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<th>Coinvariant Algebras of Some Finite Groups (Groups and Combinatorics)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1997 (1997): 137-140</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/61114">http://hdl.handle.net/2433/61114</a></td>
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<td>Right</td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Coinvariant Algebras of Some Finite Groups

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Recently Y. Ito and I. Nakamura [IN2], [N2] studied the Hilbert scheme of $G$-orbits $\text{Hilb}^g(C^2)$ for a finite group $G \subset SL(2, \mathbb{C})$ and showed a direct correspondence between the representation graph of $G$ (McKay observation) and the singular fiber of the minimal resolution of $C^2/G$ (Dynkin curve). In this article, we report some attempts to extend the results to finite subgroups of $SL(3, \mathbb{C})$, which is being studied jointly with Iku Nakamura (Hokkaido Univ.) and Yasushi Gomi (Sophia Univ.). For simplicity, we take the complex number field $\mathbb{C}$ as a ground field and representations considered are complex representations.

1. Let $G$ be a finite group, $\text{Irr}(G) = \{\chi_1, \ldots, \chi_s\}$ be the set of all irreducible characters of $G$, and $\text{Irr}(G) - \{1_G\}$. Given a character $\chi$ of $G$, we can form the representation graph $\Gamma(G) = \Gamma_{\chi}(G)$ as follows: the set of vertices is $\text{Irr}(G)$ and the directed edge of weight $m_{ij}$ from $\chi_i$ to $\chi_j$ is determined by the relation

$$\chi \cdot \chi_i = \sum_{j=1}^{s} m_{ij} \chi_j, \quad i = 1, \ldots, s.$$ 

We use the convention that a pair of opposing directed edges of weight 1 is represented by a single edge and the weight $m_{ij}$ is omitted if $m_{ij} = 1$.

**Example 1.** Let $G$ be the quaternion group of order 8. Then $\text{Irr}(G)$ consists of 4 linear characters and the character $\chi$ of 2-dimensional representation. Then $\Gamma_{\chi}(G)$ is exactly the extended Dynkin diagram of type $D_4$ centered at $\chi$.

**Example 2.** Let $G$ be the alternating group of degree 5, $A_5$. Then $\text{Irr}(G) = \{1, \chi = 3_1, 3_2, 4, 5\}$, (where the characters are expressed by the degrees of the corresponding representations), and $\Gamma_{\chi}(G)$ becomes as follows:

![Diagram](attachment:image.png)

2. In [M] J. McKay stated the following which is now famous as McKay observation.

**Proposition.** Let $G$ be a finite subgroup of $SL(2, \mathbb{C})$ and $\chi$ be the character of the inclusion representation. Then $\Gamma_{\chi}(G)$ is an extended Dynkin diagram of type A, D or E.
Conversely every such extended Dynkin diagram is obtained as a representation graph of a subgroup of $SL(2, \mathbb{C})$.

Thus McKay observation establishes a bijective correspondence between subgroups $G$ of $SL(2, \mathbb{C})$ and the extended Dynkin diagram $\bar{\Gamma}_G$ of type A, D, and E.

3. There is another famous correspondence between subgroups $G$ of $SL(2, \mathbb{C})$ and the Dynkin diagram $\Gamma_G$ of type A, D and E. (The extended Dynkin diagram of $\Gamma_G$ is $\bar{\Gamma}_G$.) Let $S = \mathbb{C}^2/G$ and $p : \tilde{S} \to S$ be the minimal resolution of singularity. Then the singular fiber, $p^{-1}(0)$, is a union of projective lines, Dynkin curve of type $\Gamma_G$, having intersection matrix $-C$, where $C$ is the Cartan matrix of type $\Gamma_G$. In particular the graph obtained by Dynkin curve as follows is the Dynkin diagram $\Gamma_G$: the set of vertices is that of projective lines appearing in Dynkin curve and two lines are joined iff they meet. For details, please see a survey article of R. Steinberg[St] or P. Slodowy[Sl].

These two correspondences were famous, but relations between them had not been clear. Recently an explanation of these correspondences was given by Y. Ito and I. Nakamura[IN1], [IN2] and I. Nakamura[IN1], [N2], using Hilbert schemes.

4. Let $\text{Hilb}^n(\mathbb{C}^m)$ be the Hilbert scheme of $\mathbb{C}^m$ parametrizing all the 0-dimensional subschemes of length $n$ and let $\text{Symm}^n(\mathbb{C}^m)$ be the $n$-th symmetric product of $\mathbb{C}^m$, that is, the quotient of $n$-copies of $\mathbb{C}^m$ by the natural action of the symmetric group of degree $n$. There is a canonical morphism $\pi$ from $\text{Hilb}^n(\mathbb{C}^m)$ to $\text{Symm}^n(\mathbb{C}^m)$ associating to each 0-dimensional subscheme of $\mathbb{C}^m$ its support. Let $G$ be a finite subgroup of $SL(m, \mathbb{C})$. The group $G$ acts on $\mathbb{C}^m$ so that it acts naturally on both $\text{Hilb}^n(\mathbb{C}^m)$ and $\text{Symm}^n(\mathbb{C}^m)$. Since $\pi$ is $G$-equivariant, $\pi$ induces a morphism from the $G$-fixed point set $\text{Hilb}^n(\mathbb{C}^m)^G$ to the $G$-fixed point set $\text{Symm}^n(\mathbb{C}^m)^G$.

Now consider the special situation that $n$ is the order of the group $G$ and $m = 2$. Then $\text{Symm}^n(\mathbb{C}^2)^G$ is isomorphic to the quotient space $\mathbb{C}^2/G$ and there is a unique irreducible component of $\text{Hilb}^n(\mathbb{C}^2)^G$ dominating $\text{Symm}^n(\mathbb{C}^2)^G$, which we denote by $\text{Hilb}^G(\mathbb{C}^2)$ and call it the Hilbert scheme of $G$-orbits, following the notation and the definition by I. Nakamura. Notice that we have a morphism $p : \text{Hilb}^G(\mathbb{C}^2) \to \mathbb{C}^2/G$ induced by $\pi$. The following theorem is proved in a unified way.

**Theorem.** [IN2]. $\text{Hilb}^G(\mathbb{C}^2)$ is nonsingular and $p : \text{Hilb}^G(\mathbb{C}^2) \to \mathbb{C}^2/G$ is a minimal resolution of singularity.

5. Let $R = \mathbb{C}[x, y]$ be the ring of regular functions on $\mathbb{C}^2$ and $M$ be the maximal ideal corresponding to the origin, that is $M = (x, y)$. For a finite group $G \subset SL(2, \mathbb{C})$ of order $n$, let $R_G$ be the invariant algebra of $G$ and $N$ be the ideal of $R$ generated by invariant homogeneous polynomials of positive degree which generate $R_G$. The ring $R_G = R/N$ is called the coinvariant algebra of $G$.

We identify a $G$-invariant 0-dimensional subscheme with its defining ideal of $R$. For $I \in \text{Hilb}^G(\mathbb{C}^2)$ with support origin, put $V(I) = I/(MI + N)$. Then $V(I)$ is a $G$-module and we denote its character by $\chi_V(I)$. Let $E$ be the exceptional set of $p$ and $\text{Irr}(E)$ be
the set of irreducible components of $E$. For $\chi \in \text{Irr}(G)^d$, define

$$E(\chi) = \{ I \in E | (\chi, \chi_{V(I)})_G \neq 0 \}$$

where $(\cdot, \cdot)_G$ is the usual inner product on functions on $G$. Then by verifying every case the following theorem is obtained.

**Theorem.** [IN2],[N2].

$$E = \{ I \mid G\text{-invariant ideal of } R, N \subset I \subset M, R/I \simeq \mathbb{C}G \}$$

and the map $\chi \mapsto E(\chi)$ gives a bijective correspondence between $\text{Irr}(G)^d$ and $\text{Irr}(E)$.

6. Let $G$ be a subgroup of $SL(3, \mathbb{C})$. $R$, $R^G$, $R_G$, $M$ and $N$ are defined similarly for $\mathbb{C}^3$ and $G$ as in 5. Now theorem 5 suggests the necessity to study

$$F_G := \{ I \mid G\text{-invariant ideal of } R, N \subset I \subset M, R/I \simeq \mathbb{C}G \},$$

which would be a fiber of the origin of the quotient space $\mathbb{C}^3/G$ in the Hilbert scheme of $G$-orbits. For that purpose we need detailed structures of the coinvariant algebras $R_G$.

What we have mainly obtained so far are

- decomposition of $R_G$ (or its overalgebra) into irreducible components, particularly for groups of orders $60(A_5)$, $168(PSL(2,7))$, $108, 180, 216, 504, 648$, and $1080$,

- explicit determination of basis for each irreducible component above for $A_5$ and $PSL(2,7)$.

As an outcome of these calculations we can show that $F_{A_5}$ is a union of projective lines whose graph is given by

```
3_2
   /
  /
4
   /
  /
5
  /
3_1
```

and a graph for $PSL(2,7)$ also can be given. Details will appear in [GNS].

**References**


