Primitive idempotents of the Grothendieck ring of Mackey functors

小田 文仁 FUMIHITO ODA *

Abstract

We study the Grothendieck ring of the category of Mackey functors for a finite group and determine the primitive idempotents of the ring.

1 Preliminaries

Let $G$ be a finite group and an $R$ a commutative ring. We denote by $S(G)$ the set of all subgroups of $G$ and let $C(G)$ be the set of representatives of conjugacy classes of $S(G)$. For $H \in S(G)$ and $g \in G$ let $^{g}H = gHg^{-1}$, $H^{g} = g^{-1}Hg$. If $H \in S(G)$ and $L, K \in S(G)$ let $[L \setminus H/K]$ be a set of representatives of cosets $LhK$ with $h \in H$. If $L \leq H \leq G$ let $H/L$ be a set of representatives of cosets $hL$ with $h \in H$.

A Mackey functor for $G$ over $R$ is a mapping

$$M : S(G) \rightarrow R\text{-mod}$$

with morphisms

$$I_{K}^{H} : M(K) \rightarrow M(H) \quad \text{(induction)}$$

$$R_{K}^{H} : M(H) \rightarrow M(K) \quad \text{(restriction)}$$

$$c_{g}^{H} : M(H) \rightarrow M(^{g}H) \quad \text{(conjugation)}$$

whenever $K \leq H$ are subgroups of $G$ and $g \in G$, such that

(M0) $I_{K}^{H}, R_{K}^{H}, c_{g}^{H} : M(H) \rightarrow M(H)$ are the identity morphisms for all subgroups $H$ and $g \in H$,

(M1) $R_{L}^{K}R_{K}^{H} = R_{L}^{H}$, $I_{K}^{H}I_{L}^{K} = I_{L}^{H}$ for all subgroups $L \leq K \leq H$,

(M2) $c_{g}^{H}c_{h}^{H} = c_{gh}^{H}$ for all subgroups $H \leq G$ and $g, h \in G$,

(M3) $R_{K}^{H}c_{g}^{H} = c_{g}^{K}R_{K}^{H}$, $I_{K}^{H}c_{g}^{K} = c_{g}^{H}I_{K}^{H}$ for all subgroups $K \leq H$ and $g \in G$,

(M4) $R_{L}^{H}I_{K}^{H} = \sum_{x \in [L \setminus H/K]} I_{L \cap x}^{L \cap x} R_{L \cap x}^{K}$ for all subgroups $L, K \leq H$.

*Department of Mathematics, Hokkaido University.
This definition was given by Green [Gr71]. Moreover, this definition is equivalent to the categorical definition given by Dress [Dr73]. The most important axiom is (M4), which is the **Mackey decomposition formula.** Note that the axioms (M0) and (M2) imply that $WH := W_G(H) := N_G(H)/H$ acts on a left $R$-module $M(H)$, so that $M(H)$ is a left $R[WH]$-module for each subgroup $H$ of $G$.

A **morphism** of Mackey functors $f : M \to N$ is a family of $R$-homomorphisms $f(H) : M(H) \to N(H)$, for $H \in S(G)$, which commute with restriction, induction, and conjugation. In particular, since $f$ commutes with conjugation, $f(H)$ is an $R[WH]$-homomorphism. We denote by $\text{Mack}_R(G)$ the category of Mackey functors for $G$ over $R$. It is easy to see that $\text{Mack}_R(G)$ is an abelian category.

We summarise some of the basic constructions of Mackey functors. We denote by

\[ \uparrow^G_{\mathcal{H}} : \text{Mack}_R(H) \to \text{Mack}_R(G), \quad \downarrow^G_{\mathcal{H}} : \text{Mack}_R(G) \to \text{Mack}_R(H), \quad \sigma^H_g : \text{Mack}_R(H) \to \text{Mack}_R(^gH), \]

the **induction**, **restriction**, and **conjugation** of Mackey functors [Sa82]. Whenever we have a normal subgroup $N$ of $G$ and a Mackey functor $L$ for $Q = G/N$ we can form the **infraction** $\text{Inf}^G_Q L$ which is a Mackey functor for $G$ defined by

\[ \text{Inf}^G_Q L = \begin{cases} L(K/N) & \text{if } K \subseteq N \\ 0 & \text{otherwise} \end{cases} \]

with zero restriction and induction morphisms $R^H_K, I^H_K$ unless $N \leq K \leq H$, in which case they are the mappings $R^H_K, I^H_K$ for $L$, and similarly with conjugations. If $M$ is a Mackey functor for $G$ over $R$ we will write

\[ \overline{M}(H) = M(H)/ \sum_{J < H} I^J_H M(J). \]

Note that $\overline{M}(H)$ is an $R[WH]$-module. We recall the **simple Mackey functor** which constructed by Thévenaz and Webb [TW89]. For an $R[G]$-module we describe a Mackey functor $S^G_{1,V}$ for $G$ as follows;

\[ S^G_{1,V}(H) = \left( \sum_{h \in H} h \right)V, \quad H \in S(G). \]

Moreover, if $H$ is any subgroup of $G$ and $V$ is a simple $R[WH]$-module we define $S^G_{H,V} = (\text{Inf}^N_{W_H} S^W_{1,V}) \uparrow^G_N H$, and this is in fact a simple Mackey functor. The $S^G_{H,V}$ so constructed constitute a complete set of representatives for the isomorphism classes of simple Mackey functors [TW89] 8.3.

**Lemma 1.1** ([TW95] 6.4) *Let $S^G_{H,V}$ be a simple Mackey functor. Then*

\[ S^G_{H,V}(L) = \begin{cases} V & \text{if } H \text{ and } K \text{ are conjugate} \\ 0 & \text{otherwise.} \end{cases} \]

A Mackey functor for $G$ over $R$ is identified as a certain finite dimensional algebra $\mu_R(G)$ which called a **Mackey algebra** [TW95].

**Lemma 1.2** ([TW95] 3.6) *Let $K$ be a field. Then $K$ is a splitting field for $\mu_K(G)$ if and only if $K$ is a splitting field for the representations of $WH$ for every subgroup $H \in S(G)$.***
The set of $G$-isomorphism classes of finite $G$-sets becomes a commutative ring $\Omega(G)$ whose name is Burnside ring, with addition defined by disjoint union and multiplication defined by cartesian product with diagonal action. The Burnside ring over $R$ of $G$ is the free $R$-module with basis the $G$-sets $G/H$ where $H$ is taken up to conjugacy. By means of induction, restriction, and conjugation of $G$-sets this gives rise to a Mackey functor denoted $\Omega^G$, which is call the Burnside functor for $G$.

Lemma 1.3 ([TW95] 8.9) Suppose that $K$ is a field which is a splitting field for $\mu_K(G)$. If $\text{char}(K) = 0$ then

$$\Omega^G \cong \bigoplus_{H \in C(G)} S_{H,K}.$$  

For three Mackey functors $M, N, L$ a pairing $M \times N \rightarrow L$ of Mackey functors is a family of $R$-bilinear maps

$$M(H) \times N(H) \rightarrow L(H) : \quad (m, n) \mapsto m \cdot n$$

such that the following axioms hold: for subgroups $H, K$ of $G$ with $H \leq K$

(P1) $R^K_H(ab) = R^K_H(a)R^K_K(b), \quad a \in M(H), b \in N(H),$

(P2) $c^K_H(ab) = c^K_H(a)c^K_H(b), \quad a \in M(H), b \in N(H),$

(P3) $I^K_H(a)b' = I^K_H(aR^K_K(b')), \quad a \in M(K), b' \in N(H),$

(P4) $a'I^K_H(b) = I^K_H(R^K_K(a')b), \quad a' \in M(H), b \in N(K).$

A Green functor $A$ is a Mackey functor with a pairing $A \times A \rightarrow A$ such that for each $H \in S(G)$ the $R$-linear map $A(H) \times A(H) \rightarrow A(H)$ makes $A(H)$ into associative $R$-algebra with unity $1_{A(H)}$ such that:

(G) $R^K_H(1_{A(H)}) = 1_{A(K)}, \quad K \leq H \leq G.$

Let $M$ be a Mackey functor and let $A$ a Green functor. If there exists a pairing $l_A : A \times M \rightarrow M$ such that $M(H)$ becomes a unitary left $A(H)$-module via the $R$-homomorphism $l_A(H) : A(H) \times M(H) \rightarrow M(H)$ then we said that $M$ is a left $A$-module [Is89], [Lu96]. One can define similarly the notion of right $A$-module with a pairing $r_A$.

Let $A$ be a Green functor for $G$. Let $M_A, A^N$ be $A$-modules and $L$ a Mackey functor for $G$ over $R$. A $A$-pairing $p : M \times N \rightarrow L$ [Is89], [Lu96] is a pairing $p : M \times N \rightarrow L$ such that the following axiom hold:

(P5) For $H \in S(G)$ diagram

$$
\begin{array}{ccc}
M(H) \times A(H) \times N(H) & \xrightarrow{1_{M(H)} \times l_A(H)} & M(H) \times N(H) \\
\downarrow & & \downarrow \quad \quad \quad \quad \downarrow p(H) \\
M(H) \times N(H) & \xrightarrow{p(H)} & L(H)
\end{array}
$$

is commutative.
2 Tensor product of Mackey functors

In this section, we recall the tensor product of Mackey functors. We refer to [Is89], [Le80], [Lu96] for detail.

Let $M$ and $N$ be Mackey functors for a finite group $G$ over a commutative ring $R$. For $H \leq G$, we put

$$T(H) = \langle 1^H_D \otimes \mu \otimes \nu \mid \mu \in M(D), \nu \in N(D), D \in S(H) \rangle \cong \bigoplus_{D \in S(H)} M(D) \otimes_R N(D),$$

where $1^H_D \otimes$ is a symbol. Let $I(H)$ be the $R$-submodule of $T(H)$ generated by the following elements:

(R1) \(1^H_D \otimes (\mu_1 + \mu_2) \otimes \nu_0 = 1^H_D \otimes \mu_1 \otimes \nu_0 + 1^H_D \otimes \mu_2 \otimes \nu_0,\)

(R2) \(1^H_D \otimes \mu_0 \otimes (\nu_1 + \nu_2) = 1^H_D \otimes \mu_0 \otimes \nu_1 + 1^H_D \otimes \mu_0 \otimes \nu_2,\)

(R3) \(1^H_D \otimes \mu_0 \alpha \otimes \nu_0 = 1^H_D \otimes \mu_0 \otimes \alpha \nu_0,\)

(R4) \(1^H_D \otimes t_D^P(\mu) \otimes \nu' = 1^H_D \otimes \mu \otimes r_D^P(\nu'),\)

(R5) \(1^H_D \otimes \mu' \otimes t_D^P(\nu) = 1^H_D \otimes r_D^P(\mu') \otimes \nu,\)

(R6) \(1^H_D \otimes ^h \mu \otimes ^h \nu = 1^H_D \otimes \mu \otimes \nu,\)

whenever $\mu_0, \mu_1, \mu_2 \in M(H), \mu \in M(D), \mu' \in M(D'), \nu_0, \nu_1, \nu_2 \in N(H), \nu \in N(D), \nu' \in N(D'), \alpha \in A(H), \ h \in H, \ D \leq D' \leq H$. Moreover, for subgroups $K \leq H$ of $G$ and an element $g$ of $G$, the linear maps restriction, induction, and conjugation defined as follows.

(T1) \(\rho^K_H : T(H) \to T(K); 1^H_D \otimes \mu \otimes \nu \mapsto \sum_{g \in [K \backslash H/D]} 1^K_D \otimes R^K_D(\theta \mu) \otimes R^K_D(\theta \nu),\)

(T2) \(\tau^K_H : T(K) \to T(H); 1^K_D \otimes \mu \otimes \nu \mapsto 1_D \otimes \mu \otimes \nu,\)

(T3) \(\sigma^H_g : T(H) \to T(g H); 1^H_D \otimes \mu \otimes \nu \mapsto 1_D \otimes c^H_g \mu \otimes c^H_g \nu.\)

For $H \leq G$, we set

$$M \otimes N(H) = T(H)/I(H).$$

A tensor product of Mackey functors $M$ and $N$ consist of $M \otimes N$ with induction, restriction, and conjugation above. Also the Mackey functor $M \otimes N$ satisfy the universality of tensor product.

Lemma 2.1 ([Is89], [Le80], [Lu96], Yoshida) There exists a unique pairing (resp. $A$-pairing) $\theta : M \times N \to M \otimes N$, such that for every pairing (resp. $A$-pairing) $\eta : M \times N \to T$, there exist a unique family of maps $\phi : M \otimes N \to T$ the diagram

$$\begin{array}{ccc}
M \times N & \xrightarrow{\theta} & M \otimes N \\
\eta & \downarrow \phi & \downarrow L \\
& &
\end{array}$$

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is commutative.

Let

$$M \times N \xrightarrow{\otimes} M \otimes_A N$$

denote the universal $A$-pairing.

**Lemma 2.2** Let $M$ be a Mackey functor for $G$ over $R$. Then there exists a $\Omega^G$-pairing

$$\theta : \Omega^G \times M \rightarrow M.$$  

**Proof.** See [Is89], [Le80], [Lu96], [TW95].

Hence for Mackey functors $M_{\Omega^G}$ and $\Omega^G N$ we have $\Omega^G$-pairing $M \times N \rightarrow M \otimes_{\Omega^G} N$.

3 GROTHENDIECK RING OF MACKEY FUNCTORS

In this section, we describe the Grothendieck ring of the category of Mackey functors for $G$ over $R$.

**Lemma 3.1** Let $M$, $N$ and $L$ be Mackey functors for $G$ over $R$. Then there exist isomorphisms of Mackey functors as follows;

(i) $M \otimes_{\Omega^G} N \cong N \otimes_{\Omega^G} M$,

(ii) $(M \otimes_{\Omega^G} N) \otimes_{\Omega^G} L \cong M \otimes_{\Omega^G} (N \otimes_{\Omega^G} L)$,

(iii) $(M \oplus N) \otimes_{\Omega^G} L \cong (M \otimes_{\Omega^G} L) \oplus (N \otimes_{\Omega^G} L)$,

(iv) $M \otimes_{\Omega^G} (N \oplus L) \cong (M \otimes_{\Omega^G} N) \oplus (M \otimes_{\Omega^G} L)$,

(v) $M \otimes_{\Omega^G} \Omega^G \cong \Omega^G \otimes_{\Omega^G} M \cong M$.

**Proof.** (i) We shall construct a family of maps $\phi : N \otimes_{\Omega^G} M \rightarrow L$, such that the next diagram

$$
\begin{array}{ccc}
M \times N & \xrightarrow{\otimes} & N \otimes_{\Omega^G} M \\
\downarrow \rho & & \downarrow \phi \\
L & & \\
\end{array}
$$

is commutative for all $\Omega^G$-pairing and a Mackey functor $L$. For $H \in S(G)$ it suffices to define

$$\phi(H) : N \otimes_{\Omega^G} M(H) \rightarrow L(H); \phi(n \times m) = \rho(m, n)$$

where $n \in N(H)$ and $m \in M(H)$. (ii)-(iv) are similar to (i).

(v) There is a family of maps $\phi : M \rightarrow L$, such that the next diagram
is commutative for all $\Omega^G$-pairing $\rho$ and a Mackey functor $L$; that is,
\[
\phi(H) : M(H) \rightarrow L(H) ; \ m \mapsto \rho(1_H, m).
\]

The next result appears also in a general form in Luca's paper [Lu96] 4.1.11, but the proof differs from ours.

**Lemma 3.2** Let $K$ be a splitting field for $\mu_K(G)$ and $\text{char}(K) = 0$.
(i) Let $M$ and $N$ be Mackey functors for $G$ over $K$. Then
\[
\overline{M \otimes N}(H) \cong \overline{M}(H) \otimes_{K} \overline{N}(H)
\]
for every subgroup $H$.
(ii) Let $A$ be a Green functor and let $M$ (resp. $N$) be a right (resp. left) $A$-module. Then
\[
\overline{M \otimes_A N}(H) \cong \overline{M}(H) \otimes_{\overline{A}(H)} \overline{N}(H)
\]
Proof. (i) We may assume $M$ and $N$ be simple Mackey functors $S^G_{P,V}, S^G_{Q,W}$ from Lemma 1.2. Let $f(H)$ be a map from $\overline{S_{P,V}}(H) \times \overline{S_{Q,W}}(H)$ to $\overline{S_{P,V} \otimes_{\Omega^G Q,W}}(H)$ defined by
\[
f(H) = \begin{cases} 
1^H_H \otimes s \otimes t & \text{if } P, Q, \text{ and } H \text{ are conjugate} \\
0 & \text{otherwise}
\end{cases}
\]
where $s \in S^G_{P,V}(H)$ and $t \in S_{Q,W}(H)$. Then $f(H)$ is a $K$-bilinear map by the definition of tensor product.

We construct a map $\phi : N \otimes_{\Omega^G} M \rightarrow L$, such that the next diagram
\[
\begin{array}{ccc}
\overline{S_{P,V}}(H) \times \overline{S_{Q,W}}(H) & \xrightarrow{f(H)} & \overline{S_{P,V} \otimes_{\Omega^G Q,W}}(H) \\
\rho \downarrow & & \phi \downarrow \\
L & & L
\end{array}
\]
is commutative for a $K$-homomorphism $\rho$ and a $K$-module $L$; that is,
\[
\phi(1^H_H \otimes p \otimes q) = \begin{cases} 
\rho(p, q) & \text{if } D \text{ and } H \text{ are conjugate} \\
0 & \text{otherwise}
\end{cases}
\]
The lemma follows by Lemma 1.1 and the universality of tensor product of $K$-modules.
Let $G_0(\text{Mack}_R(G))$ be a Grothendieck group of the category of Mackey functors $\text{Mack}_R(G)$ for a finite group $G$ over a commutative ring $R$ with addition defined by the direct sum. Then $G_0(\text{Mack}_R(G))$ has a commutative ring structure from Lemma 3.1 with multiplication defined by the tensor product of Mackey functors which we call the Grothendieck ring of Mackey functors. The Grothendieck ring $G_0(\text{Mack}_R(G))$ has a basis
\[\{S_{H,V}|H \in C(G), V \in \text{Irr}_R(WH)\}\]
from Lemma 3.1 (v) and the unit element $\Omega^G$.

The main result of this paper is the following.

**Theorem 3.3** Let $K$ be a field which is a splitting field for the representations of $WH$ for every subgroup $H \leq G$ and char$(K) = 0$. Then there is an isomorphism of rings
\[G_0(\text{Mack}_K(G)) \cong \bigoplus_{H \in C(G)} G_0(K[WH]).\]

*Proof.* We shall define a map
\[\psi : G_0(\text{Mack}_R(G)) \rightarrow \bigoplus_{H \in C(G)} G_0(K[WH])\]
by $M \mapsto (M(H))_H$. Here we use the symbol $M$ to denote also the element of $G_0(\text{Mack}_R(G))$ determined by $M$, and likewise $\overline{M}(H)$ denote the element of $G_0(K[WH])$ which this $K[WH]$-module determines. By Lemma 1.1 the matrix of $\psi$ is the identity matrix. It follows that $\psi$ is an isomorphism of abelian groups. By Lemma 1.3 $\psi$ preserves the identity. Since $K$ is a splitting field for $\mu_K(G)$ and $|G|^{-1} \in K$, we obtain the desired result from Lemma 3.2.

\[\square\]

## 4 Primitive Idempotents

Let $\text{Cl}(G)$ be the set of the representatives of conjugacy classes of $G$ and let $C_G(x)$ be the centralizer of $x \in G$ in $G$. We denote by $\text{Irr}_K(G)$ the irreducible characters of $G$ over a field $K$. We need the next lemma.

**Lemma 4.1** For an element $x$ of $G$, we put
\[\epsilon_{G,x} = |C_G(x)|^{-1} \sum_{\chi \in \text{Irr}_K(G)} \chi(x^{-1})\chi.\]
Then $\{\epsilon_{G,x}|x \in \text{Cl}(G)\}$ is the set of primitive idempotents of the character ring of $G$ over $K$.

For $H \leq G$ and $x \in WH$, we set
\[E_{H,x} = |C_{WH}(x)|^{-1} \sum_{\chi \in \text{Irr}_K(WH)} \chi(x^{-1})S_{H,V}^{WH}\]
in $G_0(\text{Mack}_K(G))$, where $V_\chi$ is irreducible $K[WH]$-module corresponding $\chi$.

**Corollary 4.2** There exist the set of primitive idempotents
\[\{E_{H,x}|x \in \text{Cl}(WH), H \in C(G)\}\]
of the Grothendieck ring $C \otimes G_0(\text{Mack}_K(G))$.

*Proof.* Let $\psi$ be the isomorphism in Theorem 3.3. It is easy to see that $\psi(E_{H,x}) = \epsilon_{WH,x}$ from Lemma 1.1. Thus we obtain the corollary by Lemma 4.1 and Theorem 3.3.

\[\square\]
参考文献


