Augmenting Edge-Connectivity and Vertex-Connectivity Simultaneously

ISHII, Toshimasa; NAGAMOCHI, Hiroshi; IBARAKI, Toshihide

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Augmenting Edge-Connectivity and Vertex-Connectivity Simultaneously

ISHII Toshimasa, NAGAMOCHI Hiroshi and IBARAKI Toshihide

Kyoto University, Kyoto, Japan 606-01

Abstract Given an undirected multigraph \( G = (V, E) \) and requirement functions \( \{r_{\lambda}(x, y) \in \mathbb{Z}^+ | x, y \in V \} \) and \( \{r_{\kappa}(x, y) \in \mathbb{Z}^+ | x, y \in V \} \) (where \( \mathbb{Z}^+ \) is the set of nonnegative integers), the edge and vertex-connectivities augmentation problem asks to augment \( G \) by adding the smallest number of new edges to \( G \) so that for every \( x, y \in V \), the edge-connectivity and vertex-connectivity between \( x \) and \( y \) are at least \( r_{\lambda}(x, y) \) and \( r_{\kappa}(x, y) \), respectively in the resulting graph \( G' \). In this paper, we show that if \( r_{\kappa}(x, y) = 2 \) holds for every \( x, y \in V \), then the problem can be solved in polynomial time.

1 Introduction

Let \( G = (V, E) \) stand for an undirected multigraph with a set \( V \) of vertices and a set \( E \) of edges, where an edge with end vertices \( u \) and \( v \) is denoted by \( (u, v) \). A singleton set \( \{x\} \) may be simply denoted by \( x \). For two disjoint subsets of vertices \( X, Y \subseteq V \), we denote by \( E_{G}(X, Y) \) the set of edges, one of whose end vertices is in \( X \) and the other is in \( Y \), and also denote \( c_{G}(X, Y) = |E_{G}(X, Y)|. \) In particular, \( E_{G}(u, v) \) implies the set of edges with end vertices \( u \) and \( v \). We denote \( n = |V| \) and \( e = |E| \). For a subset \( V' \subseteq V \) in \( G \), \( G - V' \) denotes the subgraph induced by \( V - V' \). A cut is defined as a subset \( X \subseteq V \) with \( \emptyset \neq X \neq V \), and the size of a cut \( X \) is denoted by \( c_{G}(X, V - X) \), which may also be written as \( c_{G}(X) \). A cut with the minimum size is called a (global) minimum cut, and its size, denoted by \( \lambda(G) \), is called the edge-connectivity of \( G \). The local edge-connectivity \( \lambda_{G}(x, y) \) for two vertices \( x, y \in V \) is defined to be the minimum size of a cut in \( G \) that separates \( x \) and \( y \) (i.e., \( x \) and \( y \) belong to different sides of \( X \) and \( V - X \)), or equivalently the maximum number of edge-disjoint paths between \( x \) and \( y \) by Menger's theorem [4].

For a subset \( X \subseteq V \), \( \{v \in V - X | (u, v) \in E \} \) for some \( u \in X \) is called the neighbor set of \( X \), denoted by \( \Gamma_{G}(X) \). Let \( p(G) \) denote the number of components in \( G \). A separator of \( G \) is defined as a cut \( S \subseteq V \) such that \( p(G - S) > p(G) \) holds and no \( S' \subseteq S \) has this property. A separator always exists, unless \( G \) contains the complete graph \( K_{n} \). If \( G \) does not contain \( K_{n} \), then a separator of the minimum size is called a (global) minimum separator, and its size, denoted by \( \kappa(G) \), is called the vertex-connectivity of \( G \). If \( G \) contains the complete graph \( K_{n} \), we define \( \kappa(G) = n - 1 \). The local vertex-connectivity \( \kappa_{G}(x, y) \) for two vertices \( x, y \in V \) is defined to be the number of internally-disjoint paths between \( x \) and \( y \) in \( G \).

For any separator \( S \), there is the component \( X \) of \( G \) such that \( X \supseteq S \), and we call the components in \( G[X] - S \) the \( S \)-components. Let

\[
\beta(G) = \max \{p(G - S) | S \text{ is a minimum separator in } G\}. \tag{1.1}
\]

A cut \( T \subseteq V \) is called tight if \( \Gamma_{G}(T) \) is a minimum separator in \( G \) and no \( T' \subseteq T \) has this property (hence, \( G[T] \) induces a connected graph). Let \( t(G) \) denotes the maximum number of pairwise disjoint tight sets in \( G \).

In this paper, for a given function \( a : \left( \frac{V}{2} \right) \rightarrow \mathbb{R}^{+} \) (resp., \( b : \left( \frac{V}{2} \right) \rightarrow \mathbb{R}^{+} \)), we define \( R^{+} \) while denotes the set of nonnegative real numbers, we call \( G \) a-edge-connected (resp., \( b \)-vertex-connected) if \( \lambda_{G}(x, y) \geq a(x, y) \) (resp., \( \kappa_{G}(x, y) \geq b(x, y) \)) holds for every \( x, y \in V \). Given a multigraph \( G = (V, E) \) and a requirement function \( r_{\lambda} : \left( \frac{V}{2} \right) \rightarrow \mathbb{Z}^{+} \) (resp., a requirement function \( r_{\kappa} : \left( \frac{V}{2} \right) \rightarrow \mathbb{Z}^{+} \)), where \( \mathbb{Z}^{+} \) denotes the set of nonnegative integers, the edge-connectivity augmentation problem, (resp., the vertex-connectivity augmentation problem) asks to augment \( G \) by adding the smallest number of new edges so that the resulting graph \( G' \) becomes \( r_{\lambda} \)-edge-connected (resp., \( r_{\kappa} \)-vertex-connected). When the requirement function \( r_{\lambda} \) (resp., \( r_{\kappa} \)) satisfies \( r_{\lambda}(x, y) = k \in \mathbb{Z}^{+} \) for all \( x, y \in V \) (resp., \( r_{\kappa}(x, y) = \ell \in \mathbb{Z}^{+} \) for all \( x, y \in V \)), this problem is called the global \( k \)-edge-connectivity problem (resp., the global \( \ell \)-vertex-connectivity problem).

Watanabe and Nakamura [16] first proved that the global \( k \)-edge-connectivity augmentation problem can be solved in polynomial time for any given integer \( k \). Their algorithm increases edge-connectivity one by one, each time augmenting edges on the basis of structural information of the current \( G \). Currently, \( O(e + k^2 n \log n) \) time algorithm due to Gabow [6] and \( O(n^3) \) time randomized algo-
rithm due to Benczúr [1], whose deterministic running time is $O(n^3)$, are the fastest among existing algorithms. Different from the approach by Watanabe and Nakamura, Cai and Sun [2] first pointed out that the augmentation problem for a given $k$ can be directly solved by applying the Mader's edge-splitting theorem. Based on this, Frank [5] gave an $O(n^6)$ time augmentation algorithm. Afterwards, Gabow [7] and Nagamochi and Ibaraki [14] improved it to $O(mn^{2} \log(n^2/m))$ and $O(n^2(m + n \log n))$, respectively. Recently, Nagamochi and Ibaraki [15] gave an $O(n(m + n \log n) \log n)$ time algorithm. For a general requirement function $r_\kappa$, Frank [5] showed that the edge-connectivity augmentation problem can be solved in polynomial time by using Mader's edge-splitting theorem, and recently the time complexity was improved by Gabow [7] to $O(n^6 \log(n^2/m))$.

As to the vertex-connectivity augmentation problem, the problem of adding the minimum number of new edges to a $k$-vertex-connected graph to make it $(k+1)$-vertex-connected has been studied by several researchers. It is easy to see that $M(G) = \max\{\beta(G) - 1, \lceil t(G)/2 \rceil\}$ provides a lower bound on the optimal value to this problem. Eswaran and Tarjan [3] proved that the vertex-connectivity augmentation problem can be solved by adding $M(G)$ edges to $G$ for $k = 1$. Watanabe and Nakamura [17] stated the same result for $k = 2$. However, $M(G)$ can be smaller than the optimal value for general $k \geq 3$. Recently Jordán presented an $O(n^5)$ time approximation algorithm for this problem [11, 12]. The difference between the number of new edges added by his algorithm and the optimal value is at most $(k-2)/2$.

It is known that if the requirement function $r_\kappa$ satisfies $r_\kappa(x,y) = k$ for all $x,y \in V$, where $k \in \{2, 3, 4\}$, then the global $k$-vertex-connectivity augmentation problem can be solved in polynomial time due to [3, 9], [17, 8], [10], where an input graph $G$ may not be $(k-1)$-vertex-connected. However, whether there is a polynomial time algorithm for the global vertex-connectivity augmentation problem for an arbitrary $k$ is an open question (even if $G$ is $(k-1)$-vertex-connected).

In this paper, we consider the problem of augmenting the edge-connectivity and the vertex-connectivity of a given graph $G$ simultaneously by adding the smallest number of new edges. For two given functions $a : \binom{V}{2} \rightarrow R^+$ and $b : \binom{V}{2} \rightarrow R^+$, we say that $G$ is $(a,b)$-connected if $G$ is $a$-edge-connected and $b$-vertex-connected.

Given a multigraph $G = (V,E)$, and two requirement functions $r_\lambda : \binom{V}{2} \rightarrow \mathbb{Z}^+$ and $r_\kappa : \binom{V}{2} \rightarrow \mathbb{Z}^+$, the edge-and-vertex-connectivity augmentation problem, denoted by EVAP($r_\lambda, r_\kappa$), asks to augment $G$ by adding the smallest number of new edges to $G$ so that the resulting graph $G'$ becomes $(r_\lambda, r_\kappa)$-connected. Without loss of generality, $r_\lambda(x,y) \geq r_\kappa(x,y)$ is assumed for all $x,y \in V$, since if a graph is $r_\kappa$-vertex-connected then it is $r_\kappa$-edge-connected. Clearly, EVAP($r_\lambda, r_\kappa$) contains the edge-connectivity augmentation problem and the vertex-connectivity augmentation problem as its special cases.

When the requirement function $r_\kappa$ satisfies $r_\kappa(x,y) = \ell \in \mathbb{Z}^+$ for all $x,y \in V$, this problem is denoted by EVAP($r_\lambda, \ell$), if no confusion arises. In this paper, we consider this problem in case $r_\kappa(x,y) = 2$ holds for every $x,y \in V$ (but $r_\lambda(x,y)$ are arbitrary). We first present a lower bound on the number of edges that is necessary to make a given graph $G$ $(r_\lambda, 2)$-connected. We then show that this problem can be solved in polynomial time, by actually presenting a polynomial time algorithm that adds a new edge set whose size is equal to this lower bound.

In Section 2, after introducing basic definitions and the concept of edge-splitting, we derive a lower bound on the number of edges that are necessary to make a given graph $G$ $(r_\lambda, r_\kappa)$-connected. In Section 3, we outline our algorithm for making a given graph $G$ $(r_\lambda, 2)$-connected by adding a new edge set whose size is equal to the above lower bound. In Sections 4–7, we prove the correctness of each step in our algorithm.

## 2 Preliminaries

### 2.1 Definitions

For a multigraph $G = (V,E)$, its vertex set $V$ and edge set $E$ may be denoted by $V[G]$ and $E[G]$, respectively. For a subset $V' \subseteq V$ (resp., $E' \subseteq E$) in $G$, $G[V']$ (resp., $G[E']$) denotes the subgraph induced by $V'$ (resp., $E'$). For $V' \subseteq V$ (resp., $E' \subseteq E$) in $G$, we denote $G[V-V']$ (resp., $G[E-E']$) simply by $G-V'$ (resp., $G-E'$). For an edge set $F$ with $F \cap E = \emptyset$, we denote $G = (V,E \cup F)$ by $G + F$. A partition $X_1, \ldots, X_r$ of vertex set $V$ means a family of nonempty disjoint subsets of $V$ whose union is $V$, and a subpartition of $V$ means a partition of a subset of $V$.

We say that a cut $X$ separates two disjoint subsets $Y$ and $Y'$ of $V$ if $Y \subseteq X$ and $Y' \subseteq V-X$ (or $Y \subseteq V-X$ and $Y' \subseteq X$) hold. In particular, a cut $X$ separates $x$ and $y$ if $x \in X$ and $y \in V-X$ (or $x \in V-X$ and $y \in X$) hold. A cut $X$ crosses another cut $Y$ if none of subsets $X \cap Y$, $X-Y$, $Y-X$ and $V-(X \cup Y)$ is empty. We say that a separator $S \subseteq V$ separates two disjoint subsets $Y$ and $Y'$ of $G-S$ if no two vertices $x \in Y$ and $y \in Y'$ are connected in $G-S$. In particular, a separator $S$ separates vertices $x$ and $y$ in $V-S$ if $x$ and $y$ are contained in different components of $G-S$.

### 2.2 Edge-Splitting

In this section, we introduce an operation of transforming a graph, called edge-splitting, which is helpful to solve the edge-connectivity augmentation problem.
Given a multigraph $G = (V, E)$, a designated vertex $x \in V$, vertices $u, v \in \Gamma_{G}(s)$ (possibly $u = v$) and a nonnegative integer $\delta \leq \min\{c_{G}(s, u), c_{G}(s, v)\}$, we construct graph $G' = (V, E')$ from $G$ by deleting $\delta$ edges from $E_{G_{\cdot}}(s, u)$ and $E_{G_{\cdot}}(s, v)$, respectively, and adding new $\delta$ edges to $E_{G_{\cdot}}(u, v)$:

- $c_{G'}(s, u) := c_{G}(s, u) - \delta,$
- $c_{G'}(s, v) := c_{G}(s, v) - \delta,$
- $c_{G'}(u, v) := c_{G}(u, v) + \delta,$  
- $c_{G'}(x, y) := c_{G}(x, y)$ for all other pairs $x, y \in V$.

In case of $u = v$, we interpret that $c_{G'}(s, u) := c_{G}(s, u) - 2\delta$, $c_{G'}(u, u) := c_{G}(u, u) + 2\delta$, and $c_{G'}(x, y) := c_{G}(x, y)$ for all other pairs $x, y \in V$, where an integer $\delta$ is chosen so as to satisfy $0 \leq \delta \leq \frac{1}{2} c_{G}(s, u)$. We say that $G'$ is obtained from $G$ by splitting $\delta$ pair of edges $(s, u)$ and $(s, v)$ (or by splitting $(s, u)$ and $(s, v)$ by size $\delta$), and denote the resulting graph $G'$ by $G'(u, v; \delta)$. A sequence of splittings is complete if the resulting graph $G'$ does not have any neighbor of $s$.

The following theorem is proven by Mader [13].

**Theorem 2.1** [13] Let $G = (V, E)$ be a multigraph with a designated vertex $x \in V$ with $c_{G}(s) \neq 1, 3$ and $\lambda_{G}(x, y) \geq 2$ for all pairs $x, y \in V$. Then for any edge $(s, u) \in E$ there is an edge $(s, v) \in E$ such that $\lambda_{G_{\cdot}}(u, v; 1) (x, y) = \lambda_{G}(x, y)$ holds for all pairs $x, y \in V - s$.\[\Box\]

This says that if $c_{G}(s)$ is even, there always exists a complete splitting at $s$ such that the resulting graph $G'$ satisfies $\lambda_{G_{\cdot}}(s, x, y) = \lambda_{G}(x, y)$ for every pair of $x, y \in V - s$.

### 2.3 Lower Bound

In this section, we consider problem EVAP$(r_{\lambda}, r_{\kappa})$, and give a lower bound on the number of edges that is necessary to make a graph $G$ $(r_{\lambda}, r_{\kappa})$-connected, where $r_{\lambda}$ and $r_{\kappa}$ are given requirement functions. Define

- $r_{\lambda}(X) \equiv \max\{r_{\lambda}(u, v) \mid u, v \in X \in V - X\}$ for each cut $X$,
- $r_{\kappa}(X) \equiv \max\{r_{\kappa}(u, v) \mid u, v \in X \in V - X - \Gamma_{G}(X)\}$ for each cut $X$ with $V - X - \Gamma_{G}(X) \neq \emptyset$, where see Section 1 for the definition of $\Gamma_{G}(X)$.

To make a graph $G$ $r_{\lambda}$-edge-connected, it is necessary to add

1. at least $r_{\lambda}(X) - c_{G}(X)$ edges between $X$ and $V - X$ for each cut $X$.

Also, to make a graph $G$ $r_{\kappa}$-vertex-connected, it is necessary to add

2. at least $r_{\kappa}(X) - |\Gamma_{G}(X)|$ edges between $X$ and $V - X - \Gamma_{G}(X)$ for each cut $X$ with $V - X - \Gamma_{G}(X) \neq \emptyset$.

For a separator $S$ of $G$, let $T_{1}, \ldots, T_{q}$ denote all components of $G - S$. Now we consider a graph $H_{S} = ((T_{1}, \ldots, T_{q}), \mathcal{E})$ in which we regard each $T_{i}$ as one vertex of $H_{S}$ and the edge set $\mathcal{E}$ is defined as follows:

- $T_{i}, T_{j}$: $T_{i} \in \mathcal{E}$ if $x \in T_{i}$ and $y \in T_{j}$ with $r_{\kappa}(x, y) \geq |S| + 1$.

In a $r_{\kappa}$-vertex-connected graph, any pair of vertices $x, y \in V$ with $r_{\kappa}(x, y) \geq |S| + 1$ cannot be separated by such separator $S$. Hence, if there is a pair of vertices $x \in T_{i}$ and $y \in T_{j}$ with $r_{\lambda}(x, y) \geq |S| + 1$, then we must add at least one edge between $T_{i}$ and $T_{j}$ (i.e., the number of $S$-components must become at most $p(H_{S})$), in order to make $G$ $r_{\kappa}$-vertex-connected. Therefore in this case, it is necessary to add

(3) at least $p(G - S) - p(H_{S})$ edges to connect components of $G - S$ for a separator $S$.

(See Section 1 for the definition of $p(G - S)$.) Now define $\delta(G) = \max\{p(G - S) - p(H_{S}) \mid S$ is a separator in $G\}$.

Given a subpartition $\{X_{1}, \ldots, X_{p}, X_{p+1}, \ldots, X_{q}\}$ of $V$ such that $q \geq p \geq 0$ and $V - X_{i} - \Gamma_{G}(X_{i}) \neq \emptyset$, we need to add $\sum_{i=1}^{p} (r_{\lambda}(X_{i}) - c_{G}(X_{i}))$ edges for each $X_{i}$, $i = 1, \ldots, p$, and to add $\sum_{i=p+1}^{q} (r_{\lambda}(X_{i}) - |\Gamma_{G}(X_{i})|)$ edges for each $X_{i}$, $i = p + 1, \ldots, q$, based on observations (1) and (2). Now note that adding one edge to $G$ can contribute to the requirements of at most two $X_{i}$, Therefore, we need to add $[\frac{\alpha(G)}{2}]$ new edges to make $G$ $(r_{\lambda}, r_{\kappa})$-edge-connected, where

\[
\alpha(G) = \max \left\{ \sum_{i=1}^{p} (r_{\lambda}(X_{i}) - c_{G}(X_{i})) \right\} + \sum_{i=p+1}^{q} (r_{\kappa}(X_{i}) - |\Gamma_{G}(X_{i})|) \quad (2.1)
\]

and the max is taken over all subpartitions $\{X_{1}, \ldots, X_{p}, X_{p+1}, \ldots, X_{q}\}$ of $V$ such that $q \geq p \geq 0$ and $V - X_{i} - \Gamma_{G}(X_{i}) \neq \emptyset$, $i = p + 1, \ldots, q$. On the other hand, from observation (3), to make $G$ $r_{\kappa}$-vertex-connected, at least $\max\{p(G - S) - p(H_{S}) \mid S$ is a separator in $G\}$ new edges are necessarily added to $G$. Consequently, we have the next lemma.

**Lemma 2.1 (Lower Bound)** To make a given graph $G$ $(r_{\lambda}, r_{\kappa})$-connected, at least

\[
\gamma(G) \equiv \max\{[\frac{\alpha(G)}{2}], \delta(G)\}
\]

new edges must be added.\[\Box\]

Now we specialize this lower bound to problem EVAP$(r_{\lambda}, 2)$ based on which we give a polynomial time algorithm for solving EVAP$(r_{\lambda}, 2)$ in the next section.

In problem EVAP$(r_{\lambda}, 2)$, we can assume $r_{\lambda}(x, y) \geq r_{\kappa}(x, y) = 2$ for all $x, y \in V$. Now the $\alpha(G)$ in (2.1) can be simplified to
\[ \alpha(G) = \max \left\{ \sum_{i=1}^{p} (r_{i}(X_{i}) - c_{G}(X_{i})) + \sum_{i=p+1}^{q} (2 - |\Gamma_{G}(X_{i})|) \right\}, \]  

where the maximization is taken over all subpartitions \{X_{1}, \cdots, X_{p}, X_{p+1}, \cdots, X_{q}\} of \( V \) such that \( q \geq p \geq 0 \) and \( V - X_{i} - \Gamma_{G}(X_{i}) \neq \emptyset \) for \( i = p + 1, \cdots, q \).

Also we specialize the second lower bound \( \delta(G) \). Now, to derive \( \delta(G) \), the maximization is taken over all separators \( S \) that satisfy \( |S| \leq 1 \), since each pair of vertices \( x, y \in V \) satisfy \( r_{s}(x, y) = 2 \). Note that \( p(H_{S}) = 1 \) holds for any separator \( S \) with \( |S| \leq 1 \), since any pair of \( S \)-components \( T_{i} \) and \( T_{j} \) has a pair of vertices \( x \in T_{i} \) and \( y \in T_{j} \) where \( r_{s}(x, y) = 2 > |S| \). Hence this lower bound can be rewritten by

\[ \max\{p(G - S) - 1 \mid S \text{ is a separator with } |S| \leq 1\}. \] (2.3)

A vertex \( v \) is called a cut vertex in \( G = (V, E) \) if \( S = \{v\} \) is a minimum separator in \( G \). If \( G \) has a cut vertex \( v \in V \), then \( p(G - v) > p(G) \) holds from the definition of a separator; otherwise \( p(G - v) = p(G) \) holds for all \( v \in V \). Hence the lower bound in (2.3) can be simplified to

\[ \max_{v \in V\{p(G - v) - 1\}}. \]

Also note that if \( \kappa(G) \leq 1 \) holds, then (1.1) in Section 1 satisfies \( \beta(G) = \max_{v \in V\{p(G - v) \}} \) and the lower bound in (2.3) can be simplified to \( \beta(G) - 1 \). In case of \( \kappa(G) \geq 2 \), the lower bound in (2.3) is not defined but \( \max_{v \in V\{p(G - v) - 1\}} = 0 \) holds. Therefore, in Problem EVAP(\( r_{1} \)), we can define the lower bound in (2.3) by \( \max_{v \in V\{p(G - v) - 1\}} \) without confusion. This means that we can define

\[ \beta(G) = \max_{v \in V\{p(G - v)\}}. \] (2.4)

and the lower bound in (2.3) becomes

\[ \beta(G) - 1. \]

Now define \( \gamma(G) = \max\{[\alpha(G)/2], \beta(G) - 1\} \). From the above discussion, a set of new edges gives an optimal solution to EVAP(\( r_{1} \)) if its size is equal to \( \gamma(G) \) and the graph obtained by adding \( \gamma(G) \) edges to \( G \) is \((r_{1}, 2)\)-connected. We now show that this is always possible, by presenting a polynomial time algorithm in the next section for making \( G \) \((r_{1}, 2)\)-connected by adding \( \gamma(G) \) new edges.

**Lemma 2.2** If \( \kappa(G) = 1 \) (i.e., \( G \) is connected and has a cut vertex), then any two tight sets \( X \) and \( Y \) in \( G \) are disjoint. \( \square \)

### 3 A Polynomial Time Algorithm for EVAP(\( r_{1} \), 2)

We now present a polynomial time algorithm, based on the argument in the previous section. Call an edge \( e = (u, u') \) admissible with respect to a vertex \( v \) if \( v \) is a cut vertex such that \( v \neq u, u' \) and \( p(G - v) = p((G - e) - v) \). For a subset \( F \) of edges in a graph \( G \), we say that two edge \( e_{1} = (u_{1}, w_{1}) \) and \( e_{2} = (u_{2}, w_{2}) \) are switched in \( F \) if we delete \( e_{1} \) and \( e_{2} \) from \( F \), and add edges \( (u_{1}, u_{2}) \) and \( (w_{1}, w_{2}) \) to \( F \). Our algorithm for solving the EVAP(\( r_{1} \), 2), denoted by Algorithm EVA(\( r_{1} \), 2), consists of the following four major steps.

**Algorithm EVA(\( r_{1} \), 2)**

**Input:** An undirected multigraph \( G = (V, E) \), and a requirement function \( \{r_{1}(x, y) \in Z^{+} \mid x, y \in V\} \).

**Output:** An undirected multigraph \( G' = G + F \) with \( \lambda_{G'}(x, y) \geq \gamma_{G'}(x, y) \) for every \( x, y \in V \) and \( \kappa(G') \geq 2 \) where the size of new edge set \( F \) is the minimum.

**Step I. (Addition of vertex \( s \) and associated edges):**

After adding a new vertex \( s \), add a set \( F' \) of a sufficiently large number of edges between \( s \) and \( V \) so that the resulting graph \( G' = (V \cup \{s\}, E \cup F') \) satisfies

\[ c_{G'}(X) \geq r_{1}(X) \] (3.1)

for all \( X \) with \( \emptyset \neq X \subset V \),

\[ |\Gamma_{G'}(X \cup s)| \geq 2 \] (3.2)

for all \( X \) with \( \emptyset \neq X \subset V \) and \( V - X - \Gamma_{G'}(X) \neq \emptyset \). (This can be done for example by adding \( \max\{r_{1}(x, y) \mid x, y \in V\} \) edges between \( s \) and each vertex \( v \in V \).)

Next, to make \( F' \) minimal we discard new edges in \( F' \), one by one, as long as (3.1) and (3.2) remain valid. Denote the resulting set of new edges by \( F_{1} \) and the resulting graph by \( G_{1} = (V \cup \{s\}, E \cup F_{1}) \), where \( F_{1} = E_{G_{1}}(s, V) \) If the resulting graph by \( G_{1} \) is not connected, then \( \kappa_{G_{1}}(x, y) \geq 2 \) cannot be attained for some \( x, y \in V \), since a subset \( X \subset V \) which induces a component \( G[X] \) of \( G \) satisfies \( \Gamma_{G_{1}}(X) = \emptyset \) or \( \{s\} \), and hence \( \kappa_{G_{1}}(x, y) \leq 1 \) for \( x \in X \) and \( y \in V - X \).

**Property 3.1** In the above step, it is possible to choose a subset \( F_{1} \) for which \( |F_{1}| = \alpha(G) \) holds. \( \square \)
Step II. (Edge-splitting): If $e_{G_{1}}(s)$ is odd, then we add one edge $(s, w)$ to $G$ by choosing vertex $w \in V$ which is not a cut vertex in $G$.

Next we find a complete edge-splitting at $s$ in $G_{1} = (V \cup \{s\}, E \cup F_{1})$ which preserves condition (3.1) (i.e., the $r_{1}$-edge-connectivity). By Mader’s theorem, there always exists such a complete edge-splitting at $s$, and it can be computed in polynomial time. Let $G_{2} = (V, E \cup F_{2})$ denote the graph obtained by such a complete edge-splitting, ignoring the isolated vertex $s$. The next is immediate from Mader’s theorem.

Property 3.2 There is a complete edge-splitting at $s$ of $G_{1}$, so that the resulting graph $G_{2}$ is $r_{1}$-edge-connected. $\square$

If $G_{2}$ is also 2-vertex-connected, then we are done because $|F_{2}| = |F_{1}|/2 = \lceil \alpha(G)/2 \rceil$ implies that $G_{2}$ is optimally augmented by lower bound $\lceil \alpha(G)/2 \rceil$. Otherwise, go to Step III.

Step III. (Switching edges): Now $G_{2}$ has cut vertices. Then, by property (3.2) for $G_{1}$, $G_{2}$ satisfies

$$G_{2}[X \cup \{v\}] \text{ contains at least one edge in } F_{2} \text{ for any cut vertex } v \quad (3.3)$$

and its $v$-component $X$.

Property 3.3 Assume that $G_{2}$ has an admissible edge $e_{1} \in F_{2}$ with respect to a cut vertex $v$. Let $X$ be a $v$-component with $e_{1} \notin E[G_{2}[X \cup \{v\}]]$, and $e_{2}$ be chosen arbitrarily from $F_{2} \cap E[G_{2}[X \cup \{v\}]]$. Then switching $e_{1}$ and $e_{2}$ decreases the number of $v$-components in $G_{2}$ at least by one while preserving the $r_{1}$-edge-connectivity. Moreover, the resulting graph $G_{2}'$ from switching $e_{1}$ and $e_{2}$ still satisfies (3.3), and $\kappa_{G_{2}}(x, y) \geq 2$ holds for any pair of vertices $x$ and $y$ with $\kappa_{G_{2}}(x, y) \geq 2$. $\square$

Property 3.4 If $G_{2}$ has two cut vertices $v_{1}$ and $v_{2}$, then there are $v_{1}$-component $X_{1}$ and $v_{2}$-component $X_{2}$ such that $X_{1} \cap X_{2} = \emptyset$. Let edge $e_{1}$ be arbitrarily chosen from $F_{2} \cap E[G_{2}[X_{1} \cup \{v\}]]$. Then $e_{1}$ is admissible with respect to $v_{2}$. $\square$

Based on Property 3.3, Step III repeats switching pairs of edges in $F_{2}$ until the resulting graph has no admissible edge in $F_{2}$.

Let $G_{3} = (V, E \cup F_{3})$ be the resulting graph obtained by such a sequence of switching edges in $F_{2}$, where $F_{3}$ denotes the final $F_{2}$. Then Property 3.4 implies that, if there are at least two cut vertices, then $G_{3}$ has an admissible edge in $F_{3}$, which is a contradiction. Hence $G_{3}$ has the following property.

Property 3.5 $G_{3}$ has at most one cut vertex. $\square$

If $G_{3}$ has no cut vertex, then we are done, since $|F_{3}| = \lceil \alpha(G)/2 \rceil$ implies that $G_{3}$ is optimally augmented. Otherwise, go to Step IV.

Step IV. (Edge augmentation): Now $G_{3}$ has exactly one cut vertex $v$. Then $G_{3}$ and $v$ satisfy the following property.

Property 3.6 For the graph $G_{3}$ and its cut vertex $v$, it holds $p(G_{3} - v) = p(G - v) - \lceil \alpha(G)/2 \rceil$. $\square$

Now let $T_{1}, \ldots, T_{q}$ be all $v$-components in $G_{3}$, where $q = p(G_{3} - v)$. We can make $G_{3}$ 2-vertex-connected by adding one edge between $T_{i}$ and $T_{i+1}$ for each $i = 1, \ldots, q - 1$ (i.e., $p(G_{3} - v)$ edges in total). Let $F_{4}$ denote a set of these $p(G_{3} - v) - 1$ edges added. Note that $p(G_{3} - v) = p(G - v) - \lceil \alpha(G)/2 \rceil \leq \beta(G) - \lceil \alpha(G)/2 \rceil$ holds from Property 3.6 and $\beta(G) \geq p(G - v)$ (see (2.4)). Also note that $|F_{4}| \geq p(G_{3} - v) - 1 + \lceil \alpha(G)/2 \rceil$ holds since $\beta(G) - 1$ is a lower bound on the number of edges that must be added to make $G$ $(r_{1}, 2)$-connected. These imply $|F_{4}| = \beta(G) - 1 - \lceil \alpha(G)/2 \rceil$. Therefore we have the following property.

Property 3.7 There is a set of $\beta(G) - 1 - \lceil \alpha(G)/2 \rceil$ new edges $F_{4}$ obtained for $G_{3}$ such that the resulting graph $G_{4} = (V, E \cup F_{3} \cup F_{4})$ is 2-vertex-connected.

Finally, we are done since $|F_{3}| + |F_{4}| = \beta(G) - 1$ implies that $G_{4}$ is optimally augmented by lower bound $\beta(G) - 1$.

We shall explain in the subsequent sections that the required properties (summarized as Properties 3.1 – 3.7) always hold. Together with these proofs, this algorithm establishes the next theorem, which is the main goal of this thesis.

Theorem 3.1 Given a requirement function $r_{\lambda}(x, y) \in Z^{+} \mid x, y \in V \}$, a multigraph $G$ can be made $(r_{\lambda}, 2)$-connected by adding $\gamma(G) = \max \{\lceil \alpha(G)/2 \rceil, \beta(G) - 1\}$ new edges in $O(n^{3}m \log \frac{n^{2}}{m})$ time. $\square$

4 Correctness of Step I

We give a proof of Property 3.1 in order to prove the correctness of Step I.

Proof of Property 3.1: It is clear that $\lambda_{G_{1}}(x, y) \geq r_{\lambda}(x, y) \geq 2$ holds for all $x, y \in V$ by (3.1).

First, we show $|F_{1}| \geq \alpha(G)$. Let $F^{*} = \{X_{1}, \ldots, X_{p}, X_{p+1}, \ldots, X_{q}\}$ be a subpartition of $V$
with $V - X' - \Gamma_{G_{1}}(X') \neq \emptyset$ for $i = p + 1, \ldots, q$ that attains the maximum of (2.2); i.e., $\alpha(G) = \sum_{i=1}^{p}(r_{\lambda}(X_{i}) - c_{G}(X_{i})) + \sum_{i=p+1}^{q}(2 - |\Gamma_{G}(X_{i})|)$. If $|F_{1}| < \alpha(G)$ holds, then there must be at least one cut $X_{i} \in F^{*}$ that violates (3.1) or (3.2), contradicting construction of $G_{1}$.

Now we prove the converse, $|F_{1}| \leq \alpha(G)$, through five claims.

A cut $X \subset V$ is called critical in $G_{1}$ if $s \in \Gamma_{G_{1}}(X)$ holds and the removal of any edge $e \in E_{G_{1}}(s, X)$ violates (3.1) or (3.2). Clearly, a subset $X \subset V$ with $s \in \Gamma_{G_{1}}(X)$ is critical if and only if $X$ satisfies at least one of the following conditions:

1. $c_{G_{1}}(X) = r_{\lambda}(X)$.
2. $c_{G_{1}}(s, X) = 1, |\Gamma_{G_{1}}(X) - s| = 1$, and $V - X = \Gamma_{G_{1}}(X) \neq \emptyset$.
3. $\{s\} = \Gamma_{G_{1}}(X)$, $|\Gamma_{G_{1}}(s) \cap \Gamma_{G_{1}}(X)| = 2$, and there is a vertex $v \in \Gamma_{G_{1}}(s) \cap X$ with $c_{G_{1}}(s, v) = 1$.

We call a critical cut $X$ $v$-minimal if $v \in \Gamma_{G_{1}}(s) \cap X$ and there is no critical cut $X'$ with $\{v\} \subset X' \subset X$. A subset $X$ is called critical of type (1) (resp., (2), (3)) if it satisfies (1) (resp., (2), (3)).

We will prove that $G_{1}$ has a set of critical cuts $X_{1}, \ldots, X_{q}$ only of type (1) and (2) such that

$$X_{i} \cap X_{j} = \emptyset, 1 \leq i < j \leq q$$

and

$$\Gamma_{G_{1}}(s) \subseteq X_{1} \cup \cdots \cup X_{q}. \quad (4.1)$$

This implies that

$$|F_{1}| = \sum_{i=1}^{p}(r_{\lambda}(X_{i}) - c_{G}(X_{i})) + \sum_{i=p+1}^{q}(2 - |\Gamma_{G}(X_{i})|)$$

where $X_{i}, i = 1, \ldots, p$ is of type (1) and $X_{i}, i = p + 1, \ldots, q$ is of type (2), from which $|F_{1}| \leq \alpha(G)$ by definition of $\alpha(G)$.

Claim 4.1 Any critical cut $X$ of type (3) is also critical of type (1).

By this claim, we can regard critical cuts of type (3) as those of type (1). The next property is known in [5].

Claim 4.2 Let $X$ and $Y$ be critical cuts of type (1) in $G_{1}$. Then at least one of the following statements holds.

(i) Both $X \cap Y$ and $X \cup Y$ are critical.

(ii) Both $X - Y$ and $Y - X$ are critical, and $c_{G_{1}}(X \cap Y, (V \cup \{s\}) - (X \cup Y)) = 0$.

An analogous property holds for type (2) critical cuts.

Claim 4.3 Let $X$ and $Y$ be critical cuts of type (2). If $Y$ is $v$-minimal for some $v \in V - X$, then they do not cross each other.

Claim 4.4 Let $X$ be a critical cut of type (1), and $Y$ be a critical cut of type (2) such that $\Gamma_{G_{1}}(s) \cap (Y - X) \neq \emptyset$. If $X$ and $Y$ cross each other, then $c_{G_{1}}(X \cap Y, s) = 0$ holds and cut $Y - X$ is critical of type (1).

Now we are ready to prove that $G_{1}$ has a set of critical cuts $X_{1}, \ldots, X_{q}$ that satisfies (4.1). Let $N_{1} \subseteq \Gamma_{G_{1}}(s)$ be the set of neighbors of $u$ such that there is a critical cut $X$ of type (1) with $u \in X$. Let us choose a critical cut $X_{u}$ of type (1) with $u \in X_{u}$ for each $u \in N_{1}$ so that $\sum_{X \in \{X_{u} \mid u \in N_{1}\}}|X|$ is minimized.

Claim 4.5 $F = F_{1} \cup F_{2}$ consists of disjoint critical cuts whose union contains $\Gamma_{G_{1}}(s)$.

Proof. Let $F_{1} = \{X_{1}, \ldots, X_{p}\}$ and $F_{2} = \{X_{p+1}, \ldots, X_{q}\}$ with each $\emptyset \neq X_{i} \subset V$. Clearly, $\Gamma_{G_{1}}(s) \subseteq \cup_{X_{i} \in F_{1}}X_{i}$ holds from construction of $F$.

We show that $X_{i}$ and $X_{j}$ are pairwise disjoint for each $X_{i}, X_{j} \in F_{1}$. Assume that $F_{1}$ contains $X_{i}$ and $X_{j}$ which are not pairwise disjoint. Note that $X_{i} \cap X_{j}$ does not hold from construction of $F_{1}$. If $X_{i}$ and $X_{j}$ cross each other, then Claim 4.2 implies that at least one of the following statements holds:

(i) Both $X_{i} \cap X_{j}$ and $X_{i} \cup X_{j}$ are critical.

(ii) Both $X_{i} - X_{j}$ and $X_{j} - X_{i}$ are critical, and $c_{G_{1}}(X \cap Y_{i}, (V \cup \{s\}) - (X \cup Y)) = 0$.

If the statement (i) holds, then $F'_{1} = (F_{1} - X_{i} - X_{j}) \cup \{X_{i} \cup X_{j}\}$ would satisfy $N_{1} \subseteq F'_{1}$ and $\sum_{X \in F_{1}}|X| < \sum_{X \in F_{1}}|X|$, contradicting the minimality of $\sum_{X \in F_{1}}|X|$. If the statement (ii) holds, then $F''_{1} = (F_{1} - X_{i} - X_{j}) \cup \{X_{i} - X_{j}, X_{j} - X_{i}\}$ satisfies $\sum_{X \in F_{1}}|X| < \sum_{X \in F_{1}}|X|$ and $N_{1} \subseteq F''_{1}$ (by $c_{G_{1}}(X \cap Y_{i}, (V \cup \{s\}) - (X \cup Y)) = 0$). This again contradicts the minimality of $\sum_{X \in F_{1}}|X|$. Therefore $X_{i}$ and $X_{j}$ are pairwise disjoint for each $X_{i}, X_{j} \in F_{1}$.

Claim 4.3 implies that $X_{i}$ and $X_{j}$ are pairwise disjoint for each $X_{i} \in F_{1}$. Finally, we show that $X_{i}$ and $X_{j}$ are pairwise disjoint for each $X_{i} \in F_{1}$ and $X_{j} \in F_{2}$. Note that $\Gamma_{G_{1}}(s) \cap X_{j} \neq \emptyset$ holds from definition of $N_{1}$. Then $X_{j} \subset X_{i}$ does not hold. Also note that $X_{i} \subset X_{j}$ does not hold, otherwise $\Gamma_{G_{1}}(s) \cap X_{i} \neq \emptyset$ and $\Gamma_{G_{1}}(s) \cap (X_{j} - X_{i}) \neq \emptyset$ imply $c_{G_{1}}(X_{j}, s) \geq c_{G_{1}}(X_{i}, s) + 1 \geq 2$, contradicting that $X_{i}$ is of type (2). Assume that $X_{i}$ and $X_{j}$ cross each other. Now $\Gamma_{G_{1}}(s) \cap (X_{j} - X_{i}) \neq \emptyset$ holds. Therefore Claim 4.4 implies that $c_{G_{1}}(s \cap X_{j}, s) = 0$ holds and $X_{j} - X_{i}$ is a critical cut of type (1). This implies that any vertex in $X_{j}$ cannot belong to $N_{2}$, contradicting $X_{j} \in F_{2}$.

Clearly $F$ is a subpartition of $V$ by Claim 4.5. Since $\Gamma_{G_{1}}(s) \subseteq X_{1} \cup \cdots \cup X_{q}$ with $X_{i} \in F$ holds, it
holds
$$|F_1| = \sum_{i=1}^{p} (r_x(X_i) - c_G(X_i)) + \sum_{i=p+1}^{q} (2 - |\Gamma_G(X_i)|),$$
for $F_1 = \{X_1, \ldots, X_p\}$ and $F_2 = \{X_{p+1}, \ldots, X_q\}$. From definition of $\alpha(G)$, we have $|F_1| \leq \alpha(G)$. $\Box$

5 Correctness of Step II

Let $G_1 = (V \cup \{s\}, E \cup F_1)$ be the graph obtained from a given graph $G = (V, E)$ after Step I. In this section, we describe about the correctness of Property 3.2 and the purpose of operations in case where $c_G(s)$ is odd.

In Step II, a graph $G_2 = (V, E \cup F_2)$ is constructed from $G_1$ by a complete edge-splitting at $s$. Then the correctness of Property 3.2 is immediate from Mader's theorem (see Theorem 2.1).

In this step, a non cut vertex $w$ is chosen when we add an extra edge $(s, w)$ to $G_1$ if $c_G(s)$ is odd. Such choice of $w$ will be used for the correctness of Step IV in Section 7 (i.e., by this choice, of $w$, we will be able to make $G(r_3, 2)$-connected by adding $\beta(G) - 1$ new edges in case of $\beta(G) - 1 > \lceil \alpha(G)/2 \rceil$).

6 Correctness of Step III

Let $G_2 = (V, E \cup F_2)$ be the graph obtained in Step II. Now $G_2$ is 2-edge-connected but has cut vertices.

In order to justify Step III, we now prove Property 3.3 in Step III.

Proof of Property 3.3: We prove Property 3.3 via two claims.

Claim 6.1 Let $v \in V$ denote a cut vertex in $G_2$. Assume that a $v$-component $T$ contains an admissible edge $e = (u, u')$ with respect to $v$. Then $G_2[T] - e$ contains a path $P$ between $u$ and $u'$.

Claim 6.2 Let $e_1 = (u_1, w_1)$ and $e_2 = (u_2, w_2)$ be the edges in the statement of Property 3.3. Then the graph $G'_2 = (V, E \cup F'_2)$ obtained by switching $e_1$ and $e_2$, where $F'_2 = F_2 \cup \{ (u_1, w_2), (w_1, u_2)\} - \{e_1, e_2\}$, satisfies the following:

(i) $\lambda_{G'_2}(x, y) \geq \lambda_G(x, y)$ for every $x, y \in V$.
(ii) $p(G'_2) - v < p(G_2) - v$.
(iii) $\kappa_{G'_2}(x, y) \geq 2$ for every pair of vertices $x$ and $y$ that satisfies $\kappa_{G_2}(x, y) \geq 2$.

(The statements (ii) and (iii) and Lemma 2.2 imply that switching $e_1$ and $e_2$ decreases the number $t(G_2)$ of tight sets in $G_2$ by at least one if $e_1$ or $e_2$ is contained in a tight set in $G_2$.)

Proof. (i) We assume that there is a cut $X$ such that $c_{G_2}(X) \leq r_1(X) - 1$ holds. Note that $c_{G_2}(X) \leq c_{G_2}(X)$ holds if cut $X$ does not separate $\{u_1, u_2\}$ and $\{w_1, w_2\}$ in $G_2$. Since $c_{G_2}(X) \geq r_1(X)$ originally holds, cut $X$ separates $\{u_1, u_2\}$ and $\{w_1, w_2\}$ and hence $c_{G_2}(X) = c_{G_2}(X) - 2$ holds. Since the cut $X$ crosses both $v$-components $T_1$ and $T_2$ in $G_2$, either $G_2[X]$ or $G_2[V - X]$ consists of at least two components. Without loss of generality, assume that $G_2[X]$ consists of at least two components. There are vertices $x' \in X$ and $y' \in V - X$ such that $r_1(x', y') = r_1(X) \geq c_{G_2}(X) + 1$. Without loss of generality, assume that $x' \in X \cap T_1$. Note that $c_{G_2}(X \cap T_2) \geq r_3(X \cap T_2) \geq 2$ and $c_{G_2}(X \cap T_1) \geq r_3(x' \cap T_1) \geq r_3(x', y') \geq c_{G_2}(X) + 1$ hold. This implies $c_{G_2}(X) = c_{G_2}(X \cap T_1) + c_{G_2}(X \cap T_2) \geq (c_{G_2}(X) + 1) + 2$, contradicting $c_{G_2}(X) = c_{G_2}(X)$ - 2.

(ii) It is sufficient to show that $G_2[T_1 \cup T_2]$ is connected. Since the removal of the admissible edge $e_1$ does not increase the number of $v$-components, $T_1$ remains a $v$-component in $G_2 - v$. If $T_2$ remains a $v$-component in $G_2 - e_2$, then $G[T_1]$ and $G[T_2]$ are joined by the edges $(u_1, w_2)$ and $(w_1, u_2)$ obtained by switching $e_1$ and $e_2$ in $G_2$. If $T_2$ consists of two components $T_2'$ and $T_2''$ in $G_2 - e_2$, then $u_2 \neq v \neq w_2$ holds and $u_2$ and $w_2$ are separated by $T_2'$. Assume $u_2 \in T'_2$ and $w_2 \in T''_2$ without loss of generality. Now $T'_2$ (resp., $T''_2$) and $T_1$ are joined by the edges $(u_1, w_2)$ (resp., $(w_1, u_2)$). This implies that $G_2[T_1 \cup T_2]$ is a component since $T_1$ remains a $v$-component in $G_2 - e_1$. Therefore if $v$ remains a cut vertex in $G_2$, then $T_1 \cup T_2$ is a $v$-component (otherwise, clearly, $p(G_2 - v) = 1$).

(iii) Assume that there are vertices $x, y \in V$ such that $\kappa_{G_2}(x, y) = 2$ but $\kappa_{G_2}(x, y) = 1$. Let $v' \in V$ denote a cut vertex in $G'_2$ that separates $x$ and $y$. Clearly, $v' \neq v$ (because $v = v'$ would imply $\kappa_{G_2}(x, y) = 1$). Let $W_1, W_2, \ldots, W_q$ ($q \geq 2$) be the $v'$-components of $G'_2$, where $x \in W_1$ and $y \in W_2$. Since a cut vertex $v'$ does not separate $x$ and $y$ in $G_2$, $e_1 \in E_{G_2}(W_1, W_2)$ or $e_2 \in E_{G_2}(W_1, W_2)$ holds. Also note that no edge other than $e_1$ and $e_2$ cannot belong to $E_{G_2}(W_1, W_2)$. We can easily see that $G_2[W_1 \cup W_2 \cup \{v'\}]$ contains $u_1, w_2$, and $w_2$. Then note that $u_i \in W_i$ cannot hold for any $i, j$ with $1 \leq i \leq j \leq 2$. Otherwise (assume $u_1 \in W_1$ without loss of generality) then $e_2 \in E_{G_2}(W_1, W_2)$ holds (assume $u_2 \in W_1$ and $w_2 \in W_2$ without loss of generality). Now $(w_1, w_2) \in E_{G_2}(W_1, W_2)$ holds and $G'_2[W_1] \cup G'_2[W_2]$ are both connected from definition of $W_1$ and $W_2$, contradicting that cut vertex $v'$ separates $x$ and $y$ in $G'_2$. Therefore, for each $i = 1, 2$, we have $e_i = (u_i, w_i) \in E_{G_2}(W_1, W_2)$ or $u_i = v'$ or $w_i = v'$.

We first consider the case of $e_1 \in E_{G_2}(W_1, W_2)$. Then $v' \in T_1$ holds since $G_2[T_1 - e_1]$ is connected by Claim 6.1. Hence $e_2 \in E_{G_2}(W_1, W_2)$ holds since $v' \in T_1$ implies $u_2 \neq v' \neq u_2$. Let $v' \notin W_2$ and $u_1, u_2 \in W_1$ without loss of generality. Now $G'_2(T_2 \cap W_2) \cap (T_2 - W_2) = \emptyset$ holds since $v'$ is a cut vertex of $G'_2$ and $v' \notin T_2$ hold. Note that $E_{G_2}(T_2 \cap W_2, V - W_2)$.
\[(T_2 \cap W_2) = \{(w_1, w_2)\}\] since \(T_2\) is a \(v\)-component of \(G_2\) and \(w_2 \in W_1\). This implies \(G_2(T_2 \cap W_2) = \{w_2\}\) holds and hence \(e_2\) is a bridge of \(G_2\) from \(E_{G_2}(W_1, W_2) = \{e_1, e_2\}\), which contradicts \(\lambda(G_2) \geq 2\).

We then consider the case of \(e_1 \notin E_{G_2}(W_1, W_2)\) holds, i.e., \(v' = w_1 \in T_1\) or \(v' = w_1 \in T_1\) holds. This implies that \(e_2 \in E_{G_2}(W_1, W_2)\) holds and \(v' \notin T_2\). Therefore, this clearly leads to a contradiction, in a similar way to above case of \(e_1 \in E_{G_2}(W_1, W_2)\). \(\square\)

From the above claim, Property 3.3 is proved.

7 Correctness of Step IV

Let \(G_3 = (V, E \cup F_3)\) be obtained from \(G_2\) after Step III. Now clearly \(|F_3| = \lceil \alpha(G) / 2 \rceil\) This \(G_3\) has exactly one cut vertex \(v\).

The correctness of Step IV clearly follows if we prove Property 3.6. The proof is now given below via two claims.

Claim 7.1 \(G_3\) has no edge in \(F_3\) incident to the cut vertex \(v\). \(\square\)

Claim 7.2 \(p(G - v) = p(G_3 - v) + |F_3|\) holds. That is, deleting any edge \(e \in F_3\) increases the number of \(v\)-components in \(G_3\).

Proof. If \(p(G - v) < p(G_3 - v) + |F_3|\) holds, then there is at least one edge \(e \in F_3\) with \(p((G_3 - e) - v) = p(G_3 - v)\). Then \(e\) is admissible with respect to \(v\) since Claim 7.1 implies that any edge in \(F_3\) is not incident to \(v\), contradicting construction of \(G_3\). \(\square\)

This claim implies that since \(G_3\) has no edge in \(F_3\) incident to the cut vertex \(v\), a graph \(H = (W, F_3)\) is a forest, where a vertex set \(W\) of \(H\) is obtained by removing the cut vertex \(v\) and contracting each component of \(G - v\) to one vertex. Now Claim 7.2 implies Property 3.6 since \(|F_3| = \lceil \alpha(G) / 2 \rceil\) holds from construction.

References


