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Variable Coefficients A-stable Explicit
Runge-Kutta Methods (II)
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§1. Introduction
We study the numerical method for solving the stiff initial value problem
\[
\dot{y} = f(x, y), \quad y(x_0) = y_0.
\]
The method which we propose is the variable coefficients Runge-Kutta methods (abbr: R-K methods);
\[
y_{n+1} = y_n + h \sum_{i=1}^{r} b_i k_i,
\]
\[
k_1 = f(x_n, y_n),
\]
\[
k_i = f(x_n + c_i h, y_n + h \sum_{j=1}^{i-1} a_{ij} k_j),
\]
\[
c_i = \sum_{j=1}^{i-1} a_{ij}, \quad (i = 2, \ldots, r).
\]

§2. Derivation of Methods for a single equation of the first order
In this section, we consider two-stage first order Runge-Kutta formulae.

1-st order condition of (1.1) is
\[
\sum_{i} b_i = 1. \quad (2.1)
\]
Let us now apply the 2-stage first order R-K methods (1.1) to the test function
\[
\dot{y} = \lambda y, \quad Re(\lambda) < 0,
\]
we have
\[
y_{n+1} = (1 + z + b_2 a_{21} z^2) y_n, \quad (z = h \lambda).
\]
If we set the coefficient \( b_2 a_{21} \) in the form
\[
b_2 a_{21} = \frac{1}{1 - z + z^2}, \quad (2.4)
\]
then (2.3) reduces to
\[
y_{n+1} = \frac{1}{1 - z + z^2} y_n. \quad (2.5)
\]
which is an A-stable algorithm.

In determining the coefficients from the order condition (2.1), we set \( b_2 \) as a free parameter defined in the form
\[
b_2 a_{21} = \frac{y_n}{y_n - 3h k_1 + 2h k_2}. \quad (2.6)
\]
It is easily seen that the value of (2.6) applied to the test function (2.2) is the same as that of (2.4). Solving order condition (2.1) with (2.6), we have the coefficients
\[
b_2 = \frac{y_n}{a_{21}(y_n - 3h k_1 + 2h k_2)}, \quad b_1 = 1 - b_2, \quad a_{21} : \text{free parameter}. \quad (2.7)
\]
§3. Numerical integration for s-systems of equations of the first order

We consider numerical integration for s-systems differential equation:

\[ \dot{Y} = F(Y), \]  

(3.1)

with

\[ Y = (y_1, y_2, \ldots, y_s), \quad F(Y) = (f_1(Y), f_2(Y), \ldots, f_s(Y)). \]

We consider r-stage explicit R-K methods for s-systems equation (3.1)

\[ \begin{align*}
\dot{y}_{n+1} &= \dot{y}_n + h \sum_{i=1}^{r} b_i k_i, \\
\dot{k}_1 &= f(x_n, y_n), \\
\dot{k}_i &= f(x_n + \beta_i h, y_n + \sum_{j=1}^{i-1} a_{ij} k_j, y_n + \sum_{j=1}^{i-1} a_{ij} k_j), \quad (i = 1, 2, \ldots, s); \\
\end{align*} \]

(3.2)

\[ \dot{b}_i \text{ varies in each steps and with the component-wise.} \]

Introducing the vector notations

\[ Y_n = (y_{n,1}, y_{n,2}, \ldots, y_{n,s}), \quad B_i = \text{diag}[b_{i,1}, b_{i,2}, \ldots, b_{i,s}], \quad K_i = [k_{i,1}, k_{i,2}, \ldots, k_{i,s}] (i = 1, 2, \ldots, s), \]

we may write (3.2) in the form;

\[ Y_{n+1} = Y_n + \sum_{i=1}^{s-1} B_i K_i. \]

(3.3)

As the same reason stated in §2, we may set the coefficients \( i b_2a_{21} \) of i-th component in the form

\[ i b_2a_{21} = \frac{i y_n}{y_n - 3h k_1 + 2h k_2}. \]

(3.4)

From order condition (2.1) with (3.4), we have the coefficients of i-th component in the forms

\[ i b_{2}a_{21} = \frac{i y_n}{y_n - 3h k_1 + 2h k_2}, \quad i b_1 = 1 - i b_2, \quad a_{21} \text{ free parameter.} \]

(3.5)

The stability results for system equation based on the test function (2.2) can not to be wa-
tertight when applied to the variable coefficient methods, so it is essential to use \( \dot{Y} = AY \) as test function, and not \( \dot{y} = \lambda y \). We state by attacking the simple matrix \( A \) defined by

\[ Z = \Lambda Z \quad \text{with} \quad \Lambda = HAH^{-1} \quad \text{for some} \ H \quad \text{and} \quad \Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_s\}. \]

(3.6)

Cleary the numerical processes \{\( Z_n \)\} derived by (3.2) with (3.5) is stable on the above diagram,

the transformation \( \tilde{Z}_n = HY_n \) does not equivalent to the numerical processes \( Z_n \). We study the stability of numerical processes \( \tilde{Z}_n \) on the most simple differential equation:

\[ \dot{Y} = AY \quad \text{with} \quad A = \begin{pmatrix} p & q \\ q & p \end{pmatrix}, \quad (p + q < 0, p - q < 0). \]

(3.7)

Integrating the differential equation (3.7), we have

\[ Y_{n+1} = (I + hA + h^2CA^2)Y_n, \]

(3.8)
with

\[\begin{align*}
    y_{n+1}^1 &= y_n + h(p^1 y_n + q^2 y_n) + h^2(b^2_{2a21}(p^1 f_n + q^2 f_n)), \\
    y_{n+1}^2 &= y_n + h(q^1 y_n + p^2 y_n) + h^2(b^2_{2a21}(q^1 f_n + p^2 f_n)), \\
    f_n &= p^1 y_n + q^2 y_n, \\
    b^2_{2a21} &= \frac{y_n}{(y_n - 3h^1 k_1 + 2h^1 k_2)}, \\
    b^2_{2a21} &= \frac{y_n}{(y_n - 3h^2 k_1 + 2h^2 k_2)}.
\end{align*}\]

From (3.8), we have

\[
\tilde{Z}_{n+1} = (I + h\Lambda + h^2(I - h\Lambda + (h\Lambda)^2)^{-1})\tilde{Z}_n
+ h^2H\{C - (I - h\Lambda + (h\Lambda)^2))^2H^{-1}\tilde{Z}_n.
\]

Setting K by

\[
K = I + h\Lambda + h^2(I - h\Lambda + (h\Lambda)^2)^{-1}
= \begin{pmatrix}
1 - \lambda_2 + \lambda_2^2 & 0 \\
0 & 1 - \lambda_1 + \lambda_1^2
\end{pmatrix}^{-1},
\]

\[
\lambda_1 = h(p + q), \lambda_2 = h(p - q)
\]

and Q_n by

\[
Q_n = H\{C - (I - h\Lambda + (h\Lambda)^2))\}^{-1}A^2H^{-1}
= \begin{pmatrix}
(s_1 + s_2) & -(s_2 - s_1) \\
(s_2 - s_1) & (s_1 + s_2)
\end{pmatrix} - (I - h\Lambda + (h\Lambda)^2)^{-1}A^2,
\]

\[
(s_1 = y^1_{2a21}, s_2 = y^2_{2a21}).
\]

We may write (3.9) in the form

\[
\tilde{Z}_{n+1} = (K + Q_n)Z_n, \quad \tilde{Z}_1 = \Theta(\tau), \quad (n = 1, 2, \ldots).
\]

Using the standard technique, we obtain the following Lemmas.

**Lemma 1**

\[
\tilde{Z}_{n+1} = K^n\Theta(\tau) + \sum_{\nu=0}^{n-1}K^{n-(\nu+1)}Q_\nu\tilde{Z}_\nu.
\]

**Lemma 2**

\[
\tilde{Z}_{n+1} = K^n\Theta(\tau) + \Pi_{\nu=0}^{n-1}S_{n-\nu}Q_\nu\Theta(\tau) + \sum_{\mu=1}^{n}(\Pi_{\nu=0}^{\mu}S_{n-\nu})(\Pi_{\nu=0}^{\mu}S_{n-\nu})Q_\nu K^\mu,
\]

with \(S_\nu = K + Q_\nu\),

where \(S_\nu = S_\nu(r_1, r_2)\) and \(S_\nu(r_1, r_2) = K + Q(r_1(\nu), r_2(\nu))\) with

\[
Q_\nu(r_1(\nu), r_2(\nu)) = \begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22}
\end{pmatrix},
\]

\[
c_{11} = p_1 + \frac{1}{1 - \lambda_1 + \lambda_1^2}, \quad c_{22} = p_2 + \frac{1}{1 - \lambda_2 + \lambda_2^2},
\]

\[
c_{12} = \frac{1}{1 - \lambda_1 + \lambda_1^2}, \quad c_{21} = \frac{1}{1 - \lambda_2 + \lambda_2^2}.
\]
\[ c_{12} = c_{21} = \frac{1}{2} (p_{1} - p_{2}), \]
with \( p_{1} = -h * (p + qr_{1}) / \{1 - h(p + qr_{1}) + h^{2}(p^{2} + q^{2}) + 2pqh^{2}r_{1}\} \),
\[ p_{2} = -h * (p + qr_{2}) / \{1 - h(p + qr_{2}) + h^{2}(p^{2} + q^{2}) + 2pqh^{2}r_{2}\}, \]
\[ r_{1}(\nu) = \frac{y_{\nu}}{y_{\nu}}/y_{\nu}, r_{2}(\nu) = \frac{y_{\nu}}{y_{\nu}} \]
(3.11)

Using Lemma [1,2], we have

**Theorem 1**

If there exists the constant \( T \) such that \( \|I_{l=0}^{n}S(r_{1}(l), r_{2}(l))\| \leq T \) for all \( m,n(m \leq n) \), then \( \{\tilde{Z}_{n}\} \) stable.

**Numerical Example**

Setting \( r_{1} = r_{2} = 1 \) or \( r_{1} = r_{2} = -1 \) in (3.11), we compute the following differential equation
\[ \dot{Y} = AY \]
with
\[ A = \begin{pmatrix} -550.5 & q \\ q & -500.5 \end{pmatrix}, \]
\[ Y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \]

The datas are the absolute error of numerical solution with step size \( h = 1 \)

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<th>( x )</th>
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<tr>
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\[ \lambda_{1} = -500.5 + q, \lambda_{2} = -500.5 - q. \]

**References**