<table>
<thead>
<tr>
<th>Title</th>
<th>THE ALPERIN AND DADE CONJECTURES FOR SOME FINITE GROUPS (Groups and Combinatorics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>An, Jianbei</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1997), 991: 28-35</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/61128">http://hdl.handle.net/2433/61128</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
THE ALPERIN AND DADE CONJECTURES 
FOR SOME FINITE GROUPS

Jianbei An

Department of Mathematics
University of Auckland
Auckland, New Zealand

1. Alperin’s Weight Conjecture

Let $G$ be a finite group, $p$ a prime, and $O_p(G)$ the largest normal $p$-subgroup of $G$. In addition, let $B$ be a $p$-block, $R$ a $p$-subgroup of $G$, $\varphi$ an irreducible ordinary character of the factor group $N(R)/R$. Then a pair $(R, \varphi)$ is called a $B$-weight (character version) if the $p$-defect of $\varphi$ is 0 and if the block $B(\varphi)$ of the normalizer $N(R)$ containing $\varphi$ induces the block $B$ (in the sense of Brauer), where $\varphi$ is also viewed as a character of $N(R)$ and the $p$-defect of $\varphi$ is the largest integer $a$ such that $p^a$ divides $\frac{|G|}{\varphi(1)}$. A weight is always identified with its conjugates in $G$.

Alperin’s Weight Conjecture (1987): The number of $B$-weights equals the number of irreducible Brauer characters in the block $B$.

In 1989, Knörr and Robinson translated the conjecture into one involving only ordinary irreducible characters.

A $p$-subgroup chain

$$C : Q_0 < Q_1 < \ldots < Q_n$$

of length $|C| = n$ is called a normal $p$-chain if each subgroup $Q_i$ is a proper normal subgroup of $Q_n$ for $1 \leq i \leq n - 1$. Let $\mathcal{N}$ be the set of all normal $p$-chains. Then $G$ acts on $\mathcal{N}$ by conjugation, and the stabilizer

$$N(C) = \cap_{i=1}^{n} N(Q_i)$$

of the chain $C$ in $G$ is called the normalizer of $C$. Denote by $k(N(C), B)$ the number of irreducible characters $\psi$ of $N(C)$ such that the block $B(\psi)$ of $N(C)$ containing $\psi$ induces the given block $B$. Alperin’s weight conjecture is equivalent to the following.
The Knörr-Robinson Form (1989): Whenever $G$ is a finite group and $B$ is a $p$-block, we have
\[ \sum_{C}(-1)^{|C|}k(N(C), B) = 0, \]
where $C$ runs over a set $N/G$ of representatives for $G$-orbits in $N$.

2. Dade’s Ordinary Conjecture

A $p$-subgroup $R$ of $G$ is called a radical subgroup if $R$ is the largest normal $p$-subgroup of its normalizer $N(R)$, that is, $R = O_p(N(R))$.

A $p$-subgroup chain
\[ C : P_0 < P_1 < \ldots < P_u \]
is called a radical $p$-chain if it satisfies the following two conditions

(a) $P_0 = O_p(G)$.

(b) $P_k$ is a radical subgroup of the subgroup $\cap_{\ell=0}^{k-1} N(P_\ell)$

for each $1 \leq k \leq u$. Let $\mathcal{R} = \mathcal{R}(G)$ be the set of all radical $p$-chains of $G$.

Given a non-negative integer $d$, a $p$-block $B$ of $G$ and a radical $p$-chain $C$, let $k(N(C), B, d)$ be the number of irreducible characters $\psi$ of the normalizer $N(C)$ such that

$B(\psi)$ induces $B$ and the defect of $\psi$ is $d$.

Dade’s Ordinary Conjecture (1990): If $O_p(G) = 1$ and $B$ is a block with non-trivial defect groups, then for any integer $d$,
\[ \sum_{C}(-1)^{|C|}k(N(C), B, d) = 0 \quad (2.1) \]

where $C$ runs over a set $\mathcal{R}/G$ of representatives for the $G$-orbits in $\mathcal{R}$.

It was shown by Dade [D1] that
\[ \sum_{C \in \mathcal{N}/G} (-1)^{|C|}k(N(C), B, d) = \sum_{C \in \mathcal{R}/G} (-1)^{|C|}k(N(C), B, d). \]

Thus Dade’s ordinary conjecture implies the Knörr-Robinson form of Alperin’s weight conjecture. It is also mentioned in Dade’s paper [D1] that the ordinary
conjecture is equivalent to the final conjecture if the group $G$ has both trivial Schur multiplier $\text{Mult}(G)$ and trivial outer automorphism group $\text{Out}(G)$. These conditions are satisfied by the following 11 sporadic simple groups:

$$J_1, J_4, M_{11}, M_{23}, M_{24}, Ly, Co_2, Co_3, Fi_{23}, Th, M.$$  

3. Dade’s Invariant Conjecture

Suppose the center $Z(G)$ of $G$ is trivial. Then we can identify $G$ with its inner automorphism group $\text{Inn}(G)$. So the automorphism group $A = \text{Aut}(G)$ acts naturally on each $p$-chain $C$, and moreover, the stabilizer $N_A(C)$ of $C$ in $A$ acts on each irreducible character $\psi$ of $N_G(C)$. So $N_G(C)$ is a normal subgroup of the stabilizer $N_A(C, \psi)$ of $\psi$ in $N_A(C)$. The factor group $N_A(C, \psi)/N_G(C)$ is isomorphic to the subgroup of an outer automorphism group $O = \text{Out}(G)$ of $G$.

Given a radical $p$-chain $C$, a $p$-block $B \in \text{Blk}(G)$, a non-negative integer $d$, and a subgroup $U$ of $O = \text{Out}(G)$, let $k(N(C), B, d, U)$ be the number of irreducible characters $\psi$ of $N_G(C)$ such that the block of $N(C)$ containing $\psi$ induces the block $B$, the defect of $\psi$ is $d$, and $N_A(C, \psi)/N_G(C) = U$. The Dade invariant conjecture is stated as follows:

**Dade’s Invariant Conjecture** [D3]: If $Z(G) = O_p(G) = 1$ and $B$ is a $p$-block of $G$ with non-trivial defect group, then for any integer $d \geq 0$ and any subgroup $U \leq \text{Out}(G)$,

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} k(N(C), B, d, U) = 0,$$

where $\mathcal{R}/G$ is a set of representatives for the $G$-orbits in $\mathcal{R}$.

Dade’s invariant conjecture is equivalent to his final conjecture whenever $G$ has trivial Schur multiplier $\text{Mult}(G)$ and an outer automorphism group all of whose Sylow subgroups are cyclic. A lot of finite simple groups satisfy these conditions, for example,

$$He, HN, R_1(q), R_2(q), ^3D_4(q), G_2(q) \text{ (with } q \neq 3, 4), F_4(q) \text{ (with } q \neq 4), E_8(q).$$
4. Current Works

1. Alperin's weight conjecture has been verified for the following groups and blocks:

**Blacks:**

(a) Cyclic and tame blocks (by Dade, Uno).
(b) Abelian defect blocks with small inertial index (by Puig and Usami).
(c) Abelian defect principal 2-blocks (by Fong and Harris).
(d) Abelian defect unipotent blocks of a finite reductive group (by Broué, Malle and Michel).

**Groups:**

(a) $p$-solvable groups (by Okuyama, Isaacs, Navarro, Gres, Barker).
(b) Groups of Lie type in the defining characteristic (by Alperin, Cabanes, and reproved by Lehrer and Thévenaz).
(c) $S_n$ (by Alperin and Fong).
(d) Classical groups in non-defining characteristics (by Alperin, Fong, Conder and An). In this case, the numbers of irreducible Brauer characters for blocks of symplectic and even-dimensional orthogonal groups are unknown (when $p \neq 2$).
(e) $S_2(2^{2n+1})$, $^2G_2(q^2)$, $^2F_4(q^2)$, $G_2(q)$, $^3D_4(q)$ (by Dade, An).
(f) $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$, $M_{24}$, $He$, $Co_3$, $J_1$ (by Dade, Conder, An).
(g) The covering groups of $S_n$ and $A_n$ ($p \neq 2$) (by Michler and Olsson).
(h) Wrath product groups $G \wr S_n$ provided the conjecture holds for that finite group $G$ (by Ewert)

2. Dade's final conjecture has been verified for the following cases:

(a) 10 sporadic simple groups:
$M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$, $M_{24}$, $He$, $J_1$, $J_2$, $J_3$, $Co_3$ (by Dade, Huang, Kotlica, Schwartz, Conder, An).
(b) $L_2(q)$, $L_3(q)$ ($p|q$), $S_2(q)$, $^2G_2(3^{2n+1})$ ($p \neq 3$), $G_2(q)$ ($p \neq 3$ and $p \not| q \neq 4$), the Tits group (by Dade, An).
(c) Cyclic blocks (by Dade).
3. The invariant conjecture has been verified for all tame blocks, and for the group McL \((p \neq 2)\) (by Uno, Murray).

4. The ordinary conjecture has been verified for the following cases:
   (a) \(\text{GL}_n(q)\) \((p|q), 2F_4(2^{2n+1})\) \((p \neq 2)\), \(G_2(q)\) \((p \equiv q)\) (by Olsson, Uno, An).
   (b) \(S_n\) (by Olsson and Uno when \(p\) odd, An when \(p = 2\)).
   (c) \(Ru\) (by Dade).
   (d) Unipotent abelian defect blocks (by Broué, Malle and Michel).
   (e) Abelian defect principal 2-blocks (by Fong and Harris).
   (f) All abelian defect blocks with small inertia index (by Usami).

References


[AF] J. L. Alperin and P. Fong, Weights for symmetric and general linear groups, \(J.\ Algebra 131\) (1990), 2-22.


