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<thead>
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<th>Title</th>
<th>An introduction to the piecewise algebraic curve</th>
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</thead>
<tbody>
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An introduction to the piecewise algebraic curve

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Abstract

In this paper, we review the recent development of our research on piecewise algebraic curves.

Keywords piecewise algebraic curve, spline function, conformality condition.

1 Introduction

Let us recall the formulation of splines at first. Let $D$ be a bounded polygonal domain of $\mathbb{R}^2$ and we partition $D$ with irreducible algebraic curves into cells $\Delta_i, i = 1, \ldots, N$. The partition is denoted by $\Delta$. A function $f(x)$ defined on $D$ is a spline function if $f(x) \in C^r(D)$ and $f(x)|_{\Delta_i} = p_i \in P_k$, which is expressed for short as follows:

$$f(x) \in S_n^r(D, \Delta).$$

In [1] R. H. Wang got the following basic results:

Let $\Delta_i$ and $\Delta_j$ be two adjacent cells with partitioning curve $l_{ij} = 0$. $f(x) \in c^r(\Delta_i \cup \Delta_j)$ if and only if

$$p_i - p_j = l_{ij}^{r+1} \ast q_{ij},$$

where $q_{ij} \in P_{k-(\mu+1)d_{ij}}$ is called a smooth cofactor of the partitioning curve $l_{ij}$ and $d_{ij}$ is the degree of $l_{ij}$.

Further, $f(x) \in S_n^r(D, \Delta)$, if and only if there exists a smooth cofactor on each interior partitioning curve and

$$\sum_{l_{ij} \in L_k} l_{ij}^{r+1} \ast q_{ij} \equiv 0.$$ 

where $L_k$ is the set of partitioning curves sharing the same interior vertex.

Algebraic curve $\Gamma$ is defined as follows

$$(*) \quad \Gamma = \{(x, y)|p(x, y) = 0, p \in P\}.$$ 

The so-called piecewise algebraic curve is defined by using the piecewise polynomial or polynomial spline function $s(x, y)$ to replace the polynomial $p(x, y)$ in $(*)$, we have

$$\Gamma = \{(x, y) \ s(x, y) = 0\}. $$
Let $\Gamma: s(x, y) = 0$ and $\gamma: t(x, y) = 0$ be two piecewise algebraic curves. $\gamma$ is called a local branch of $\Gamma$, if there exists a union of cells in $\Delta$ 

$$\Omega = \bigcup \delta_i$$

such that $\gamma$ is a branch of $\Gamma$ on $\Omega$.

Why do we have to study piecewise algebraic curves? Let us consider the following interpolation problem: Let $d = dim S^\mu_k(\Delta)$. How can we choose a set of knots $K = \{(x_i, y_i)\}_{i=1}^d$ such that for any given values $z_1, \ldots, z_d$, there exists a unique $s \in S^\mu_k(\Delta)$ satisfying

$$s(x_i, y_i) = z_i, i = 1, \ldots, d$$

According to the theory on bivariate spline mentioned above, the interpolation problem is a linear algebraic problem. Therefore there is a unique solution if and only if the linear homogeneous equations

$$s(x_i, y_i) = 0, i = 1, \ldots, d$$

has only a trivial solution, that is, if and only if $K$ does not lie on any piecewise algebraic curve $\Gamma: s(x, y) = 0, s \in S^\mu_k(\Delta)$. Denote by $p_i(x, y) \in P_k$ the polynomial defined by $s(x, y) \in S^\mu_k(\Delta)$ on $\Delta_i$. Because there is the possibility that

$$\{(x, y) | p_i(x, y) = s|\Delta_i = 0 \} \cap \overline{\Delta_i} = \emptyset$$

it is difficult to derive the piecewise algebraic curve.

2 Some Examples

**Example 1** $D = R^2, \Delta: x = 0, 2$ cells

$$R^2_- \quad \quad R^2_+$$

$$x = 0$$

$$R^2_- = \{(x,y) \in R^2 : x < 0\}$$

$$R^2_+ = \{(x,y) \in R^2 : x \geq 0\}$$

Define $s \in S^0_1(\Delta)$ as follows

$$s(x, y) = \begin{cases} 
  x - 1 & (x, y) \in R^2_-, \\
  -x - 1 & (x, y) \in R^2_+.
\end{cases}$$
The piecewise algebraic curve $\Gamma : s(x,y) = 0$ is empty.

Example 2 \( s \in S_0^0(\Delta) \) is defined by

\[
s(x, y) = \begin{cases} 
  x - 1 & (x,y) \in R_2^-; \\
  3x - 1 & (x,y) \in R_2^+.
\end{cases}
\]

The piecewise algebraic curve $\Gamma : s(x,y) = 0$ is $s - \frac{1}{3} = 0$.

Example 3 \( s \in S_1^0(\Delta) \) is defined as follows

\[
s(x, y) = \begin{cases} 
  x - y & (x,y) \in R_2^-, \\
  2x - y & (x,y) \in R_2^+.
\end{cases}
\]

Example 4 \( D = R^2, \Delta : x = 0, y = 0 \), \( s \in S_1^1(\Delta) \) is defined as follows

\[
s(x, y) = \begin{cases} 
  3x^2 + 3y^2 - 1 & (x,y) \in D_1, \\
  x^2 + 3y^2 - 1 & (x,y) \in D_2, \\
  x^2 - 1 & (x,y) \in D_3, \\
  3x^2 - 1 & (x,y) \in D_4.
\end{cases}
\]
$D = R^2, \Delta : x = 0$

$\Gamma : s(x, y) = xy - y^2 - yx+ = 0, s \in S^0_2(\Delta)$

$\gamma : t(x, y) = x - y = 0, t \in S^1_1(\Delta)$

$\gamma$ is a local branch of $\Gamma$ on $R^2$.

3 Intersection of piecewise algebraic curves

Denote by $\text{Inter}(\Gamma_1, \Gamma_2)$ the intersection set of the two piecewise algebraic curves $\Gamma_1 : s_1(x, y) = 0$ and $\Gamma_2 : s_2(x, y) = 0$. The number

$$BN(m_1, r_1; m_2, r_2)$$

$$:= \max \{\text{Card Inter}(\Gamma_1, \Gamma_2) < \infty; \Gamma_i : s_i(x, y) = 0, s_i \in S^i_{m_i}(\Delta), i = 1, 2\}$$

is called the Bezout number of $S^1_{m_1}$ and $S^1_{m_2}$. It is obvious that

$$BN(m_1, r_1; m_2, r_2) \leq Nm_1m_2,$$

where $N$ is the number of cells in $\Delta$. 
X.Q. Shi and R.H. Wang discussed the Bezout number of $S_m^0(\Delta)$ and $S_n^0(\Delta)$. We find that the Bezout number $BN(m,0;n,0)$ depends on some property of the triangulation $\Delta$.

A triangulation $\Delta$ is called to be 2-signs, if one can mark $-1$ or $1$ on each vertex of $\Delta$ such that the numbers marked on 3 vertices of any cell in $\Delta$ are not the same one. A triangulation $\Delta$ is called to be 3-signs, if one can mark $-1$, $0$, or $1$ on each vertex of $\Delta$ such that the numbers marked on 3 vertices of any cell in $\Delta$ are totally different.

Let $v$ be an interior vertex of $\Delta$. Denote by $d(v)$ the number of boundary vertices of the star $st(v)$. $d(v)$ is called the degree of $v$. An interior vertex is called to be even(odd) if $d(v)$ is even(odd). A triangulation $\Delta$ is called to be even, if all of its interior vertices are even.

Proposition  The even triangulation of a simple connected domain is of 3-signs. X.Q. Shi and R.H. Wang proved

Theorem 1  If $\Delta$ is a triangulation of a simple connected domain, then

1° $BN(1,0;1,0) = t$,  if $\Delta$ is even;

2° $BN(1,0;1,0) \leq T - [(V_{odd} + 2)/3]$,  otherwise,

where $T$ is the number of cells in $\Delta$, $V_{odd}$ is the number of odd vertices of $\Delta$, and $[x]$ denotes the maximum integer less than or equal to $x$.

Denote by $\delta = [v_1, v_2, v_3]$ the triangle with vertices $v_1, v_2$ and $v_3$. Let $f, g \in S_1^0(\Delta)$, and

$$f_i = f(v_i), g_i = g(v_i), i = 1, 2, 3.$$
Then the piecewise algebraic curves $f = 0$ and $g = 0$ can be represented on $\delta$ as follows

$$f_1u_1 + f_2u_2 + f_3u_3 = 0,$$

and

$$g_1u_1 + g_2u_2 + g_3u_3 = 0$$

respectively, where $(u_1, u_2, u_3)$ are the barycentric coordinates of a point $v$ with respect to the triangle $\delta$. Suppose that $(u_1^*, u_2^*, u_3^*)$ are the barycentric coordinates of the intersection point of $f_1u_1 + f_2u_2 + f_3u_3 = 0$ and $g_1u_1 + g_2u_2 + g_3u_3 = 0$, then

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = u_1^* \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} + u_2^* \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} + u_3^* \begin{pmatrix} f_3 \\ g_3 \end{pmatrix}$$

$$u_i^* \geq 0, u_1^* + u_2^* + u_3^* = 1$$

Lemma 1 Suppose (1) and (2) have only one intersection point $p$. Then the point $p$ is an interior point of the triangle $\delta = [v_1, v_2, v_3]$ if and only if the origin $(0,0)$ is an interior point of triangle $\delta^* = [\omega_1, \omega_2, \omega_3]$, where $\omega_i, i = 1, 2, 3$ are defined by

$$\omega_i = (f_i, g_i),$$

Note: $u_i^* > 0, i = 1, 2, 3.$

Lemma 2 Let $v$ be an interior vertex of the triangulation $\Delta$, and $f = 0, g = 0$ have only finite intersection points on $st(v)$, where $f, g \in S^0_1(\Delta)$. Then $f = 0$ and $g = 0$ have at most $N$ intersection points:

$$N = \begin{cases} d(v), & \text{if } d(v) \text{ is even}, \\ d(v) - 1, & \text{if } d(v) \text{ is odd}. \end{cases}$$

Proof Assume $d(v) = 2m$, and $v_0, v_1, ..., v_{2m}$ are the vertices of $st(v)$, where $v_0 = v$. Let $\omega_0, \omega_1$ and $\omega_2$ be some points on $R^2$ such that the origin is an interior point of the triangle $[\omega_0, \omega_1, \omega_2]$, for example,

$$\omega_0 = (-1, -1), \omega_1 = (1,0), \omega_2 = (0,1).$$

Now we define two piecewise linear curves $f = 0, g = 0$ on $st(v)$ by using the following values

$$(f(v_0), g(v_0)) = \omega_0 = (-1, -1),$$

$$(f(v_{2i-1}), g(v_{2i-1})) = \omega_1 = (1,0), (i = 1, ..., m)$$

$$(f(v_{2i}), g(v_{2i})) = \omega_2 = (0,1).$$
Because the origin is inside the triangle $[\omega_0, \omega_1, \omega_2]$, By Lemma 1, $N = d(v)$.

Now assume $d(v) = 2m + 1$. For two piecewise linear curves $f = 0$ and $g = 0$, suppose

$$\omega'_0 = (f(v_0), f(v_0)), \omega'_i = (f(v_i), g(v_i)),$$

$$i = 1, \ldots, 2m + 1, v_0 = v$$

Lemma 1 shows that $f = 0$ and $g = 0$ have an intersection point inside the triangle $[v_0, v_i, v_{i+1}]$ if and only if the origin is inside the triangle $[\omega'_0, \omega'_i, \omega'_{i+1}]$ ($i = 1, \ldots, 2m+1, \omega'_1 = \omega'_{2m+2}$).

By joining the origin and $\omega'_0$, we obtain a straight line $L$. According to Lemma 1, two piecewise algebraic curves have a unique intersection point inside the triangle $[\omega'_0, \omega'_i, \omega'_{i+1}]$, if and only if the vertices $\omega'_i$ and $\omega'_{i+1}$ are located at two different sides of the straight line $L$. So it is obvious that

$$N \leq d(v) - 1,$$

where $d(v)$ is odd. Moreover, if we take

$$\omega'_0 = (-1, -1), \omega'_{2i} = (0, 1), \omega'_{2i+1} = (1, 0),$$

$$i = 1, \ldots, m$$

then $f = 0$ and $g = 0$ have $2m = d(v) - 1$ intersection points.

The proof of Theorem 1:
Let $f, g \in S_1^0(\Delta)$ be defined by

$$(f(v), g(v)) = \omega_i, v \in \Delta,$$

where $v$ is marked by $i, i = -1, 0, 1$,

$$\omega_{-1} = (-1, -1), \omega_0 = (1, 0) \text{ and } \omega_1 = (0, 1).$$

According to Lemma 1, the piecewise linear curves $f = 0$ and $g = 0$ have just an intersection point in each triangle of $\Delta$, i.e. if $\Delta$ is even, then

$$BN(1, 0; 1, 0) = T.$$ 

Similarly, we can prove $2^v$ in Theorem 1.

Note: One can find some triangulations satisfying

$$BN(1, 0; 1, 0) = T - [(V_{odd} + 2)/3].$$

Lemma 3 If the triangulation $\delta$ is of 2-signs, then

$$BN(1, 0; 2, 0) = 2T.$$

where $T$ is the number of triangles in $\Delta$.

Let $f \in S_1^0(\Delta)$ be defined as follow

$$f(v) = \begin{cases} 
1 & \text{if } v \in \Delta \text{ is marked by } 1 \\
-1 & \text{if } v \in \Delta \text{ is marked by } -1
\end{cases} \quad (4)$$

Assuming that $\delta = [v_1, v_2, v_3] \in \Delta$ is a triangle, and $f(u_1, u_2, u_3) = f|_\delta = u_1 + u_2 - u_3$, where $(u_1, u_2, u_3)$ are the barycentric coordinates of $(x, y) \in \delta$ with respect to $\delta$.

Define $g(x, y) \in S_2^0(\Delta)$ by using the following way

$$g(x, y)|_\delta = g(u_1, u_2, u_3)$$

$$= u_1^2 + u_2^2 + u_3^3 - \frac{3}{2}(u_1 u_2 + u_2 u_3 + u_3 u_1) \quad (5)$$

for any $\delta \in \Delta$.

It is no difficult to check that the piecewise algebraic curves $f(u_1, u_2, u_3) = 0$ and $g(u_1, u_2, u_3) = 0$ have two intersection points in $\delta$. So

$$BN(1, 0; 2, 0) = 2T.$$ 

Lemma 4 If the triangulation $\Delta$ is of 2-signs, then

$$BN(1, 0; 3, 0) = 3T,$$

where $T$ is the number of triangles in $\Delta$.

Proof. Let $f \in S_1^0(\Delta)$ be defined as in Lemma 3, and let

$$g(u_1, u_2, u_3) = u_1^3 + u_2^3 + u_3^3 + au_1^2 u_2 + u_2^2 u_3 + u_3^2 u_1$$

$$+ bu_1 u_2^2 + u_2 u_3^2 + u_3^2 u_1 + u_1 u_2 u_3. \quad (6)$$
To find the conditions such that \( f(u_1, u_2, u_3) = 0 \) and \( g(u_1, u_2, u_3) = 0 \) have 3 intersection points in the triangle \( \delta \), take \( u_1 + u_2 = u_3 = \frac{1}{2} \) and consider

\[
g(u_1, u_2, \frac{1}{2}) = 0, u_1 + u_2 = \frac{1}{2},
\]

If there are 3 real constants \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) such that

\[
g(u_1, u_2, \frac{1}{2}) = g(u_1, u_2, u_1 + u_2)
\]

\[
= u_1^3 + u_2^3 + 2(u_1 + u_2)^3 + au_1^2 u_2 + (u_1^2 + u_2^2)(u_1 + u_2) + bu_1 u_2^2 + u_1 u_2 (u_1 + u_2)
\]

\[
= 4u_1^3 + 4u_2^3 + (8 + a)u_1^2 u_2 + (8 + b)u_1 u_2^2
\]

\[
= 4(u_1 + \alpha_1 u_2)(u_1 + \alpha_2 u_2)(u_1 + \alpha_3 u_2).
\]

then

\[
\begin{align*}
a &= 4(\alpha_1 + \alpha_2 + \alpha_3) - 8, \\
b &= 4(\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1) - 8, \\
\alpha_1 \alpha_2 \alpha_3 &= 1.
\end{align*}
\]

Choose \( \alpha_1, \alpha_2, \alpha_3 > 0 \) satisfying (7). one can obtain a special \( g(u_1, u_2, u_3) \) by (7) such that \( g(u_1, u_2, u_3) = 0 \) and \( f(u_1, u_2, u_3) = 0 \) have 3 intersection points in the interval \( u_1 \in (0, \frac{1}{2}), u_1 + u_2 = \frac{1}{2} \).

This shows that \( f(x, y) = 0 \) and \( \overline{g}(x, y) = 0 \) have 3T intersection points, where \( \overline{g}(x, y) \) is defined by

\[
\overline{g}(x, y) = g(u_1, u_2, u_3), \forall \delta \in \Delta.
\]

**Theorem 2** If \( \Delta \) is a 2-signs triangulation, and \( \max \{ m, n \} \geq 2 \), then the Bezout number of the spaces \( S^0_m(\Delta) \) and \( S^0_n(\Delta) \) is \( mnT \), i.e.

\[
BN(m, 0; n, 0) = nmT,
\]

where \( T \) is the number of triangles in \( \Delta \).

**Proof.** Let \( f(x, y) \) and \( g(x, y) \) be defined by (4) and (5), \( \overline{g}(x, y) \) be defined as (8), and \( m \geq n \).

If \( m \geq 2 \) is even, we define

\[
f_1 = f^n, g_1 = g^{m/2},
\]

then

\[
f_1(x, y) \in S^0_n(\Delta), g_1(x, y) \in S^0_m(\Delta),
\]

moreover, the piecewise algebraic curves \( f_1(x, y) = 0, g_1(x, y) = 0 \) have \( mnT \) intersection points in \( \Delta \). This means that

\[
BN(m, 0; n, 0) = nmT.
\]

If \( m \geq 3 \) is odd, we define

\[
f_1 = f^n, g_1 = g^{\frac{m-3}{2}},
\]
then theorem 2 can be also proved.

It seems that the Bezout number depends on whether the triangulation is of 2-signs or not. Based on many examples, however, the following conjecture may be right.

Conjecture Any triangulation is of 2-signs.

For the $c^1$ – smoothness cases

Let $s_i \in S_{m_i}^1(\Delta), i = 1, 2$ be two bivariate splines. We are going to consider the problem on intersection of two piecewise curves $s_1(x, y) = 0$ and $s_2(x, y) = 0$.

A partition $\Delta$ of $D$ is called a proper partition, if all angles of the intersection determined by any two adjacent edges of $\Delta$ are less than $\pi/2$.

By using the resultant of two bivariate splines $s_1$ and $s_2$ with respect to $\rho$ in the polar coordinate($\rho, \theta$), R.H.Wang and G.H.Zhao proved

Theorem 3 Let $\Gamma_i : s_i(x, y) = 0, i = 1, 2$ be two piecewise algebraic curves, where $s_i \in S_{m_i}^1(\Delta), i = 1, 2$. For any given interior vertex of $\Delta$, the cardinality $\Lambda$ of the intersection set $\delta$ of $\Gamma_1$ and $\Gamma_2$ on $st(v)$ is upper bounded by

$$n_i(m_1m_2 - 1) + 1$$

except that the cardinality of $\delta$ is infinite, where $n_i$ is the number of edges passing through $v$.

References


