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The Terwilliger Algebra for Bipartite P- and Q-polynomial Schemes

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Extended Abstract.

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote a symmetric association scheme with $D \geq 3$. Suppose Y is bipartite P- and Q-polynomial, and fix any $x \in X$. Let $T = T(x)$ denote the Terwilliger algebra for Y with respect to x. The algebra T acts on the vector space $V = \mathbb{C}^X$ by matrix multiplication, and V is referred to as the standard module for T. V is equipped with the standard inner product on $\mathbb{C}^X$. It is known that T is a semisimple matrix algebra, and so by the Wedderburn-Artin theorem, V decomposes into a direct sum of irreducible T-modules. We study the action of T on these modules.

Let $E_0, E_1, ..., E_D$ denote the primitive idempotents for Y and let $E_0^*, E_1^*, ..., E_D^*$ denote the dual primitive idempotents for Y with respect to x. Fix any irreducible T-module $W \subseteq V$, and let r, d, t, and d^* respectively denote the endpoint, diameter, dual-endpoint and dual-diameter of W. In other words, set

$$
\begin{aligned}
r &:= min\{i \mid E_i^* W \neq 0\}, && (1)\\
d &:= |\{i \mid E_i^* W \neq 0\}| - 1, && (2)\\
t &:= min\{i \mid E_i W \neq 0\}, && (3)\\
d^* &:= |\{i \mid E_i W \neq 0\}| - 1. && (4)
\end{aligned}
$$

We prove the following theorem.

Theorem. With the above notation, let W denote any irreducible T-module for Y. Then

(i) W must satisfy each of the following

$$
\begin{aligned}
d &= d^*, && (5)\\
2r + d &\geq D, && (6)\\
2t + d &= D. && (7)
\end{aligned}
$$

(ii) W is thin and dual-thin.

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(iii) For any nonzero $v \in E_t W$,

$$E^*_r v, E^*_{r+1} v, \ldots, E^*_{r+d} v \quad \text{is an orthogonal basis for } W.$$

(iv) For any nonzero $v \in E^*_r W$,

$$E_t v, E_{t+1} v, \ldots, E_{t+d} v \quad \text{is an orthogonal basis for } W.$$

We describe the action of T on these bases by generalizing the intersection and dual-intersection numbers of Y. These constants are then computed from the eigenvalues and dual-eigenvalues of Y. Using these expressions, we prove that the isomorphism class of W is determined by two parameters, r and d, the endpoint and diameter of W, and we obtain simple expressions for the square-norms of our basis vectors for W. In addition, we show how to recursively compute the multiplicities with which the irreducible T-modules occur in the Wedderburn decomposition of V. Finally, we carry out all of the above computations for the bipartite schemes of type I.

References.

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