

where $d(v)$ is odd. Moreover, if we take

$$\omega'_0 = (-1, -1), \omega'_{2i} = (0, 1), \omega'_{2i+1} = (1, 0),$$

$i = 1, \ldots, m$, then $f = 0$ and $g = 0$ have $2m = d(v) - 1$ intersection points.

The proof of Theorem 1:
Let \( f, g \in S^0_1(\Delta) \) be defined by
\[
(f(v), g(v)) = \omega_i, v \in \Delta,
\]
where \( v \) is marked by \( i, i = -1, 0, 1 \),
\[
\omega_{-1} = (-1, -1), \omega_0 = (1, 0) \quad \text{and} \quad \omega_1 = (0, 1).
\]
According to Lemma 1, the piecewise linear curves \( f = 0 \) and \( g = 0 \) have just an intersection point in each triangle of \( \Delta \), i.e. if \( \Delta \) is even, then
\[
BN(1, 0; 1, 0) = T.
\]
Similarly, we can prove \( 2^o \) in Theorem 1.

Note: One can find some triangulations satisfying
\[
BN(1, 0; 1, 0) = T - [(\text{odd} + 2)/3].
\]

Lemma 3 If the triangulation \( \delta \) is of 2-signs, then
\[
BN(1, 0; 2, 0) = 2T.
\]
where \( T \) is the number of triangles in \( \Delta \).

Let \( f \in S^0_1(\Delta) \) be defined as follow
\[
f(v) = \begin{cases} 
1 & \text{if } v \in \Delta \text{ is marked by 1} \\
-1 & \text{if } v \in \Delta \text{ is marked by -1}
\end{cases}
\]
Assuming that \( \delta = [v_1, v_2, v_3] \in \Delta \) is a triangle, and \( f(u_1, u_2, u_3) = f|_{\delta} = u_1 + u_2 - u_3 \),
where \((u_1, u_2, u_3)\) are the barycentric coordinates of \((x, y) \in \delta\) with respect to \( \delta \).

Define \( g(x, y) \in S^0_2(\Delta) \) by using the following way
\[
g(x, y)|_{\delta} = g(u_1, u_2, u_3) = u_1^3 + u_2^3 + u_3^3 - 3/2(u_1u_2 + u_2u_3 + u_3u_1)
\]
for any \( \delta \in \Delta \).

It is no difficult to check that the piecewise algebraic curves \( f(u_1, u_2, u_3) = 0 \) and \( g(u_1, u_2, u_3) = 0 \) have two intersection points in \( \delta \). So
\[
BN(1, 0; 2, 0) = 2T.
\]

Lemma 4 If the triangulation \( \Delta \) is of 2-signs, then
\[
BN(1, 0; 3, 0) = 3T,
\]
where \( T \) is the number of triangles in \( \Delta \).

Proof. Let \( f \in S^0_1(\Delta) \) be defined as in Lemma 3, and let
\[
ge(u_1, u_2, u_3) = u_1^3 + u_2^3 + u_3^3 - au_1^2u_2 + u_2^2u_3 + u_3^2u_1 \\
+ bu_1u_2^2 + u_2u_3^2 + u_3u_1^2 + u_1u_2u_3.
\]
To find the conditions such that \( f(u_1, u_2, u_3) = 0 \) and \( g(u_1, u_2, u_3) = 0 \) have 3 intersection points in the triangle \( \delta \), take \( u_1 + u_2 = u_3 = \frac{1}{2} \) and consider

\[
g(u_1, u_2, \frac{1}{2}) = 0, \quad u_1 + u_2 = \frac{1}{2}.\]

If there are 3 real constants \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) such that

\[
g(u_1, u_2, \frac{1}{2}) = g(u_1, u_2, u_1 + u_2) = u_1^3 + u_2^3 + 2(u_1 + u_2)^3 + au_1^2u_2 + (u_1^2 + u_2^2)(u_1 + u_2) + bu_1u_2^2 + u_1u_2(u_1 + u_2) = 4u_1^3 + 4u_2^3 + (8 + a)u_1^2u_2 + (8 + b)u_1u_2^2 = 4(u_1 + \alpha_1u_2)(u_1 + \alpha_2u_2)(u_1 + \alpha_3u_2).
\]

then

\[
\begin{align*}
  a &= 4(\alpha_1 + \alpha_2 + \alpha_3) - 8, \\
  b &= 4(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1) - 8, \\
  \alpha_1\alpha_2\alpha_3 &= 1.
\end{align*}
\]

Choose \( \alpha_1, \alpha_2, \alpha_3 > 0 \) satisfying (7). One can obtain a special \( g(u_1, u_2, u_3) \) by (7) such that \( g(u_1, u_2, u_3) = 0 \) and \( f(u_1, u_2, u_3) = 0 \) have 3 intersection points in the interval \( u_1 \in (0, \frac{1}{2}), u_1 + u_2 = \frac{1}{2} \).

This shows that \( f(x, y) = 0 \) and \( g(x, y) = 0 \) have \( 3T \) intersection points, where \( g(x, y) \) is defined by

\[
g(x, y)|_\delta = g(u_1, u_2, u_3), \forall \delta \in \Delta.
\]

Theorem 2 If \( \Delta \) is a 2-signs triangulation, and \( \max\{m, n\} \geq 2 \), then the Bezout number of the spaces \( S^0_m(\Delta) \) and \( S^0_n(\Delta) \) is \( mnT \), i.e.

\[
BN(m, 0; n, 0) = mnT,
\]

where \( T \) is the number of triangles in \( \Delta \).

Proof. Let \( f(x, y) \) and \( g(x, y) \) be defined by (4) and (5), \( g(x, y) \) be defined as (8), and \( m \geq n \).

If \( m \geq 2 \) is even, we define

\[
f_1 = f^n, \quad g_1 = g^{m/2},
\]

then

\[
f_1(x, y) \in S^0_n(\Delta), \quad g_1(x, y) \in S^0_m(\Delta),
\]

moreover, the piecewise algebraic curves \( f_1(x, y) = 0, g_1(x, y) = 0 \) have \( mnT \) intersection points in \( \Delta \). This means that

\[
BN(m, 0; n, 0) = mnT.
\]

If \( m \geq 3 \) is odd, we define

\[
f_1 = f^n, \quad g_1 = g^{\frac{m-1}{2}},
\]
then theorem 2 can be also proved.

It seems that the Bezout number depends on whether the triangulation is of 2-signs or not. Based on many examples, however, the following conjecture may be right.

Conjecture Any triangulation is of 2-signs.

For the $c^1$ – smoothness cases

Let $s_i \in S_{m_i}^1(\Delta), i = 1, 2$ be two bivariate splines. We are going to consider the problem on intersection of two piecewise curves $s_1(x, y) = 0$ and $s_2(x, y) = 0$.

A partition $\Delta$ of $D$ is called a proper partition, if all angles of the intersection determined by any two adjacent edges of $\Delta$ are less than $\pi/2$.

By using the resultant of two bivariate splines $s_1$ and $s_2$ with respect to $\rho$ in the polar coordinate($\rho, \theta$), R.H.Wang and G.H.Zhao proved

Theorem 3 Let $\Gamma_i : s_i(x, y) = 0, i = 1, 2$ be two piecewise algebraic curves, where $s_i \in S_{m_i}^1(\Delta), i = 1, 2$. For any given interior vertex of $\Delta$, the cardinality $\Lambda$ of the intersection set $\delta$ of $\Gamma_1$ and $\Gamma_2$ on $st(v)$ is upper bounded by

$$n_i(m_1m_2 - 1) + 1$$

except that the cardinality of $\delta$ is infinite, where $n_i$ is the number of edges passing through $v$.

References