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<th>A Map from the Lower-Half of the n-Cube onto the (n-1)-Cube which Preserves Intersecting Antichains</th>
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Kyoto University
A Map from the Lower-Half of the $n$-Cube onto the $(n - 1)$-Cube which Preserves Intersecting Antichains

Masahiro Miyakawa
(宮川正弘)
Tsukuba College of Technology
4-12 Kasuga, Tsukuba
Ibaraki 305, Japan
Grant Pogosyan
(G. ポゴシャン)
International Christian University
3-10-2, Òsawa, Mitaka-shi
Mitaka-shi, Tokyo 181, Japan

Akihiro Nozaki
(野崎昭弘)
Otsuma Women’s University
2-7-1 Karakida, Tama-shi
Tama-shi, Tokyo 206, Japan
Ivo G. Rosenberg
(I.G. ローゼンバーグ)
Université de Montréal
C.P. 6128, Succ. “Centre-ville”
Montréal P.Q. H3C 3J7, Canada

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Abstract

We prove that there is a 1-1 correspondence between the set of intersecting antichains in the lower-half of the $n$-cube and the set of intersecting antichains in the $(n - 1)$-cube. This reduces the enumeration of intersecting antichains contained in the former set to that in the latter.

1. Introduction

Enumeration of intersecting antichains in the lower-half of the $n$-cube plays a central role in determining the number of so-called $n$-ary clique Boolean functions[7]. It is time consuming and is barely feasible already for $n = 7$. In this note we present a bijection between the lower-half of the $n$-cube and the $(n - 1)$-cube which preserves intersecting antichains and therefore reduces the enumeration of intersecting antichains in the former set to the latter set whereby the dimension of the space is reduced by 1. However, under current computer power the enumeration for $n = 8$ seems still to be beyond the reach.

2. Definitions and Preliminaries

Let $E = \{0, 1\}$ and $n$ a positive integer. The cartesian power $E^n$ is the $n$-dimensional cube. Let $a = (a_1, \ldots, a_n) \in E^n$ and $b = (b_1, \ldots, b_n) \in E^n$. We write $a \triangleleft b$ if $a_i \leq b_i$ for all $i$, $1 \leq i \leq n$. We call $A \subseteq E^n$ an antichain if $a \triangleleft a'$ holds for no $a, a' \in A$, i.e., no two elements from $A$ are comparable; e.g., every $A$ with $|A| \leq 1$ is an antichain. Two vectors $a = (a_1, \ldots, a_n) \in E^n$ and $b = (b_1, \ldots, b_n) \in E^n$ are intersecting (at the $i$-th coordinate) provided $a_i = b_i = 1$ for some $1 \leq i \leq n$. A subset $A$ of $E^n$ is intersecting if all pairs $a, b \in A$ are intersecting. Note that a singleton set $\{a\}$ is intersecting unless $a$ is the zero vector $a = (0, \ldots, 0)$. Also note that the empty set $\emptyset$ is an intersecting antichain.

For $a = (a_1, \ldots, a_n) \in E^n$ set $w(a) := a_1 + \ldots + a_n$ (i.e., $w(a)$ is the number of $1$s in $a$). For $t = 0, \ldots, n$ denote by $B_t$ the $t$-th layer $w^{-1}(t)$ of the hypercube $E^n$.

Set $n' := \lfloor \frac{1}{2} n \rfloor$ and notice that $n' = \frac{1}{2} n$ if $n$ is even and $n' = \frac{1}{2} (n - 1)$ if $n$ is odd. For even $n$ the layer $B_{n'}$ is called the midlayer. For even $n$ and $i \in E$ set $C_i := \{(a_1, \ldots, a_n) \in B_{n'} : a_i = 1\}$ and call $C_1$ and $C_0$ the upper and lower halves of the midlayer. For even $n$ define the upper and lower...
halves of $E^n$ as

$$L_n := B_0 \cup \ldots \cup B_{n'-1} \cup C_0, \quad U_n := C_1 \cup B_{n'+1} \cup \ldots \cup B_n,$$

while for odd $n$

$$L_n := B_0 \cup \ldots \cup B_n, \quad U_n := B_{n'+1} \cup \ldots \cup B_n.$$

3. A map between $L_n$ and $E^{n-1}$

As usual set $0 := 1$ and $1 := 0$. Define a map $\varphi$ from $L_n$ into $E^{n-1}$ by setting

$$\varphi((a_1, \ldots, a_n)) = \begin{cases} (a_2, \ldots, a_n) & \text{if } a_1 = 0, \\ (a_1, \ldots, a_n) & \text{if } a_1 = 1. \end{cases}$$

(1)

**Theorem 1.** The map $\varphi$ is 1-1 and onto, and preserves intersecting antichains.

**Claim.** The map $\varphi$ is injective.

**Proof of the claim.** Let $a = (a_1, \ldots, a_n) \in L_n$ and $b = (b_1, \ldots, b_n) \in L_n$ satisfy $\varphi(a) = \varphi(b)$. We show that $a_1 = b_1$. Suppose to the contrary that $a_1 \neq b_1$. We can choose the notation so that $a_1 = 0$ and $b_1 = 1$. Then $b_1 = a_1$ and from $\varphi(a) = \varphi(b)$ and from (1) we obtain $b_1 = a_1$ for all $i = 2, \ldots, n$. As $a, b \in L_n$ clearly $w(a), w(b) \leq n'$ and

$$n' \geq w(b) = b_1 + \ldots + b_n = a_1 + \ldots + a_n = 1 - a_1 + \ldots + 1 - a_n = n - w(a) \geq n - n'.$$

Thus $n' \geq n - n'$ and hence $n$ is even, $n = 2n'$ and we have equality in (2) proving $w(a) = w(b) = n'$. However, now $b \in C_1$ in contradiction to $b \in C_0$. Thus $a_1 = b_1$ and from $\varphi(a) = \varphi(b)$ and (1) we obtain $a = b$ proving the claim. □

**Proof.** We show that $\varphi$ induces a bijection from the set of intersecting antichains in $L_n$ onto the set of intersecting antichains in $E^{n-1}$.

Let $W$ be an intersecting antichain in $L_n$. For $i = 0, 1$ set $W_i = \{(a_1, \ldots, a_n) \in W : a_i = 1\}$. From the definition of $\varphi$ clearly $\varphi$ is an order preserving map on $W_0$ and so $\varphi(W_0)$ is an antichain. It is immediate that $\varphi(W_0)$ is intersecting. Similarly $\varphi$ is an order reversing map on $W_1$ and so $\varphi(W_1)$ is also an antichain in $E^{n-1}$. To show that $\varphi(W_1)$ is intersecting let $(1, a_2, \ldots, a_n), (1, b_2, \ldots, b_n) \in W_1$. We prove that $a_i = b_i = 0$ for some $1 < i \leq n$. Indeed, if not then $a_i + b_i \geq 1$ for all $j = 2, \ldots, n$ and $n + 1 \leq w(a) + w(b)$. As $a, b \in W_1 \subseteq L_n$ we get $w(a), w(b) \leq n'$ leading to the contradiction $n + 1 \leq w(a) + w(b) \leq 2n' \leq n$. Thus $a_i = b_i = 0$ for some $1 < i \leq n$, hence both $\varphi(a)$ and $\varphi(b)$ have the $i$-th coordinate $\bar{a}_i = \bar{b}_i = 1$ and $\varphi(W_1)$ is intersecting.

We show that the images of every $a = (0, a_2, \ldots, a_n) \in W_0$ and of each $b = (1, b_2, \ldots, b_n) \in W_1$ are incomparable. As $a$ and $b$ belong to the intersecting set $W$ we have $a_i = b_i = 1$ for some $1 < i \leq n$ and therefore $a_i = 1 > 0 = b_i$. Suppose to the contrary that $a_k \geq b_k$ for all $k = 2, \ldots, n$. Then $a_k + b_k \geq 1$ for all $k$ and $w(a) + w(b) \geq 1 + n - 1 = n$. From $a, b \in L_n$ we see that $n' \geq w(a)$ and $n' \geq w(b)$. Consequently

$$2n' \geq w(a) + w(b) \geq n.$$ (3)

Thus $n$ is even, we have equality in (3) and so $w(a) = w(b) = n'$. However, this leads to the contradiction $b \in C_0 \cap W_1 = \emptyset$. Thus $\varphi(a)$ and $\varphi(b)$ are incomparable. Finally we show that the images of $a$ and $b$ are intersecting. As $a$ and $b$ belong to the antichain $W$, clearly $a_1 = 1 > 0 = b_1$ for some $1 < j \leq n$. Now $\varphi(a) \wedge \varphi(b) \neq 0$ due to $a_1 = 1 = b_1$. We have proved that $\varphi$ maps intersecting antichains in $L_n$ into intersecting antichains in $E^{n-1}$.

To show that this map of antichain is surjective let $W'$ be an intersecting antichain in $E^{n-1}$. Set

$$W_0 := \{(0, a_1, \ldots, a_n) : a = (a_1, \ldots, a_{n-1}) \in W', \ w(a) \leq n'\}$$

$$W_1 := \{(1, a_1, \ldots, a_{n-1}) : a = (a_1, \ldots, a_{n-1}) \in W', \ w(a) > n'\}.$$  

Clearly $W_0 \subseteq L_n$. For $c = (1, a_1, \ldots, a_n) \in W_1$ we have $w(c) = 1 + n - 1 - w(a) = n - w(a) < n - n'$ and so $w(c) \leq n'$ if $n$ is odd and $w(c) < n'$ if $n$ is even. This shows that $c \in L_n$ and $W_1 \subseteq L_n$. As above it can be verified that $W_0$ and $W_1$ are intersecting antichains. Consider $a = (0, a_1, \ldots, a_{n-1}) \in W_0$ and $b = (1, b_1, \ldots, b_{n-1}) \in W_1$. As $(a_1, \ldots, a_{n-1})$ and $(b_1, \ldots, b_{n-1})$ belong to the intersecting antichain $W'$, clearly $a_i = b_i = 0$ and $a_1 = 1, b_1 = 0$ for some $1 \leq i, j \leq n - 1$. Now $a_j = b_1 = 1$ and $a_1 = 1, b_1 = 0$. Thus $W_0 \cup W_1$ is an intersecting antichain and $\varphi(W_0 \cup W_1) = W'$. In particular, for every $b \in E^{n-1} \setminus \{a\}$ the antichain $\{b\} \in E^{n-1}$ is the image of some antichain $\{a\} \in L_n$. Moreover, as $\varphi((0, \ldots, 0)) = (0, \ldots, 0)$ clearly $\varphi$ is surjective. This concludes the proof. □

**Example.** Let $W = \{01001, 01010, 01100, 11000\}$ (where “abcd” stands for $(a,b,c,d,e)$). It can be checked that $W$ is an intersecting antichain in $L_5$ whose image is the intersecting antichain in $E^4$: $\varphi(W) = \{1001, 1010, 1100, 0111\}$. □

To show an application first we state a result from [PMNR97].
Theorem 2. \( \binom{n-1}{\frac{n}{2}-1} \) is the maximum size of an intersecting antichain in \( L_n \).

As a corollary of the above two theorems we have the following.

Corollary 3. \( \binom{n}{\frac{n}{2}} \) is the maximum size of an intersecting antichain in \( E^n \).

Proof. Replace \( n \) by \( n+1 \) in the formula given in Theorem 2. \( \square \)

The maximum size \( m \) of an antichain in \( L_n \) seems to be not yet considered in the literature. We show that \( m \) coincides with the maximum size of intersecting antichain in \( E^n \).

Theorem 4. \( \binom{n}{\frac{n}{2}} \) is the maximum size of an antichain in \( L_n \).

Proof. From [Spe28] for \( n \) odd the largest antichain in \( E^n \) is of size \( \binom{n}{n/2} \). Thus let \( n \) be even and let \( A \) be an antichain in \( L_n \) of size \( m \). Again from [Spe28] we obtain that \( A \subseteq B_{n'-1} \cup C_0 \). We show that \( A \) coincides with the antichain \( M = C_0 \cup \{ (a_1, \ldots, a_n) \in B_{n'-1} : a_1 = 1 \} \). For \( i = 0, 1 \) we have:

\[
A_i := \{ (a_1, \ldots, a_n) \in A \cap B_{n'-1} : a_1 = i \}.
\]

Also set

\[
D := \{ d \in C_0 : d > a \text{ for some } a \in A_0 \}
\]

and

\[
\alpha := |\{ (a, d) : a \in A_0, d \in D, a < d \}|.
\]

For a fixed \( a = (0, a_2, \ldots, a_n) \in A_0 \) we have \( w(a) = n'-1 \) and there are exactly \( n' \) elements \( d \in D \) with \( d > a \). Similarly for each fixed \( d = (0, d_2, \ldots, d_n) \in D \) due to \( w(d) = n' \) there are exactly \( n' \) elements \( b \in B_{n'-1} \) with \( b < d \). Together \( n'|\{A_0| \leq \alpha \leq n'|D| \) and

so \( |A_0| \leq |D| \). Now \( A \) being an antichain, \( D \) is disjoint from \( A \cap C_0 \) and \( D \cup (A \cap C_0) \cup A_1 \) is an antichain in \( L_n \). Then \( D \cup (A \cap C_0) \cup A_1 \) is a subset of the antichain \( M \) and from maximality \( A \) coincides with \( M \). Thus

\[
m = \binom{n-1}{n'} + \binom{n-1}{n'-2} = \binom{n-1}{n-1-n'} + \binom{n-1}{n'-2} = \binom{n-1}{n'-1} + \binom{n-1}{n'-2} = \binom{n}{n'-1}.
\]

\( \square \)

Lastly we note that the numbers of antichains in \( L_n \) and \( E^{n-1} \) do not coincide (as the proof indicates, the condition of intersecting is necessary for \( 1 \)-1 correspondence). This can be seen in Table 1 and Table 2, where we list the numbers of antichains in \( L_n \) and in \( E^n \). The numbers of intersecting antichains in \( L_n \) (and hence in \( E_{n-1} \)) is given in [PMNR97] for up to \( n = 7 \).

References


Table 1. Numbers $b(r, n)$ of size $r$ antichains in $L_n$; the maximum size is $m = \left(\frac{n}{2}\right)$.

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Table 2. Numbers $a(r, n)$ of size $r$ antichains in $L_n$; the maximum size is $\left(\frac{n}{2}\right)$.

The numbers given in total correspond to the Dedekind numbers of $n$-ary monotone functions.

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