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Variable Coefficients A-stable Explicit Runge-Kutta Methods (II)

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§1. Introduction

We study the numerical method for solving the stiff initial value problem

$$\dot{y} = f(x, y), y(x_0) = y_0.$$

The method which we propose is the variable coefficients Runge-Kutta methods (abb;R-K methods);

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{i=1}^{r} b_i k_i, \\ k_1 &= f(x_n, y_n), \\ k_i &= f(x_n + c_i h, y_n + h\sum_{j=1}^{i-1} a_{ij}k_j), \\ c_i &= \sum_{j=1}^{i-1} a_{ij}, \qquad (i = 2, .., r). \end{aligned} \tag{1.1}$$

§2. Derivation of Methods for a single equation of the first order

In this section, we consider two-stage first order Runge-Kutta formulae.

1-st order condition of (1.1) is

$$\sum_i b_i = 1. \tag{2.1}$$

Let us now apply the 2-stage first order R-K methods (1.1) to the test function

$$\dot{y} = \lambda y, \quad Re(\lambda) < 0, \tag{2.2}$$

we have

$$y_{n+1} = (1 + z + b_2 a_{21} z^2) y_n\, , (z = h\lambda). \tag{2.3}$$

If we set the coefficient $b_2 a_{21}$ in the form

$$b_2 a_{21} = \frac{1}{1 - z + z^2}, \tag{2.4}$$

then (2.3) reduces to

$$y_{n+1} = \frac{1}{1 - z + z^2} y_n. \tag{2.5}$$

which is an A-stable algorithm.

In determing the coefficients from the order condition (2.1), we set b_2 as a free parameter defined in the form

$$b_2 a_{21} = \frac{y_n}{y_n - 3hk_1 + 2hk_2}. \tag{2.6}$$

It is easily seen that the value of (2.6) applied to the test function (2.2) is the same as that of (2.4). Solving order condition (2.1) with (2.6), we have the coefficients

$$b_2 = \frac{y_n}{a_{21}(y_n - 3hk_1 + 2hk_2)}, b_1 = 1 - b_2,\ a_{21} : free \quad parameter. \tag{2.7}$$

§3. Nnmerical integration for s-systems of equations of the first order

We consider numerical integration for s-systems differential equation;

$$\dot{Y} = F(Y), \tag{3.1}$$

with

$$Y = ({}^1y, {}^2y, ..., {}^sy), F(Y) = ({}^1f(Y), {}^2f(Y), ..., {}^sf(Y)).$$

We consider r-stage explicit R-K methods for s-systems equation (3.1)

$$\begin{aligned} {}^ly_{(n+1)} &= {}^ly_n + h\sum_{i=1}^{r} {}^lb_i {}^lk_i, \\ {}^lk_1 &= {}^lf(x_n, {}^1y_n, {}^2y_n, .., {}^sy_n), \\ {}^lk_i &= {}^lf(x_n + c_ih, {}^1y_n + \sum_{i+1}^{j-1} a_{i,j}{}^1k_i, .., {}^qy_n + \sum_{i=1}^{j-1} a_{ij}{}^qk_j),\ (l = 1, 2, .., s), \end{aligned} \tag{3.2}$$

${}^lb_i$: varies in each steps and with the component-wise.

Introducing the vector notations

$$Y_n = ({}^1y_n, {}^2y_n, ..., {}^sy_n), B_i = diag[{}^1b_i, {}^2b_i, .., {}^sb_i], K_i = [{}^1k_i, {}^2k_i, ...{}^sk_i]\ (i = 1, 2, .., s-1),$$

we may write (3.2) in the form;

$$Y_{n+1} = Y_n + \sum_{i=1}^{s-1} B_iK_i. \tag{3.3}$$

As the same reason stated in §2, we may set the coefficients ${}^ib_2a_{21}$ of i - th component in the form

$${}^ib_2a_{21} = \frac{{}^iy_n}{{}^iy_n - 3h{}^ik_1 + 2h{}^ik_2}. \tag{3.4}$$

From order condition (2.1) with (3.4), we have the coefficients of i-th component in the forms

$${}^ib_2a_{21} = \frac{{}^iy_n}{{}^iy_n - 3h{}^ik_1 + 2h{}^ik_2}.\quad {}^ib_1 = 1 - {}^ib_2,\ a_{21}; free\quad parameter. \tag{3.5}$$

The stability results for system equation based on the test function (2.2) can not to be watertight when applied to the variable coefficient methods, so it is essential to use $\dot{Y} = AY$ as test function, and not $\dot{y} = \lambda y$. We state by attacking the simple matrix A defined by

$$\dot{Z} = \Lambda Z \quad with \quad \Lambda = HAH^{-1} \quad for\ some\ H \quad and \quad \Lambda = diag\{\lambda_1, .., \lambda_s\}. \tag{3.6}$$

$$\begin{array}{ccc} \dot{Y} = AY & \xrightarrow{\text{Z=HY}} & \dot{Z} = \Lambda Z \\ \downarrow & \text{numerical integration} & \downarrow \\ Y_n & \longrightarrow & Z_n \end{array}$$

Cleary the numerical processes $\{Z_n\}$ derived by (3.2) with (3.5) is stable on the above diagram, the transformation $\tilde{Z}_n = HY_n$ does not equivalent to the numerical processes Z_n. We study the stability of numerical processes $\tilde{Z}_n$ on the most simple differential qeuation:

$$\dot{Y} = AY \quad with \quad A = \begin{pmatrix} p & q \\ q & p \end{pmatrix}, \qquad (p + q < 0, p - q < 0). \tag{3.7}$$

Integrating the differential equatioh (3.7), we have

$$Y_{n+1} = (I + hA + h^2CA^2)Y_n, \tag{3.8}$$

with

$$ {}^1y_{n+1} = {}^1y_n + h(p\,{}^1y_n + q\,{}^2y_n) + h^2({}^1b_2a_{21}(p\,{}^1f_n + q\,{}^2f_n)), $$
$$ {}^2y_{n+1} = {}^2y_n + h(q\,{}^1y_n + p\,{}^2y_n) + h^2({}^2b_2a_{21}(q\,{}^1f_n + p\,{}^2f_n)), $$
$$ {}^1f_n = p\,{}^1y_n + q\,{}^2y_n,\ {}^2f_n = q\,{}^1y_n + p\,{}^2y_n, $$
$$ {}^1b_2a_{21} = \frac{{}^1y_n}{({}^1y_n - 3h\,{}^1k_1 + 2h\,{}^1k_2)}, $$
$$ {}^2b_2a_{21} = \frac{{}^2y_n}{({}^2y_n - 3h\,{}^2k_1 + 2h\,{}^2k_2)}, $$

From (3.8), we have

$$ \begin{aligned} \tilde{Z}_{n+1} &= \{I + h\Lambda + h^2(I - h\Lambda + (h\Lambda)^2)^{-1})\Lambda^2\}\tilde{Z}_n \\ &+ h^2H\{C - (I - hA + (hA)^2)\}A^2H^{-1}\tilde{Z}_n, \end{aligned} \tag{3.9} $$

Setting K by

$$ \begin{aligned} K &= I + h\Lambda + h^2\{I - h\Lambda + (h\Lambda)^2\}^{-1} \\ &= \begin{pmatrix} 1 - \lambda_2 + \lambda_2^2 & 0 \\ 0 & 1 - \lambda_1 + \lambda_1^2 \end{pmatrix}^{-1}, \end{aligned} $$
$$ \lambda_1 = h(p+q), \lambda_2 = h(p-q) $$

and Q_n by

$$ \begin{aligned} Q_n &= H\{C - (I - hA + (hA)^2\}^{-1}A^2H^{-1} \\ &= \{\frac{1}{2}\begin{pmatrix} (s_1 + s_2) & -(s_2 - s_1) \\ -(s_2 - s_1) & (s_1 + s_2) \end{pmatrix} - (I - h\Lambda + (h\Lambda)^2)^{-1}\}\Lambda^2, \end{aligned} $$
$$ (s_1 = {}^1b_2a_{21}, s_2 = {}^2b_2a_{21}). $$

We may write (3.9) in the form

$$ \tilde{Z}_{n+1} = (K + Q_n)Z_n, \quad \tilde{Z}_1 = \Theta(\tau),\ (n = 1, 2, \ldots). \tag{3.10} $$

Using the standard thchnique, we obtain the following Lemmas.

Lemma 1

$$ \tilde{Z}_{n+1} = K^n\Theta(\tau) + \sum_{\nu=0}^{n-1} K^{n-(\nu+1)}Q_\nu\tilde{Z}_\nu. $$

Lemma 2

$$ \tilde{Z}_{n+1} = K^n\Theta(\tau) + \Pi_{\nu=0}^{n-1}S_{n-\nu}Q_0\Theta(\tau) + \sum_{\mu=1}^{n}(\Pi_{\nu=0}^{n}S_{n-\nu})(\Pi_{\nu=0}^{\mu}S_\nu^{-1})Q_\mu K^\mu $$

with $S_\nu = K + Q_\nu$,

where $S_\nu = S_\nu(r_1, r_2)$ and $S_\nu(r_1, r_2) = K + Q(r_1(\nu), r_2(\nu))$ with

$$ Q_\nu(r_1(\nu), r_2(\nu)) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, $$

$$ c_{11} = p_1 + \frac{1}{1 - \lambda_1 + \lambda_1^2},\ c_{22} = p_2 + \frac{1}{1 - \lambda_2 + \lambda_2^2}, $$

$$c_{12} = c_{21} = \frac{1}{2}(p_1 - p_2),$$

with $p_1 = -h * (p + qr_1)/\{1 - h(p + qr_1) + h^2(p^2 + q^2) + 2pqh^2 r_1\}$,

$$p_2 = -h * (p + qr_2)/\{1 - h(p + qr_2) + h^2(p^2 + q^2) + 2pqh^2 r_2\}, r_1(\nu) = {}^2y_\nu/{}^1y_\nu, r_2(\nu) = {}^1y_\nu/{}^2y_\nu \quad (3.11)$$

Using Lemma [1,2], we have

Theorem 1

If there exists the constant T such that $||\Pi_{l=m}^{n} S(r_1(l), r_2(l))|| \leq T$ for all m,n($m \leq n$), then $\{\tilde{Z}_n\}$ *stable.*

Numerical Example

Setting $r_1 = r_2 = 1$ or $r_1 = r_2 = -1$ in (3.11) , we compute the following differential equation $\dot{Y} = AY$ with

$$A = \begin{pmatrix} -550.5 & q \\ q & -500,5 \end{pmatrix}, Y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

The datas are the absolute error of numerical solution with step size h =1

x		1	3	
q=0	${}^1y_n$	0	0	
	${}^2y_n$	0.796E-5	0,126E-6	
q=1	${}^1y_n$	0.210E+1	0.158E-4	
	${}^2y_n$	0.210E+1	0.158E-4	
x		1	3	5
q=50	${}^1y_n$	0.121E+3	0.487E-1	0.195E-4
	${}^2y_n$	0.121E+3	0.487E-1	0.195E-4
q=499.5	${}^1y_n$	0.332E+7	0.366E+6	0.540E+1
	${}^2y_n$	0.332E+7	0.366E+6	0.540E+1

λ_1 = -500.5 + q, λ_2 = -500.5 - q.

References

[1] M. Nakashima, Variable Coefficients A-stable Explicit Runge-Kutta Methods
[2] P.j. van der Houwen, Construction of Integration Formulas for Initial Value Problem.