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The Flow and Stability of a Thin Liquid Film on the Surface of a Rotating Disc

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§1. Introduction

In this paper we investigate the flow and stability of a thin liquid film on the surface of a rapidly rotating disc. The formulation is more complex than that for a slowly rotating disc given by Needham & Merkin [1], but their results are shown to apply in a restricted region. The axisymmetric steady-state flow problem is first considered, and an asymptotic solution valid at large radii is found, which is compared to a numerical solution valid for all radii. The stability of the asymptotic steady-state solution to small perturbations is investigated, and the local evolution of fully nonlinear disturbances is shown to be analogous to disturbances to the flow down a vertical wall.

§2. Formulation of the problem

We model the problem by considering the flow of an incompressible Newtonian fluid over the surface of a rotating horizontal disc, the fluid being ejected onto the disc as plug flow from a distributor rotating with the disc at its centre, see Figure 1. The horizontal length scale a is taken to be the radius at which transient behaviour near the inlet is left behind, and then a vertical length scale h is defined as the film thickness at this radius, independent of the exact inflow conditions at the distributor. Thus, following Needham & Merkin [1], we introduce a small dimensionless parameter $\varepsilon = h/a$ into the problem. The component of the velocity in the radial direction is scaled using the radial outward flow implied by the volumetric flow rate Q at radius a, i.e. $\mathcal{U}_0 = Q/2\pi a h$. It is convenient to take $\mathcal{U}_0 = \mathcal{V}_0$ and thus we may expect $v \ll u$ for a realistic solution. The continuity equation requires $\mathcal{W}_0 = \varepsilon\mathcal{U}_0$, and the pressure and time variables are non-dimensionalised with $\mathcal{P}_0 = \rho\mathcal{U}_0^2$, $\mathcal{T}_0 = a/\mathcal{U}_0$ respectively. The full Navier-Stokes equations in dimensionless varibles are thus

$$\frac{Du}{Dt} - (G^2 r + 2Gv + \frac{v^2}{r}) = -P_r + GEu_{zz} - 2\frac{GE}{r^2}v_\theta + \varepsilon^2 GE(\nabla_{\parallel}^2 u - \frac{u}{r^2})\,; \tag{1}$$

$$\frac{Dv}{Dt} + 2Gu + \frac{uv}{r} = -\frac{1}{r}P_\theta + GEv_{zz} + 2\frac{GE}{r^2}u_\theta + \varepsilon^2 GE(\nabla_{\parallel}^2 - \frac{v}{r^2})\,; \tag{2}$$

$$\varepsilon^2\frac{Dw}{Dt} = -P_z - \frac{\varepsilon}{F^2} + \varepsilon^2 GEw_{zz} + \varepsilon^4 GE\nabla_{\parallel}^2 w\,; \tag{3}$$

$$(ru)_r + v_\theta + rw_z = 0\,, \tag{4}$$

where

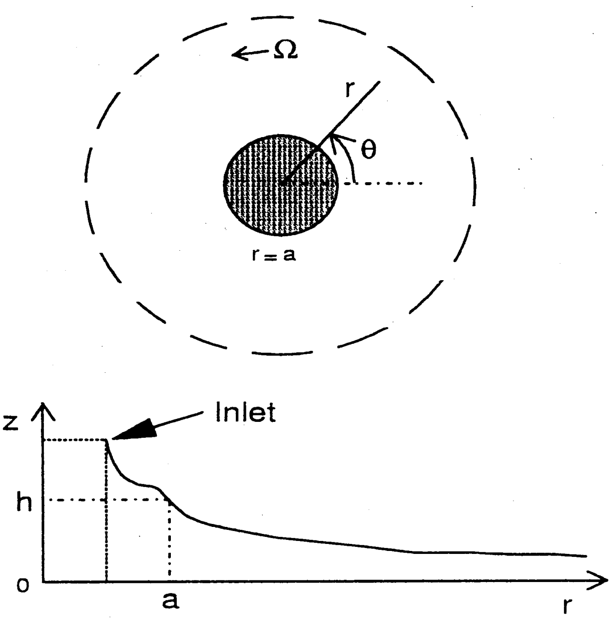

Figure 1: The coordinate system.

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + (G + \frac{v}{r})\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}, \quad \text{and} \quad \nabla_{\parallel}^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Here we have introduced the dimensionless parameters $G = \Omega\, a/\mathcal{V}_0$, the dimensionless rotation speed of the disc (which has the form of an inverse Rossby number for the flow); $E = \nu/\Omega h^2$, the Ekman number; $F = \mathcal{U}_0/\sqrt{g\,a}$, a Froude number; and $W = \Gamma/ah\rho g$, a Weber number, where Γ is the surface tension. A Reynolds number for the flow may be defined as $Re = 1/GE = h^2\mathcal{U}_0/\nu a$, however this quantity cannot usefully be used to characterise the flow, since it does not depend upon the rotation speed of the disc, which in practice is one of the most important factors. The boundary conditions comprise the usual no slip conditions at the disc surface

$$u = v = w = 0 \qquad \text{on} \qquad z = 0\,; \tag{5}$$

on the free surface $z = H(r, \theta, t)$ we have the kinematic condition plus two tangential and one normal stress condition,

$$H_t + uH_r + (G + \frac{v}{r})H_\theta = w\,; \tag{6}$$

$$u_z = O(\varepsilon^2), \qquad v_z = O(\varepsilon^2); \tag{7}$$

$$P = 2\varepsilon^2 GE\left(w_z - H_r u_z - \frac{H_\theta}{r}v_z\right) - \varepsilon^2\frac{W}{F^2}\left(H_{rr} + \frac{1}{r^2}(H_{\theta\theta} + rH_r)\right) + O(\varepsilon^4). \tag{8}$$

The dimensionless groups have been chosen so that they are all $O(1)$ for the range of operating parameters we are interested in. In particular, we have $E = \nu/\Omega h^2 = O(1)$, which allows us to model rotation speeds of $\Omega \sim 100$rad/s when the fluid is water, in contrast to the work of Needham & Merkin [1], who require $E \gg 1$, and thus a much lower rotation speed.

§3. The steady-state solution

The leading order steady-state, axisymmetric problem in ε is found to be nonlinear, and no closed form solution appears to exist. An exact numerical solution has been obtained; and also an asymptotic solution for large radius r using the following scalings

$$r = \frac{R}{\varepsilon^{\lambda}}, \qquad z = \varepsilon^{\frac{2}{3}}\zeta,$$

where $\lambda > 0$. This scaling was chosen because experimental evidence suggests that the model for a slowly rotating disc is valid at large radii with $E = O(1)$. The dependent variables are also scaled to give the same leading order balance as that used in the $E \gg 1$ model,

$$u = \varepsilon^{\frac{1}{3}\lambda}\overline{u}\,, \qquad v = \varepsilon^{\frac{5}{3}\lambda}\overline{v}\,, \qquad w = \varepsilon^{\frac{6}{3}\lambda}\overline{w}$$

$$P = \varepsilon^{-\frac{6}{3}\lambda}\overline{P}\,, \qquad H = \varepsilon^{\frac{2}{3}\lambda}\overline{H}.$$

We define the parameter λ by relating the point $R = 1$ to a dimensionless radius r_0, so that

$$\lambda = -\frac{\ln r_0}{\ln \varepsilon}, \tag{9}$$

The governing equations now contain terms such as $\varepsilon^{1+\frac{8}{3}\lambda}$, and it can be seen that these terms will vary in size (relative to terms with exponent dependent only on λ) for different choices of r_0. However, because we require $\lambda > 0$ this switching of relative orders occurs only for terms at third order or smaller; the leading order behaviour is unaffected.

The asymptotic solutions for the dependent variables can be readily found, yielding for the location of the steady-state free surface

$$\overline{H}(R) = AR^{-\frac{2}{3}} + \varepsilon^{\frac{8}{3}\lambda}\frac{62A^5}{315E^2}R^{-\frac{10}{3}} - \varepsilon^{1+\frac{8}{3}\lambda}\frac{2}{9}\frac{A^2}{F^2G^2}R^{-\frac{10}{3}} + O(\varepsilon^{\frac{16}{3}\lambda})\,,$$

where the constant $A = (3E/G)^{\frac{1}{3}}$ is fixed by considering mass conservation. A comparison with the numerical solution found that two terms of the asymptotic solution give a very good approximation to the flow over all of the disc except for a very small region near the inlet (see Fig.2), and so this has been used as the starting point for an analysis of the stability of the flow.

§4. Unsteady flow

We add a disturbance to the basic state of the form $u(r, th, z, t) = \varepsilon^{\frac{1}{3}\lambda}(\overline{u}(R,\zeta) + \tilde{u}(r, R, th, \zeta, t))$, and the differential with respect to r becomes

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial r} + \varepsilon^{\lambda}\frac{\partial}{\partial R}$$

The continuity equation then requires that $\tilde{u} = \varepsilon^{\lambda}\tilde{u}$. Solving at successive orders for the perturbations to the flow field and inserting into the kinematic boundary condition

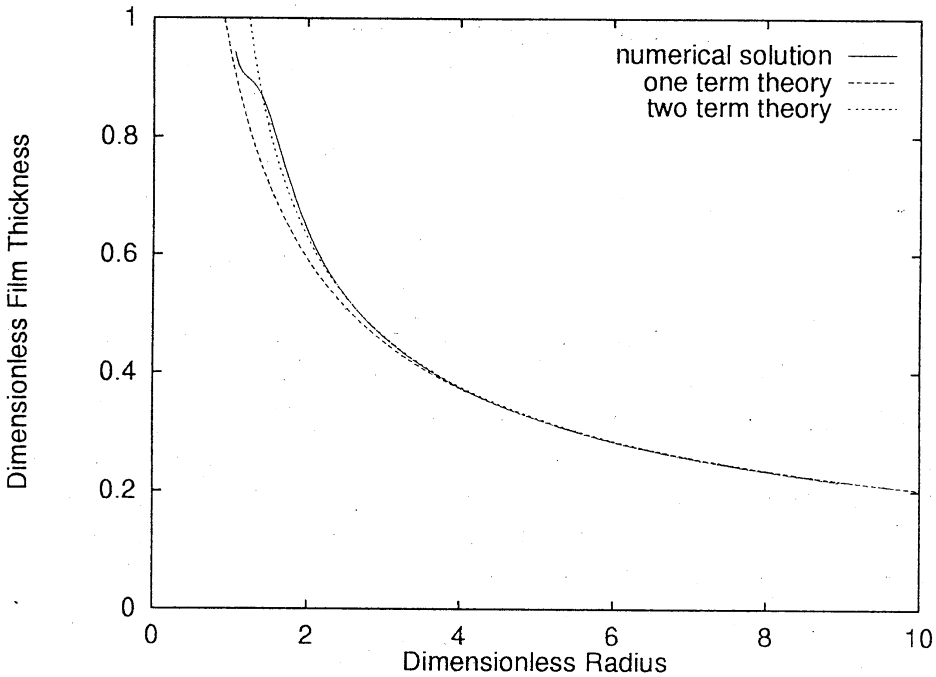

Figure 2: Comparison of numerical and asymptotic solutions for film thickness (Q=20 cc/s, Ω=40 rad/s).

at the free surface yields a nonlinear evolution equation for disturbances of arbitrary amplitude,

$$\eta_t + G\eta_\theta + \varepsilon^{\frac{1}{3}\lambda}\frac{1}{R\overline{H}_0}\left((\eta^3)_r + \varepsilon^{\lambda}(\eta^3)_R\right)$$
$$+\varepsilon^{\frac{6}{3}\lambda}\frac{1}{R^2}\left(\frac{6}{5}\frac{1}{GE}(\eta^6\eta_r)_r + \frac{\overline{W}}{G^2F^2}(\eta_{rrr}\eta^3)_r - \frac{\varepsilon}{G^2F^2}(\eta_r\eta^3)_r\right) - \varepsilon^{\frac{8}{3}\lambda}\frac{4}{5}\frac{A}{E}\frac{1}{R^{\frac{8}{3}}}(\eta^5)_\theta = O(\varepsilon^{\frac{10}{3}\lambda}) \tag{10}$$

where the perturbation to the free surface has been normalised with respect to the local basic state film thickness, $H(r,th,t) = \varepsilon^{\frac{2}{3}\lambda}\overline{H}(R)(1+\tilde{\eta}(r,R,\theta,t))$, and $\eta = 1+\tilde{\eta}(r,R,\theta,t)$. Also, $\overline{W} = \varepsilon^2 W$ is considered to be $O(1)$, so we are assuming large surface tension.

To investigate the local stability of the flow, we consider the amplitude of the disturbance to be small with respect to the local film thickness, $\tilde{\eta} \ll 1$, and set $\partial/\partial R \equiv 0$. This yields

$$\tilde{\eta}_t + G\tilde{\eta}_\theta + \varepsilon^{\frac{1}{3}\lambda}\frac{3}{R\overline{H}_0}\tilde{\eta}_r + \varepsilon^{\frac{6}{3}\lambda}\frac{1}{R^2}\left(\frac{6}{5}\frac{1}{GE}\tilde{\eta}_{rr} + \frac{\overline{W}}{G^2F^2}\tilde{\eta}_{rrrr} - \frac{\varepsilon}{G^2F^2}\tilde{\eta}_{rr}\right)$$
$$-\varepsilon^{\frac{8}{3}\lambda}4\frac{A}{E}\frac{1}{R^{\frac{8}{3}}}\tilde{\eta}_\theta = O(\varepsilon^{\frac{10}{3}\lambda})$$

For a solution in the form of a sinusoidal wave train

$$\tilde{\eta} = s e^{\psi + i\phi}$$

where s is some arbitrary constant, the phase function is of the form

$$\phi = kr + l\theta - \omega t$$

and the growth or decay of a disturbance is governed by

$$\psi = \varepsilon^{\frac{6}{3}\lambda}\frac{k^2}{R^2}\left(\frac{6}{5}\frac{1}{GE} - \frac{k^2\overline{W}}{G^2F^2} - \frac{\varepsilon}{G^2F^2}\right).$$

We could define a modified Froude number $\overline{F^2} = G^2F^2/\varepsilon = hg/(a(a\Omega^2))$, with $Re = 1/(GE)$ independent of Ω, and it can be seen that for $\overline{F}^2 = O(1)$ (low rotation speed) it is possible for the flow to be unconditionally stable. However, for $\overline{F}^2 = O(\varepsilon^{-1})$ the flow is stable only for sufficiently large radial wavenumber k. Note that the stability is independent of R; only the rate of growth or decay varies at different values of R across the disc. It is easily shown that neutral stability occurs for $k = k_c$, and the maximum growth rate is given by $k_m = k_c/\sqrt{2}$, where

$$k_c = \frac{6}{5}\frac{GF^2}{E\overline{W}}.$$

5. *Nonlinear evolution*

If now we consider an axisymmetric local disturbance ($\partial/\partial R \equiv 0$, $\partial/\partial\theta \equiv 0$) in the nonlinear evolution equation (10), and introduce the transformation

$$\eta = \frac{1}{3}\frac{\sqrt{GE}}{\overline{H}_0}\overline{\eta}$$

$$t = \varepsilon^{-\frac{1}{3}}\tau$$

we obtain an equation of the form

$$\overline{\eta}_\tau + \overline{G}\overline{\eta}^2\overline{\eta}_r + \varepsilon^{\frac{5}{3}\lambda}\left(\frac{2}{15}\overline{G}^2(\overline{\eta}^6\overline{\eta}_r)_r + S(\overline{\eta}^3\overline{\eta}_{rrr})_r\right) = 0 \tag{11}$$

where terms of $O(\varepsilon^{\frac{5}{3}\lambda+1})$ have been neglected. This equation describes the 2-D plane parallel flow down a vertical wall, and so we can expect the results for fully nonlinear waves on parallel flow (see e.g. Nakaya [2]) to also apply to flow on a disc. It is interesting to compare the definitions of $\overline{G}$ in (11) when it describes the evolution of disturbances to the two different flows. For flow down a wall, $\overline{G} \equiv$ a Reynolds number, but for flow on a rotating disc $\overline{G} = \frac{1}{27}G^2R$, a dimensionless centripetal acceleration. This suggests that holding the centripetal acceleration constant for rotating disc flow is equivalent to fixed Reynolds number in parallel flow. However, this is not a sufficient condition, since the surface tension term S is fixed for parallel flow, but for the disc we find that

$$S = \frac{1}{81}\sqrt{GE}\frac{\overline{W}}{\overline{F^2}};$$

hence the (local) nonlinear evolution of disturbances to the flow over a disc for different parameter values will be the same provided both

$$G^2R = \text{constant} \;\; \textit{and} \;\; GE \equiv 1/Re = \text{constant}. \tag{12}$$

It should also be remembered, however, that $\overline{\eta}$ in (11) has been normalised with respect to the local steady-state film thickness, and this is not invariant under (12). Hence any solutions found by expanding (11) about $\overline{\eta}=1$ are not generally applicable, since

$$\overline{\eta} = \frac{3\overline{H}_0}{\sqrt{GE}}(1+\tilde{\eta})$$

and the expansion would then be assuming that

$$\tilde{\eta} = -1 + \frac{1}{3}\frac{\sqrt{GE}}{\overline{H}_0} + \hat{\eta}$$

where $\hat{\eta} \ll 1$, which is equivalent to expanding η about

$$\eta = \frac{1}{3}\frac{\sqrt{GE}}{\overline{H}_0}$$

which is the value of the steady state free surface at only one radius,

$$R = \left(\frac{3A}{\sqrt{GE}}\right)^{\frac{3}{2}} .$$

§6. Conclusions

An asymptotic solution for the steady-state flow of a thin liquid film on a rotating disc has been found, and is in good agreement with a numerical solution. A nonlinear equation describing the evolution of an arbitrary disturbance to the flow has been derived, and the critical wavenumbers for neutral stability and maximum growth rate have been found for small amplitude disturbances. Localised, axisymmetric, large amplitude disturbances are shown to satisfy the same evolution equation as large 2-D disturbances to parallel flow down a wall, with the rôle of the Reynolds number in parallel flow being taken by the local centripetal acceleration for flow on a disc.

References

[1] D.J.Needham & J.H.Merkin, *J. Fluid Mech.* **184**(1987) 357-359

[2] C.Nakaya, *Phys. Fluids* A **1**(7)(1989) 1143-1154.