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Crossed G-sets and crossed Burnside rings

吉田知行 (YOSHIDA, Tomoyuki, 北大・理)

1 The category of crossed G-sets

(1.1) **Group action:** Let G be a finite group. By a G-set, we mean a set with left G-action. A mapping $f : X \longrightarrow Y$ between two G-sets is called a G-map if it preserves G-action, that is, $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. The category of finite G-sets and G-maps is denoted by G-**set**.

A monoid S is called a G-**monoid** if the group G acts on S as monoid homomorphisms:

$$G \times S \longrightarrow S;\ (g, s) \longmapsto {}^{g}s,$$
$${}^{gh}s = {}^{g}({}^{h}s),\ {}^{1}s = s;\ {}^{g}(st) = {}^{g}s \cdot {}^{g}t,\ {}^{g}1 = 1 \quad \text{for } s, t \in S, g, h \in G.$$

After this, G and S always denote a finite group and finite G-monoid.

(1.2) **Crossed G-sets:** A **crossed G-set** X (over S) is a G-set X equipped with a G-map called a **weight function**:

$$\|\cdot\| : X \longrightarrow S;\ x \longmapsto \|x\|.$$

So $\|gx\| = {}^{g}\|x\|$ for $g \in G$, $x \in X$. If there is no special indication, we use the symbol $\|x\|$ for the weight of an element $x \in X$. For each $s \in S$, the s-**component** is defined by

$$X[s] := \{x \in X \mid \|x\| = s\}.$$

Then we have

$$gX[s] = X[{}^{g}s], \qquad g \in G, s \in S.$$

In this paper, we study only *finite* crossed G-sets, that is, crossed G-sets in which the underlying G-sets are finite sets.

(1.3) **The category G-xset/S:** A **crossed G-map** between two crossed G-sets X and Y is a G-map $f : X \longrightarrow Y$ preserving weights: $\|f(x)\| = \|x\|$ for all $x \in X$. The set of crossed G-maps of X to Y is denoted by

$$\mathrm{XMap}_G(X, Y).$$

Finite crossed G-sets and crossed G-maps make a category

$$\textbf{G-xset}/S.$$

This category has two kinds of products, tensor product and Hadamard product, which make **G-xset**$/S$ closed monoidal categories, as stated below.

(1.4) **Tensor products:** The **tensor product** of two crossed G-sets X, Y over S is defined by

$$\begin{aligned} &X \otimes Y := X \times Y, \\ &g(x,y) := (gx, gy), \quad \|(x,y)\| := \|x\| \cdot \|y\| \quad (g \in G, x \in X, y \in Y). \end{aligned}$$

Then the category **G-xset**$/S$ becomes a monoidal category. The unit object $I = \{*\}$ is a singleton with weight $\|*\| = 1$.

(1.5) **Closeness:** The monoidal category **G-xset**$/S$ is closed, that is, the functor

$$(-) \otimes Y : \textbf{G-xset}/S \longrightarrow \textbf{G-xset}/S;\ X \longmapsto X \otimes Y$$

has a right adjoint functor called an **internal hom-functor**

$$\mathbf{hom}(Y, -) : \textbf{G-xset}/S \longrightarrow \textbf{G-xset}/S.$$

In fact, this functor is, on objects, defined by

$$\mathbf{hom}(Y, Z) := \coprod_{s \in S} \{s\} \times \{\lambda : Y \longrightarrow Z \mid \|\lambda(y)\| = s \cdot \|y\|\},$$

where λ is simply a mapping (not a G-mapping in general). The G-action and the weight are given by

$$\begin{aligned} &{}^g(s, \lambda) := ({}^g s, {}^g\lambda), \quad {}^g\lambda : y \longmapsto g\lambda(g^{-1}y), \\ &\|(s, \lambda)\| := s. \end{aligned}$$

When Y is empty, we put $\mathbf{hom}(Y, Z) := I$ (the unit object).

For crossed G-sets X, Y, Z, the natural bijection

$$\mathrm{XMap}_G(X \otimes Y, Z) \longleftrightarrow \mathrm{XMap}_G(X, \mathbf{hom}(Y, Z))$$

is given by

$$\begin{aligned} (f : X \otimes Y \longrightarrow Z) &\longmapsto (x \longmapsto (\|x\|, \widehat{x} : y \longmapsto f(x, y))); \\ (g : X \longrightarrow \mathbf{hom}(Y, Z)) &\longmapsto ((x, y) \longmapsto g'(x)(y)), \end{aligned}$$

where $g'(x)$ is defined by $g(x) = (\|x\|, g'(x))$. Furthermore, the associativity of the tensor product and the uniqueness of a right adjoint functor to a functor implies the natural isomorphisms of crossed G-sets:

$$\begin{aligned} &\mathbf{hom}(X \otimes Y, Z) \cong \mathbf{hom}(X, \mathbf{hom}(Y, Z)), \\ &\mathbf{hom}(I, Z) \cong Z. \end{aligned}$$

The first isomorphism is explicitly given by

$$(s, X \otimes Y \xrightarrow{\lambda} Z) \longmapsto (s, \widehat{\lambda} : X \longrightarrow \mathbf{hom}(Y, Z)),$$
$$\text{where } \widehat{\lambda} : x \longmapsto (s \cdot \|x\|, y(\in Y) \longmapsto \lambda(x, y)).$$

Similarly, the left adjoint functor $\mathbf{hom}'(X, -)$ to $X \otimes (-)$ is given by

$$\mathbf{hom}'(X, Z) := \coprod_{s \in S} \{s\} \times \{\lambda : X \longrightarrow Z \mid \|\lambda(x)\| = s \cdot \|x\|\}.$$

There is an isomorphism:

$$\mathbf{hom}(Y, Z) \xrightarrow{\sim} \mathbf{hom}'(Y, Z); (s, \lambda) \longmapsto (s, \ y \longmapsto \|y\| \cdot \lambda(y))$$

(1.6) **Exponentiation:** When S is commutative, there is another interesting crossed G-sets. For crossed G-sets X and Y, let Y^X be the set of all maps from X to Y. Then Y^X is a crossed G-set with G-action and weight as follows:

$$\begin{aligned} {}^g\sigma \quad &: \quad x \longmapsto g\sigma(g^{-1}x), \\ \|\sigma\| \quad &:= \quad \prod_{x \in X} \|\sigma(x)\|, \qquad \sigma \in Y^X, g \in G, x \in X. \end{aligned}$$

This functor $(X, Y) \longmapsto Y^X$ works on coproduct and tensor product well more than **hom**. In fact, the following natural isomorphisms hold:

$$\begin{aligned} X^{M \otimes N} &\cong (X^M)^N, \quad X^I \cong X \\ X^{M+N} &\cong X^M \otimes X^N, \quad X^{\emptyset} \cong I \\ (X \otimes Y)^N &\cong X^N \otimes Y^N, \quad I^N \cong I \end{aligned}$$

(1.7) **Braiding:** The monoidal category G-**xset**$/S$ is not braiding in general, and the following problem seems still open:

Problem: Characterize the pair (G, S) where G-**xset**$/S$ is symmetric or braiding.

Joyal and Street showed that associators and braidings of a abelian categorical group are constructed with abelian 3-cocycles ([JS93, Section 3]).

We give two examples. If S is abelian, then G-**xset**$/S$ is symmetric by the transposition

$$c_{XY} : X \otimes Y \longrightarrow Y \otimes X; (x, y) \longmapsto (y, x).$$

Furthermore, let G^c be the G-set G with G-action defined by conjugation:

$$G \times G^c \longrightarrow G^c; (g, s) \longmapsto {}^g s := gsg^{-1}.$$

In this case, G-**xset**$/S$ is braided by the braiding

$$c_{XY} : X \otimes Y \longrightarrow Y \otimes X; (x, y) \longmapsto (\|x\|y, x).$$

Since

$$c_{YX}{}^{-1} : X \otimes Y \longrightarrow Y \otimes X; (x, y) \longmapsto (y, \|y\|^{-1}x),$$

we have $c_{YX}{}^{-1} \neq c_{XY}$ in general, and so G-**xset** is not symmetric for a nontrivial group G.

(1.8) **Duality:** Assume that S is a group. For a crossed G-set X with weight function $\|\cdot\|$, the **dual weight function** $\|x\|' := \|x\|^{-1}$ makes X a new crossed G-set, which we denote by X^*. The crossed G-set X^* is called the **dual**(or more suitably the **antipode**) of X.

(1.9) **Groupoid:** A G-monoid S gives a groupoid (G, S), in which an object is an element of S and a morphism from s to t is a triplet of the form (t, g, s) such that $t = {}^g s$. A crossed G-set X over S is completely determined by the S-grading:

$$X[s] := \{x \in X \mid \|x\| = s\}, \quad s \in S,$$
$$\widehat{g} : X[s] \longrightarrow X[{}^g s]; x \longmapsto gx, \quad g \in G,$$

that is, the assignment $s \longmapsto X[s]$, $(t, g, s) \longmapsto \widehat{g}$ makes a functor from the groupoid (G, S) to **set**.

Viewing a crossed G-set as a functor on (G, S), the tensor product of crossed G-sets X and Y is written as

$$(X \otimes Y)[s] = \coprod_{uv=s} X[u] \times Y[v].$$

(1.10) **The category of elements:** Here is another viewpoint to the concept of crossed G-sets. Let

$$F : G\text{-}\mathbf{set} \longrightarrow \mathbf{mon}$$

be a contravariant functor from the category of finite G-sets (or another nice monoidal category) to the category of finite monoids which maps disjoint unions to direct products. Then an **element** of this functor is a pair (X, s) of a finite G-set X and an element $s \in F(X)$. A morphism $f : (X, s) \longrightarrow (Y, t)$ between elements is defined to be a G-map $f : X \longrightarrow Y$ such that $F(f)(t) = s$. Thus we have a category **Elem**(G-**set**$, F$). This category is a monoidal category with tensor product:

$$(X, s) \otimes (Y, t) := (X \times Y, F(\pi_1)[s] \cdot F(\pi_2)(t)),$$

where π_i is the i-th projection on $X \times Y$.

(1.11) **Lemma:** *Let S be a finite G-monoid. Then there is an equivalence of monoidal categories:*

$$G\text{-}\mathbf{xset}/S \cong \mathbf{Elem}(G\text{-}\mathbf{set}, \mathrm{Map}_G(-, S)),$$

where $\mathrm{Map}_G(-, S) : G\text{-}\mathbf{set} \longrightarrow \mathbf{mon}$ *is a Hom-functor in* G-**set** *and the monoid structure on* $\mathrm{Map}_G(X, S)$ *is given by the element-wise multiplication:* $(\alpha \cdot \beta)(x) := \alpha(x) \cdot \beta(x)$.

PROOF: Easy. □

One of the author would like to propose a "monomial category "as the name of $\mathbf{Elem}(G\text{-}\mathbf{set}, F)$ after monomial representations of finite groups. In fact, for the contravariant functor $X \longmapsto \mathrm{Ext}^0_{\mathbf{Z}G}(\mathbf{Z}X, \mathbf{C}^*)$ (and so $G/H \longmapsto \widehat{H}$, the group of linear characters), we have the category of monomial representations, and then taking its Grothendieck ring, we have the ring $R_+(G)$, the ring of monomial characters, which was applied to the explicit formula of Brauer induction theorem. We can develop a theory of the monomial category and its Grothendieck ring which is a partial generalization of the theory of crossed G-sets and crossed Burnside rings to a limited extent, e.g., induction-restriction maps, the fundamental theorem, idempotent formula.

(1.12) **Transitive crossed G-sets:** For a subgroup D of the group G and an element s of $C_S(D)$, the centralizer of D in S, the homogeneous G-set G/D becomes a crossed G-set with the weight function $\|gD\| := {}^g s$. We denote this crossed G-set by $(G/D)_s$.

A crossed G-set X is called **transitive** if the underlying G-set X is a transitive G-set. Such a transitive crossed G-set is isomorphic to $(G/D)_s$ for some $D \leq G$, $s \in C_G(D)$.

(1.13) **Lemma:**

(a) $(G/D)_s \cong (G/E)_t \iff D = {}^g E\,(:= gEg^{-1}),\ s = {}^g t \quad \exists g \in G$.

(b) For a crossed G-set X,

$$\mathrm{XMap}_G((G/D)_s, X) \cong X[s]^D := \{x \in X^D \mid \|x\| = s\},$$

where X^D is the set of D-fixed points of X. In particular,

$$\mathrm{XMap}_G((G/1)_s, X) \cong X[s](= \{x \in X \mid \|x\| = s\}).$$

(c) Any crossed G-set X is uniquely decomposed into a direct sum of transitive crossed G-sets.

(d) $(G/D)_s \otimes (G/E)_t \cong \displaystyle\coprod_{DgE \in D\backslash G/E} (G/D \cap {}^g E)_{s \cdot {}^g t}$.

2 The quantum double of a finite group

(2.1) **Definition:** Let F be a commutative ring. We denote by $F[G]$ the group algebra of a finite group G and by $F[G]^*$ the dual space of $F[G]$. Take a basis of $F[G]^*$ to be the characteristic functors $e(g)$ at $g \in G$, that is,

$$e(g) : F[G] \longrightarrow F;\ h(\in G) \longmapsto \begin{cases} 1 & \text{if } g = h, \\ 0 & \text{if } g \neq h. \end{cases}$$

The **quantum double** of G is now defined by

$$D(G) := F[G]^* \otimes F[G].$$

The multiplication is given by

$$(e(x) \otimes g) \cdot (e(y) \otimes h) := e(x)e(gyg^{-1}) \otimes gh.$$

The identity element is $\mathbf{1} \otimes 1 = \sum_{x \in G} e(x) \otimes 1$. The comultiplication $\Delta : D(G) \longrightarrow D(G) \otimes D(G)$, the counit $\epsilon : D(G) \longrightarrow F$, and the antipode $S : D(G) \longrightarrow D(G)$ are given as follows:

$$\Delta : e(x) \otimes g \longmapsto \sum_{uv=x} (e(u) \otimes g) \otimes (e(v) \otimes g);$$

where the summation is taken over the all pairs of $u, v \in G$ such that $uv = x$;

$$\epsilon : e(x) \otimes g \longrightarrow \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } x \neq 1; \end{cases}$$

$$S : e(x) \otimes g \longmapsto e(g^{-1}x^{-1}g) \otimes g^{-1}.$$

(2.2) **Proposition:** *Let $G^c := (G, G^c)$ be the groupoid of which object is an element of G with homomorphisms defined by G-conjugation. The path algebra of the groupoid G^c over F is isomorphic to $D(G)$.*

Proof. In general, the path algebra (or category algebra) $F[\mathcal{C}]$ of a category $\mathcal{C}$ is the path algebra of the digraph of which arrows are morphisms in $\mathcal{C}$ and of which vertices are objects of $\mathcal{C}$. Thus $F[\mathcal{C}]$ is a free F-module generated by $\mathrm{Mor}(\mathcal{C})$, the set of morphisms, equipped with multiplication

$$gf := \begin{cases} gf & \text{if } \mathrm{cod}(f) = \mathrm{dom}(g), \\ 0 & \text{else.} \end{cases}$$

(Note that $F[\mathcal{C}]$ does not possess an identity element unless $\mathrm{Mor}(\mathcal{C})$ is a finite set and that equivalent two categories do not always give isomorphic path algebras.) Thus the path algebra $F[G^c]$ is spanned by the triples (y, g, x), where $y, g, x \in G$ and $y = gxg^{-1}$. The triple (y, g, x) expresses a morphism g in G^c from x to y. Thus in $F[G^c]$, we have

$$(y, g, x) \cdot (y', g', x') = \begin{cases} (y, gg', x') & \text{if } x = y' \\ 0 & \text{else.} \end{cases}$$

Hence the correspondence

$$(y, g, x) \quad \longleftrightarrow \quad e(x) \oslash g$$

gives an isomorphism between $F[\mathcal{C}]$ and $D(G)$. □

(2.3) **Corollary** ([Whi 96, Theorem 2.2]) **:** *The category of $D(G)$-modules and the category of G^c-graded $F[G]$-modules are equivalent (as monoidal categories).*

3 Crossed Burnside rings

(3.1) **Crossed Burnside ring $\mathbf{X\Omega(G, S)}$:** The **Crossed Burnside ring** $X\Omega(G, S)$ is the Grothendieck ring of the category G-**xset**/S with respect to disjoint unions and tensor products, and so this ring is, as abelian group, generated by the isomorphism classes $[X]$ of crossed G-sets with relations:

$$[X + Y] = [X] + [Y].$$

The multiplication is defined by the tensor product of crossed G-sets:

$$[X] \cdot [Y] := [X \oslash Y].$$

Since any crossed G-set is a sum of transitive crossed G-sets, $X\Omega(G, S)$ has the following free $\boldsymbol{Z}$-basis as an abelian group:

$$\{[D, s] \mid D \leq G, s \in C_S(D)\}, \quad ([D, s] := [(G/D)_s)]).$$

Note that

$$[D, s] = [E, t] \quad \Longleftrightarrow E = {}^gD, t = {}^gs \quad \text{for } \exists g \in G$$

and that

$$[D, s] \cdot [E, t] = \sum_{DgE \in D\backslash G/E} [D \cap {}^gE, s \cdot {}^gt].$$

The identity element is $[G, 1]$.

The crossed Burnside ring is not commutative in general(see the next section). When $S = 1$, the crossed Burnside ring is the ordinary **Burnside ring** $\Omega(G)$, that is, the Grothendieck ring of G-**set** with respect to the disjoint unions and the Cartesian products. On the other hand, when $G = 1$, the crossed Burnside ring is the semigroup algebra $\boldsymbol{Z}S$.

(3.2) Proposition: The rank of the free abelian groups $X\Omega(G,S)$ is equal to

$$\sum_{(H)\in C(G)} \frac{|H/H'|}{|N_G(H)|}|C_S(H)|,$$

where $C(G)$ is the set of all conjugacy classes (H) of subgroups H of G and H' denotes the commutator subgroup of H.

(3.3) Hadamard products: The crossed Burnside ring $X\Omega(G,S)$ has another product called the Hadamard product :

$$[X] * [Y] := [X \times_S Y],$$

so that $X\Omega(G,S)$ becomes a commutative ring, which is isomorphic to the Grothendieck ring of the comma category G-**set**$/S$, and so

$$X\Omega(G,S)_{\mathrm{Had}} \cong \prod_{(s)\in S/\sim_G} \Omega(G_s),$$

where (s) runs over G-orbits of S and each component $\Omega(G_s)$ is the ordinary Burnside ring of the stabilizer G_s at s.

(3.4) The Burnside ring $\Omega(G)$: Let $\Omega(G)$ be the Burnside ring of G. Then the trivial G-monoid homomorphisms $\eta : 1 \longrightarrow S$ and $\epsilon : S \longrightarrow 1$ induce the following ring homomorphisms:

$$\begin{aligned} \eta_! &: \Omega(G) \longrightarrow X\Omega(G,S); \quad [G/D] \longmapsto [D,1] \\ \epsilon_! &: X\Omega(G,S) \longrightarrow \Omega(G); \quad [D,s] \longmapsto [G/D]. \end{aligned}$$

Clearly, $\epsilon_! \circ \eta_!$ is an identity map. We view the Burnside ring $\Omega(G)$ as a subring of the crossed Burnside ring $X\Omega(G,S)$ through the injection $\eta_!$, and so $X\Omega(G,S)$ is an $\Omega(G)$-algebra and

$$X\Omega(G,S) = \Omega(G) \oplus \mathrm{Ker}(\eta_!).$$

(3.5) induction and restriction: Various kinds of adjoint functors constructed in the preceding section induce mappings between crossed Burnside rings. For a subgroup of H and a normal subgroup N of G, the functors *Ind*, *Jnd*, *XJnd*, *Res*, *Con*, *Orb*, *Inf*, *Fix* induce the following mappings:

$$\begin{aligned} ind_H^G &: X\Omega(H,S) \longrightarrow X\Omega(G,S), \\ jnd_H^G &: X\Omega(H,S) \longrightarrow X\Omega(G,S), \\ xjnd_H^G &: X\Omega(H,S) \longrightarrow X\Omega(G,S), \\ res_H^G &: X\Omega(G,S) \longrightarrow X\Omega(H,S), \end{aligned}$$

$$\begin{aligned}
con_H^g &: X\Omega(H,S) \longrightarrow X\Omega({}^gH,S),\\
orb_G^{G/N} &: X\Omega(G,S) \longrightarrow X\Omega(G/N,S),\\
inf_G^{G/N} &: X\Omega(G/N,S) \longrightarrow X\Omega(G,S),\\
fix_G^{G/N} &: X\Omega(G,S) \longrightarrow X\Omega(G/N,S).
\end{aligned}$$

Here, when we refer to *xjnd*, the G-monoid S has to be commutative; and when we refer to *orb*, *inf*, *fix*, the G-monoid S has to be fixed elementwise by N. Furthermore, a G-monoid homomorphism $\varphi : S \longrightarrow T$ induces three maps $\varphi_!, \varphi^*, \varphi_*$:

$$\begin{aligned}
\varphi_! &: X\Omega(G,S) \longrightarrow X\Omega(G,T)\\
\varphi^* &: X\Omega(G,T) \longrightarrow X\Omega(G,S)\\
\varphi_* &: X\Omega(G,S) \longrightarrow X\Omega(G,T).
\end{aligned}$$

Among them, *ind*, *res*, *con*, *orb*, $\varphi_!, \varphi^*$ are really additive homomorphism because the corresponding functors preserve coproducts(disjoint unions) and so they can extend to additive homomorphisms of Grothendieck groups. Furthermore, *res*, $\varphi_!$ are ring homomorphisms and *con* is a ring isomorphism. However, the functors *Jnd*, *XJnd*, φ_* do not preserve coproducts, and so it is not clear that they are really extended to maps between crossed Burnside rings preserving Hadamard or tensor products. We prove the existence of the multiplicative maps *jnd*, *xjnd*, φ_* by using the fundamental theorem in the next section.

4 The fundamental theorem

(4.1) **Ghost rings:** As before, let G be a finite group and S a finite G-monoid. For any subgroup H of G, the centralizer of H in S

$$C_S(H) := \{s \in S \mid {}^h s = s \quad \forall h \in H\}$$

is a submonoid of S, on which $WH := N_G(H)/H$ acts as monoid automorphisms. The semigroup algebra $\boldsymbol{Z}[S]$ of S is a permutation G-module with G-basis S. Clearly, $\boldsymbol{Z}[C_S(H)]$ is a subring of $\boldsymbol{Z}[S]$.

The group G acts as ring automorphisms on the ring

$$\Theta(G) := \prod_{H \le G} \boldsymbol{Z}[C_S(H)]$$

by

$$\theta = \left(\sum_{s \in C_S(H)} \theta(H,s)s\right)_H \quad \Longrightarrow \quad \theta^g := \left(\sum_{s \in C_S(H)} \theta({}^gH, {}^gs)s\right)_H .$$

The subring consisting of G-fixed elements of $\Theta(G)$ is called the **crossed ghost ring**:

$$\begin{aligned} X\tilde{\Omega}(G,S) &:= \Theta(G)^G \\ &= \left\{ \left(\sum_{s\in C_S(H)} \theta(H,s)s \right)_H \,\middle|\, \theta({}^gH, {}^gs) = \theta(H,s) \quad \forall g \in G \right\}. \end{aligned}$$

The H-component $\theta(H) = \sum_s \theta(H,s)s \in Z[C_H(S)]$ of any $\theta \in X\tilde{\Omega}(G,S)$ is fixed by WH, and so

$$\begin{aligned} X\tilde{\Omega}(G,S) &= \left(\prod_{H\leq G} Z[C_S(H)]^{WH} \right)^G \\ &\cong \prod_{(H)\in C(G)} Z[C_S(H)]^{WH} \end{aligned}$$

as rings, where $C(G)$ is the set of G-conjugacy classes (H)'s of subgroups H's of G and $WH := N_G(H)/H$. Since $X\Omega(G,S)$ has a basis consisting of $[D,s]$, we have

$$\mathrm{rank}_{\mathbf{Z}} X\Omega(G,S) = \mathrm{rank}_{\mathbf{Z}} X\tilde{\Omega}(G,S).$$

(4.2) **Burnside homomorphism:** For each subgroup H of G and for any crossed G-set X over S, define

$$\varphi_H(X) := \sum_{x\in X^H} \|x\| = \sum_{s\in C_S(H)} |X[s]^H|\, s \in Z[C_S(H)]^{WH},$$

where $X^H := \{x \in X \mid hx = x \quad \forall h \in H\}$ as usual. Since the map φ_H is constant on each isomorphism class of crossed G-sets and is additive, it can extended to an additive map on the Grothendieck ring $X\Omega(G,S)$:

$$\begin{aligned} \varphi_H \;:\; X\Omega(G,S) &\longrightarrow Z[C_S(H)]^{WH} \\ ;\quad [X] &\longrightarrow \sum_{x\in X^H} \|x\| = \sum_{s\in C_S(H)} |X[s]^H|\, s. \end{aligned}$$

It is easily proved that φ_H is a ring homomorphism. The **Burnside homomorphism** is now defined by

$$\varphi := (\varphi_H) : X\Omega(G,S) \longrightarrow X\tilde{\Omega}(G,S).$$

We often write the value $\varphi_H(x)$ simply by $x(H)$. Using this notation, we have that for any $g \in G$,

$$x({}^gH) = {}^g(x(H)),$$

where we extended the action of G on S to the semigroup ring ZS. The Burnside homomorphism φ is a ring homomorphism.

The *Hadamard product* on $Z[C_S(H)]$ is defined by

$$\left(\sum_{s\in C_S(H)} \alpha_s\, s\right) * \left(\sum_{t\in C_S(H)} \beta_t\, t\right) := \sum_{s\in C_S(H)} \alpha_s\beta_s\, s.$$

Then the Burnside homomorphism φ preserves also Hadamard products.

When S is a trivial monoid, we have the Burnside homomorphism

$$\varphi : [X] \longmapsto (|X^H|)_H$$

from the classical Burnside ring $\Omega(G)$ to the ghost ring $\widetilde{\Omega}(G)$.

(4.3) **The group of obstructions:** We, furthermore, define the **group of obstructions** by

$$\mathrm{XObs}(G,S) := \prod_{[H,s]} (Z/|(WH)_s|Z)\,,$$

where $[H,s]$ runs over the isomorphism classes of transitive crossed G-sets, that is, the complete representatives of G-conjugacy classes of pairs (H,s) of $H \le G$ and $s \in C_S(H)$ under G-action ${}^g(H,s) = ({}^gH, {}^gs)$, and where $(WH)_s$ is the stabilizer at $s \in C_S(H)$ in $WH = N_G(H)/H$. For each $H \le G$ and $s \in C_S(H)$, define a linear map

$$\begin{array}{rcccl} \psi_{H,s} & : & X\widetilde{\Omega}(G,S) & \longrightarrow & Z/|(WH)_s|Z \\ & ; & \left(\displaystyle\sum_{s\in C_S(H)} \theta(H,s)s\right) & \longmapsto & \displaystyle\sum_{nH\in(WH)_s} \theta(\langle n\rangle H, s) \quad \bmod |(WH)_s|. \end{array}$$

Thus we have a linear map called a **CFB-map** after Cauchy, Frobenius and Burnside:

$$\psi := (\psi_{H,s}) : X\widetilde{\Omega}(G,S) \longrightarrow \mathrm{XObs}(G,S).$$

(4.4) **Theorem:** *The following sequence of abelian groups is exact:*

$$0 \longrightarrow X\Omega(G,S) \xrightarrow{\varphi} X\widetilde{\Omega}(G,S) \xrightarrow{\psi} \mathrm{XObs}(G,S) \longrightarrow 0.$$

The proof is similar to the one of the ordinary Burnside rings.

(4.5) **Corollary:** *For two crossed G-sets X, Y over S, the following statements are equivalent :*

(a) *X and Y are isomorphic.*

(b) *$\varphi_H(X) = \varphi_H(Y)$ for any subgroup $H \le G$.*

(c) *$|X[s]^H| = |Y[s]^H|$ for any $H \le G$ and $s \in C_S(H)$.*

(4.6) **Corollary:** *If the category of crossed G-sets is braided, then the WH-fixed point ring $Z[C_S(H)]^{WH}$ is a commutative ring.*

(4.7) **Corollary:** *Let F be a commutative ring in which $|G| \cdot 1_F$ is invertible. Then*

$$F \otimes_{\mathbf{Z}} X\Omega(G,S) \cong \prod_{(H) \in C(G)} F[C_S(H)]^{WH}.$$

In particular, $X\Omega(G,S)$ is commutative if and only if $Z[C_S(H)]^{WH}$ is commutative for any $H \leq G$.

(4.8) **The p-local crossed Burnside rings:** Let A_p be a p-local domain of characteristic 0 with the residue field of characteristic $p > 0$. We use the following notation:

$$\begin{aligned} A_p X\Omega(G,S) &:= A_p \otimes_{\mathbf{Z}} X\Omega(G,S); \\ A_p X\tilde{\Omega}(G,S) &:= \left(\prod_{H \leq G} A_p[C_S(H)] \right)^G; \\ A_p \mathrm{XObs}(G,S) &:= \prod_{[H,s]} (A_p / |(WH)_s|_p A_p), \end{aligned}$$

where $|(WH)_s|_p$ is the p-part of $|(WH)_s|$;

$$\begin{array}{rccl} \varphi^{(p)} := 1 \otimes \varphi \ : & A_p X\Omega(G,S) & \longrightarrow & A_p X\tilde{\Omega}(G,S) \\ ; & [X] & \longmapsto & (\sum_{x \in X^H} \|x\|)_H; \\ \psi^{(p)} \ : & A_p X\tilde{\Omega}(G,S) & \longrightarrow & A_p \mathrm{XObs}(G,S) \\ ; & \left(\sum_{s \in C_S(H)} \theta(H,s) s \right) & \longmapsto & \left(\sum_{nH \in (WH)_{s,p}} \overline{\theta(\langle n \rangle H, s)} \right), \end{array}$$

where $(WH)_{s,p}$ denotes a Sylow p-subgroup of $(WH)_s$.

We note that ψ_p is differ from $1_F \otimes \psi$ in general.

(4.9) **Theorem:** *The following sequence of A_p-modules is exact:*

$$0 \longrightarrow A_p X\Omega(G,S) \xrightarrow{\varphi^{(p)}} A_p X\tilde{\Omega}(G,S) \xrightarrow{\psi^{(p)}} A_p \mathrm{XObs}(G,S) \longrightarrow 0.$$

5 Idempotent formula

(5.1) **Möbius functions:** The Möbius function $\mu : P \times P \longrightarrow Z$ of a finite poset P is inductively defined by

$$\mu(x,x) = 1; \mu(x,y) = 0 \quad \text{if } x \not\leq y;$$

$$\sum_{t\leq y}\mu(x,t)=\sum_{t\geq x}\mu(t,y)=0 \quad \text{if } x<y.$$

(5.2) **Proposition** (Inversion formula): *Let*

$$\theta=\left(\sum_{s\in C_S(H)}\theta(H,s)s\right)\in Q\otimes_{\mathbf{Z}} X\tilde{\Omega}(G,S).$$

Then the inverse image of θ by $1_{\mathbf{Q}}\otimes\varphi$ is given by

$$\frac{1}{|G|}\sum_{D\leq G}|D|\sum_{(H,s)}\mu(D,H)\theta(H,s)[D,s],$$

where (H,s) runs over all pairs of $H\leq G$ and $s\in C_S(H)$, and μ is the Möbius function of the subgroup lattice of G.

(5.3) **Idempotents of a Burnisde ring:** Let $\Omega(G)$ be the Burnside ring of the finite group G. Then any primitive idempotent of $Q\times_{\mathbf{Z}}\Omega(G)$ has the form

$$e_{G,H}=\frac{1}{|N_G(H)|}\sum_{D\leq H}|D|\mu(D,H)[G/D].$$

If $\varphi=(\varphi_K):Q\otimes\Omega(G)\longrightarrow Q\otimes\tilde{\Omega}(G)$ denotes the Burniside map, then

$$\varphi_K(e_{G,H})=\begin{cases}1 & \text{if } H\sim_G K\\ 0 & \text{else.}\end{cases}$$

For a primes p, let $Z_{(p))}$ be the p-local integers a/b, $a\in Z, b\in Z-pZ$. Then the idempotent of $Z_{(p)}\otimes\Omega(G)$ has the form

$$e^p_{G,Q}:=\sum_{\substack{(H)\in C(G)\\ :O^p(H)=Q}}e_{G,H},$$

where Q is a p-perfect subgroup of G, that is, a subgroup which has no normal subgroup of index p and $O^p(H)$ is the smallest normal subgroup of H of index a power of p.

Let $\boldsymbol{a}$ be a G-functor over $Z_{(p)}$. Then $Z_{(p)}\otimes\Omega(G)$ acts on each component $\boldsymbol{a}(H)$, and so we have a decomposition

$$\boldsymbol{a}=\coprod_{(Q)}e^p_{G,Q}\cdot\boldsymbol{a},$$

in particular,

$$\boldsymbol{a}(G)=\coprod_{(Q)}e^p_{G,Q}\cdot\boldsymbol{a}(G).$$

Finally, note that if $\rho:\boldsymbol{a}\times\boldsymbol{b}\longrightarrow\boldsymbol{c}$ is a pairing of G-functors, then ρ is a $\Omega(G)$-bilinear map.

(5.4) **Lemma** (Transfer theorem):*Let* $\boldsymbol{A}$ *be a* G*-functor over* $\boldsymbol{Z}_{(p)}$ *and* Q *a* p*-perfect subgroup of* G. *Put* $N := N_G(Q)$ *and let* $e^p_{G,Q}$ *and* $e^p_{N,Q}$ *be the primitive idempotents of* $\boldsymbol{Z}_{(p)} \otimes_{\mathbf{Z}} \Omega(G)$ *and* $\boldsymbol{Z}_{(p)} \otimes_{\mathbf{Z}} \Omega(G)$*, respectively. Then there is an isomorphism of modules:*

$$e^p_{G,Q}\boldsymbol{a}(G) \cong e^p_{N,Q}\boldsymbol{a}(N).$$

This isomorphism is given by

$$e^p_{G,Q} \cdot a \longmapsto e^p_{N,Q} \cdot \mathrm{res}^G_N(a).$$

Furthermore, the isomorphism λ *is commutative with pairings.*

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