The Terwilliger Algebra for Bipartite 
$P$- and $Q$-polynomial Schemes

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Extended Abstract.

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a symmetric association scheme with $D \geq 3$. Suppose $Y$ is bipartite $P$- and $Q$-polynomial, and fix any $x \in X$. Let $T = T(x)$ denote the Terwilliger algebra for $Y$ with respect to $x$. The algebra $T$ acts on the vector space $V = \mathbb{C}^X$ by matrix multiplication, and $V$ is referred to as the standard module for $T$. $V$ is equipped with the standard inner product on $\mathbb{C}^X$. It is known that $T$ is a semisimple matrix algebra, and so by the Wedderburn-Artin theorem, $V$ decomposes into a direct sum of irreducible $T$-modules. We study the action of $T$ on these modules.

Let $E_0, E_1, ..., E_D$ denote the primitive idempotents for $Y$ and let $E_0^*, E_1^*, ..., E_D^*$ denote the dual primitive idempotents for $Y$ with respect to $x$. Fix any irreducible $T$-module $W \subseteq V$, and let $r, d, t,$ and $d^*$ respectively denote the endpoint, diameter, dual-endpoint and dual-diameter of $W$. In other words, set

$$r := \min\{i \mid E_i^* W \neq 0\}, \quad (1)$$
$$d := |\{i \mid E_i^* W \neq 0\}| - 1, \quad (2)$$
$$t := \min\{i \mid E_i W \neq 0\}, \quad (3)$$
$$d^* := |\{i \mid E_i W \neq 0\}| - 1. \quad (4)$$

We prove the following theorem.

**Theorem.** With the above notation, let $W$ denote any irreducible $T$-module for $Y$. Then

(i) $W$ must satisfy each of the following

$$d = d^*, \quad (5)$$
$$2r + d \geq D, \quad (6)$$
$$2t + d = D. \quad (7)$$

(ii) $W$ is thin and dual-thin.

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(iii) For any nonzero \( v \in E_t W \),
\[ E_r^* v, E_{r+1}^* v, \ldots, E_{r+d}^* v \]
is an orthogonal basis for \( W \).

(iv) For any nonzero \( v \in E_t^* W \),
\[ E_t v, E_{t+1} v, \ldots, E_{t+d} v \]
is an orthogonal basis for \( W \).

We describe the action of \( T \) on these bases by generalizing the intersection and dual-intersection numbers of \( Y \). These constants are then computed from the eigenvalues and dual-eigenvalues of \( Y \). Using these expressions, we prove that the isomorphism class of \( W \) is determined by two parameters, \( r \) and \( d \), the endpoint and diameter of \( W \), and we obtain simple expressions for the square-norms of our basis vectors for \( W \). In addition, we show how to recursively compute the multiplicities with which the irreducible \( T \)-modules occur in the Wedderburn decomposition of \( V \). Finally, we carry out all of the above computations for the bipartite schemes of type I.

References.

