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SPECTRAL GEOMETRY OF KÄHLER HYPERSURFACES IN THE COMPLEX GRASSMANN MANIFOLD

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§1. Introduction.
Let $M$ be a compact $C^\infty$-Riemannian manifold, $C^\infty(M)$ the space of all smooth functions on $M$, and $\Delta$ the Laplacian on $M$. Then $\Delta$ is a self-adjoint elliptic differential operator acting on $C^\infty(M)$, which has an infinite discrete sequence of eigenvalues: $\text{Spec}(M) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \uparrow \infty\}$. Let $V_k = V_k(M)$ be the eigenspace of $\Delta$ corresponding to the $k$-th eigenvalue $\lambda_k$. Then $V_k$ is finite-dimensional. We define an inner product $(\ , \ )_{L^2}$ on $C^\infty(M)$ by $(f, g)_{L^2} = \int_M fg \, dv_M$, where $dv_M$ denotes the volume element on $M$. Then $\sum_{t=}^\infty V_t$ is dense in $C^\infty(M)$ and the decomposition is orthogonal with respect to the inner product $(\ , \ )_{L^2}$. Thus we have $C^\infty(M) = \sum_{t=0}^\infty V_t(M)$ (in $L^2$-sense). Since $M$ is compact, $V_0$ is the space of all constant functions which is 1-dimensional.

In this point of view, it is one of the simplest and the most interesting problems to estimate the first eigenvalue. In [10], A. Ros gave the following sharp upper bound for the first eigenvalue of Kähler submanifold of a complex projective space.

**Theorem 1.1.** Suppose that $M$ is a complex $m$-dimensional compact Kähler submanifold of the complex projective space $\mathbb{C}P^n$ of constant holomorphic sectional curvature $c$. Then the first eigenvalue $\lambda_1$ satisfies the following inequality:

$$\lambda_1 \leq c(m + 1)$$

The equality holds if and only if $M$ is congruent to the totally geodesic Kähler submanifold $\mathbb{C}P^m$ of $\mathbb{C}P^n$.

If $M$ is not totally geodesic, J-P. Bourguignon, P. Li and S. T. Yau in [1] gave the following more sharp estimate. (See also [7].)

**Theorem 1.2.** Suppose that $M$ is a complex $m$-dimensional compact Kähler submanifold of $\mathbb{C}P^n$, which is fully immersed and not totally geodesic. Then the first eigenvalue $\lambda_1$ satisfies the following inequality:

$$\lambda_1 \leq cm \frac{n+1}{n}$$

It is unknown when the equality holds in this inequality.

Our purpose is to give the upper bound for the first eigenvalue of Kähler hypersurfaces of a complex Grassmann manifold.
Let denote by $G_r(\mathbb{C}^n)$ the complex Grassmann manifold of $r$-planes in $\mathbb{C}^n$, equipped with the Kähler metric of maximal holomorphic sectional curvature $c$. We obtain the following result which is a natural generalization of Theorem 1.1.

**Theorem A.** Suppose that $M$ is a compact connected Kähler hypersurface of $G_r(\mathbb{C}^n)$. Then the first eigenvalue $\lambda_1$ satisfies the following inequality:

$$\lambda_1 \leq c \left( n - \frac{n - 2}{r(n - r)} - 1 \right)$$

The equality holds if and only if $r = 1, n$, and $M$ is congruent to the totally geodesic complex hypersurface $\mathbb{C}P^{n-2}$ of the complex projective space $\mathbb{C}P^{n-1}$.

The 2-plane Grassmann manifold $G_2(\mathbb{C}^n)$ admits the quaternionic Kähler structure $\mathfrak{J}$. For the normal bundle $T^\bot M$ of a Kähler hypersurface $M$ of $G_2(\mathbb{C}^n)$, $\mathfrak{J}T^\bot M$ is a vector bundle of real rank 6 over $M$ which is a subbundle of the tangent bundle of $G_2(\mathbb{C}^n)$. We consider a Kähler hypersurface $M$ of $G_2(\mathbb{C}^n)$ satisfying the property that $\mathfrak{J}T^\bot M$ is a subbundle of the tangent bundle $TM$ of $M$. In the section 4, we will introduce examples satisfying this property.

For a Kähler hypersurface of $G_2(\mathbb{C}^n)$ satisfying this property, we obtain the following upper bound of the first eigenvalue.

**Theorem B.** Suppose that $M$ is a compact connected Kähler hypersurface of $G_2(\mathbb{C}^n), n \geq 4$. If $M$ satisfies the condition $\mathfrak{J} T^\bot M \subset TM$, then the following inequality holds:

$$\lambda_1 \leq c \left( n - \frac{n - 1}{2n - 5} \right)$$

The equality holds if and only if $n = 4$ and $M$ is congruent to the totally geodesic complex hypersurface $Q^3$ of the complex quadric $Q^4 = G_2(\mathbb{C}^4)$.

These two theorems are proved in the section 5. More detailed proofs of any our results are given in [8].

**Notations.** $M_{r,s}(\mathbb{C})$ denotes the set of all $r \times s$ matrices with entries in $\mathbb{C}$, and $M_r(\mathbb{C})$ stands for $M_{r,r}(\mathbb{C})$. $I_r$ and $O_r$ denote the identity $r$-matrix and the zero $r$-matrix.

**§2. Preliminaries.**

In this section, we discuss geometries of the complex $r$-plane Grassmann manifold and its first standard imbedding.

Let $M_r(\mathbb{C}^n)$ be the complex Stiefel manifold which is the set of all unitary $r$-systems of $\mathbb{C}^n$, i.e.,

$$M_r(\mathbb{C}^n) = \{ Z \in M_{n,r}(\mathbb{C}) \mid Z^*Z = I_r \} .$$

The complex $r$-plane Grassman manifold $G_r(\mathbb{C}^n)$ is defined by

$$G_r(\mathbb{C}^n) = M_r(\mathbb{C}^n)/U(r).$$
The origin $o$ of $G_r(\mathbb{C}^n)$ is defined by $\pi(Z_0)$, where $Z_0 = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$ is an element of $M_r(\mathbb{C}^n)$, and $\pi: M_r(\mathbb{C}^n) \rightarrow G_r(\mathbb{C}^n)$ is the natural projection.

The left action of the unitary group $\tilde{G} = SU(n)$ on $G_r(\mathbb{C}^n)$ is transitive, and the isotropy subgroup at the origin $o$ is

$$\tilde{K} = S(U(r) \cdot U(n-r)) = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \bigg| U_1 \in U(r), \ U_2 \in U(n-r), \ \det U_1 \det U_2 = 1 \right\}. $$

so that $G_r(\mathbb{C}^n)$ is identified with a homogeneous space $\tilde{G}/\tilde{K}$.

Set $\tilde{\mathfrak{g}} = \mathfrak{su}(n)$ and

$$\tilde{\mathfrak{g}} = \mathbb{R} \oplus \mathfrak{su}(r) \oplus \mathfrak{su}(n-r) = \left\{ +a \left( \begin{array}{cc} 0 & \sqrt{-1}\xi \\ -\sqrt{-1}\xi^* & 0 \end{array} \right) \bigg| a \in \mathbb{R}, \ \xi \in \mathfrak{M}_{n-r}(\mathbb{C}) \right\},$$

then $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{k}}$ are the Lie algebra of $\tilde{G}$ and $\tilde{K}$, respectively. Define a linear subspace $\tilde{\mathfrak{m}}$ of $\tilde{\mathfrak{g}}$ by

$$\tilde{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} \bigg| \xi \in \mathfrak{M}_{n-r}(\mathbb{C}) \right\},$$

then $\tilde{\mathfrak{m}}$ is identified with the tangent space $T_o(G_r(\mathbb{C}^n))$. The $\tilde{G}$-invariant complex structure $J$ of $G_r(\mathbb{C}^n)$ and the $\tilde{G}$-invariant Kähler metric $\tilde{g}_c$ of $G_r(\mathbb{C}^n)$ of the maximal holomorphic sectional curvature $c$ are given by

$$J \left( \begin{array}{cc} 0 & -\xi^* \\ \xi & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & -\sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0 \end{array} \right),$$

$$(2.1) \quad \tilde{g}_{c_o}(X, Y) = -\frac{2}{c} \text{tr} XY, \quad X, Y \in \tilde{\mathfrak{m}}. $$

In the case of $r = 2$, the complex 2-plane Grassmann manifold $G_2(\mathbb{C}^n)$ admits another geometric structure named the quaternionic Kähler structure $\mathfrak{J}$. $\mathfrak{J}$ is a $\tilde{G}$-invariant subbundle of $\text{End}(T(G_2(\mathbb{C}^n)))$ of rank 3, where $\text{End}(T(G_2(\mathbb{C}^n)))$ is the $\tilde{G}$-invariant vector bundle of all linear endmorphisms of the tangent bundle $T(G_2(\mathbb{C}^n))$. Under the identification with $T_o(G_r(\mathbb{C}^n))$ and $\tilde{\mathfrak{m}}$, the fiber $\mathfrak{J}_o$ at the origin $o$ is given by

$$\mathfrak{J}_o = \left\{ J_\tilde{\xi} = \text{ad}(\tilde{\xi}) \bigg| \tilde{\xi} \in \tilde{\mathfrak{q}} \right\},$$

where $\tilde{\mathfrak{q}}$ is an ideal of $\tilde{\mathfrak{k}}$ defined by

$$\tilde{\mathfrak{q}} = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & 0 \end{pmatrix} \bigg| u_1 \in \mathfrak{su}(2) \right\} \cong \mathfrak{su}(2).$$
Choose a basis \{\varepsilon_1, \varepsilon_2, \varepsilon_3\} of \mathfrak{su}(2) satisfying \([\varepsilon_i, \varepsilon_{i+1}] = 2 \varepsilon_{i+2}\), (mod 3). Set \(\tilde{\varepsilon}_i = \begin{pmatrix} \varepsilon_i & 0 \\ 0 & 0 \end{pmatrix}\) and \(J_i = J_{\varepsilon_i}\) for \(i = 1, 2, 3\), then the basis \(\{J_1, J_2, J_3\}\) is a canonical basis of \(\mathfrak{g}_0\), satisfying
\[
J_i^2 = -id_{\bar{\mathfrak{m}}} \quad \text{for} \ i = 1, 2, 3,
J_1J_2 = -J_2J_1 = J_3, \quad J_2J_3 = -J_3J_2 = J_1, \quad J_3J_1 = -J_1J_3 = J_2,
\]
and \(\tilde{g}_{c_{O}}(J_iX, J_iY) = \tilde{g}_{c_{O}}(X, Y)\), for \(X, Y \in \mathfrak{m}\) and \(i = 1, 2, 3\).

There exists an element \(\tilde{\varepsilon}_C\) of the center of \(\mathfrak{k}\) such that \(J\) is given by \(J = ad(\tilde{\varepsilon}_C)\) on \(\mathfrak{m}\). Therefore, \(J\) is commutable with \(\mathfrak{g}\).

Let \(HM(n, \mathbb{C})\) be the set of all Hermitian \((n, n)\)-matrices over \(\mathbb{C}\), which can be identified with \(\mathbb{R}^{n^2}\). For \(X, Y \in HM(n, \mathbb{C})\), the natural inner product is given by
\[
(X, Y) = \frac{2}{c} tr XY.
\]

\(GL(n, \mathbb{C})\) acts on \(HM(n, \mathbb{C})\) by \(X \mapsto BXB^*, B \in GL(n, \mathbb{C}), X \in HM(n, \mathbb{C})\). Then the action of \(SU(n)\) leaves the inner product (2.2) invariant.

The first standard imbedding \(\Psi\) of \(G_r(\mathbb{C}^n)\) is defined by
\[
\Psi(\pi(z)) = zZ^{*} \in HM(n, \mathbb{C}), \ Z \in M_r(\mathbb{C}^n).
\]
\(\Psi\) is \(SU(n)\)-equivariant and the image \(N\) of \(G_r(\mathbb{C}^n)\) under \(\Psi\) is given as follows:

\[
N = \Psi(G_r(\mathbb{C}^n)) = \{ A \in HM(n, \mathbb{C}) \mid A^2 = A, \ trA = r \}.
\]

The tangent bundle \(TN\) and the normal bundle \(T^\perp N\) are given by
\[
T_{A}N = \{X \in HM(n, \mathbb{C}) \mid XA + AX = X\} \subset HM_0,
T_{A}^\perp N = \{Z \in HM(n, \mathbb{C}) \mid ZA = ZX\}.
\]

In particular, at the origin \(A_o = \Psi(o) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\), we can obtain
\[
T_{A_o}N = \left\{ \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} \mid \xi \in M_{n-r, r}(\mathbb{C}) \right\},
T_{A_o}^\perp N = \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \mid Z_1 \in HM(r, \mathbb{C}), Z_2 \in HM(n - r, \mathbb{C}) \right\}.
\]

The complex structure \(J\) acts on \(T_{A_o}N\) as follows:
\[
J \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0 \end{pmatrix}.
\]
If \( r = 2 \), then the quaternionic Kähler structure \( \mathfrak{J} \) acts on \( T_{A_{o}}N \) as follows:

\[
J_{\xi} \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon\xi^* \\ -\xi\varepsilon & 0 \end{pmatrix}, \quad \varepsilon \in \text{su}(2).
\]

Let \( \tilde{\sigma} \) and \( \tilde{H} \) denote the second fundamental form and the mean curvature vector of \( \Psi \), respectively. Then, for \( A \in N \) and \( X, Y \in T_{A}N \), we can see

\[
(2.8) \quad \tilde{\sigma}_{A}(X, Y) = (XY + YX)(I - 2A)
\]

and \( \tilde{\sigma} \) satisfies the following:

\[
\begin{align*}
(2.9) & \quad \tilde{H}_{A} = \frac{c}{2r(n-r)}(rI - nA) \\
(2.10) & \quad \tilde{\sigma}_{A}(JX, JY) = \tilde{\sigma}_{A}(X, Y), \\
(2.11) & \quad (\tilde{\sigma}_{A}(X, Y), A) = -(X, Y).
\end{align*}
\]

§3. Examples.

One of the most simple typical examples of submanifolds of \( G_{r}(\mathbb{C}^{n}) \) is a totally geodesic submanifold. B. Y. Chen and T. Nagano in [3, 4] determined maximal totally geodesic submanifolds of \( G_{2}(\mathbb{C}^{n}) \). For arbitrary \( r \), I. Satake and S. Ihara in [11, 5] determined all (equivariant) holomorphic imbeddings of a symmetric domain into another symmetric domain. Taking a compact dual symmetric space if necessary, we obtain the complete list of maximal totally geodesic Kähler submanifolds of \( G_{r}(\mathbb{C}^{n}) \).

Since totally geodesic submanifolds of \( G_{r}(\mathbb{C}^{n}) \) are symmetric spaces, we can calculate the first eigenvalue of the Laplacian of \( M \). (cf. [14])

**Theorem 3.1.** Let \( M \) be a proper maximal totally geodesic Kähler submanifold of \( G_{r}(\mathbb{C}^{n}) \), and \( \lambda_{1} \) the first eigenvalue of the Laplace-Beltrami operator with respect to the induced Kähler metric. Then, \( M \) and \( \lambda_{1} \) are one of the following (up to isomorphism).

1. \( M_{1} = G_{r}(\mathbb{C}^{n-1}) \hookrightarrow G_{r}(\mathbb{C}^{n}), \quad 1 \leq r \leq n - 2, \quad \text{and} \quad \lambda_{1} = c(n - 1) \)
2. \( M_{2} = G_{r-1}(\mathbb{C}^{n-1}) \hookrightarrow G_{r}(\mathbb{C}^{n}), \quad 2 \leq r \leq n - 1, \quad \text{and} \quad \lambda_{1} = c(n - 1) \)
3. \( M_{3} = G_{r_{1}}(\mathbb{C}^{n_{1}}) \times G_{r_{2}}(\mathbb{C}^{n_{2}}) \hookrightarrow G_{r_{1}+r_{2}}(\mathbb{C}^{n_{1}+n_{2}}), \quad 1 \leq r_{i} \leq n_{i} - 1, \quad i = 1, 2, \quad \text{and} \quad \lambda_{1} = c \min\{n_{1}, n_{2}\} \)
4. \( M_{4} = M_{4,p} = Sp(p)/U(p) \hookrightarrow G_{p}(\mathbb{C}^{2p}), \quad p \geq 2, \quad \text{and} \quad \lambda_{1} = c(p + 1) \)
5. \( M_{5} = M_{5,p} = SO(2p)/U(p) \hookrightarrow G_{p}(\mathbb{C}^{2p}), \quad p \geq 4, \quad \text{and} \quad \lambda_{1} = c(p - 1) \)
6. \( M_{6,m} = \mathbb{C}P^{p} \hookrightarrow G_{r}(\mathbb{C}^{n}): \text{the complex projective space}, \quad r = \left( \begin{array}{c} p \\ m \end{array} \right),\quad n = \left( \begin{array}{c} p+1 \\ m \end{array} \right), \quad 2 \leq m \leq p - 1, \quad \text{and} \quad \lambda_{1} = c(p + 1) \left( \begin{array}{c} p - 1 \\ m - 1 \end{array} \right) \)
7. \( M_{7} = Q^{3} \hookrightarrow Q^{4} = G_{2}(\mathbb{C}^{4}): \text{the complex quadric}, \quad \text{and} \quad \lambda_{1} = 3c \)
8. \( M_{8} = M_{8,2l} = Q^{2l} \hookrightarrow G_{r}(\mathbb{C}^{2r}): \text{the complex quadric}, \quad r = 2^{l-1}, \quad l \geq 3, \quad \text{and} \quad \lambda_{1} = c 2^{l-2} \).
In above list, notice that $M_{4,2} = M_7$ and $M_{5,4} = M_{8,6}$.

Another one of the most simple typical examples of submanifolds of $G_r(\mathbb{C}^n)$ is a homogeneous Kähler hypersurface. K. Konno in [6] determined all Kähler C-spaces embedded as a hypersurface into a Kähler C-space with the second Betti number $b_2 = 1$.

**Theorem 3.2.** Let $M$ be a compact, simply connected homogeneous Kähler hypersurface of $G_r(\mathbb{C}^n)$, and $\lambda_1$ the first eigenvalue of the Laplace-Beltrami operator with respect to the induced Kähler metric. Then, $M$ and $\lambda_1$ are one of the following (up to isomorphism).

1. $M_9 = \mathbb{C}P^{n-2} \hookrightarrow \mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$ and $\lambda_1 = c(n-1)$
2. $M_{10} = \mathbb{Q}^{n-2} \hookrightarrow \mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$ and $\lambda_1 = c(n-2)$
3. $M_7 = \mathbb{Q}^3 \hookrightarrow \mathbb{Q}^4 = G_2(\mathbb{C}^4)$ and $\lambda_1 = 3c$
4. $M_{11} = \text{Sp}(l)/U(2) \hookrightarrow G_l(\mathbb{C}^{2l})$: Kähler C-space of type $(C_l, \alpha_2)$, $l \geq 2$ and $\lambda_1 = c(2l-1)

$M_9$ and $M_7$ are totally geodesic. $M_9$, $M_{10}$ and $M_7$ are symmetric spaces. If $l = 2$, then $M_{11}$ is congruent to $M_7$.

For each $l$ with $l > 2$, $M_{11}$ is not a symmetric space. Then, it is not easy to calculate the first eigenvalue $\lambda_1$ of $M_{11}$. We will calculate $\lambda_1$ of $M_{11}$ in the next section.

From these two theorems, we obtain the following proposition:

**Proposition 3.3.** Let $M$ be either a proper maximal totally geodesic Kähler submanifold of $G_r(\mathbb{C}^n)$ or a compact simply connected homogeneous Kähler hypersurface of $G_r(\mathbb{C}^n)$. Then, the first eigenvalue $\lambda_1$ of $M$ with respect to the induced Kähler metric satisfies the following inequality:

$$\lambda_1 \leq c(n-1).$$

Moreover, the equality holds if and only if $M$ is congruent to one of the follows:

$$M_1, \ M_2, \ M_{4,2} = M_7, \ M_9, \ M_{11}.$$
Let \( \mathfrak{t} = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A' & 0 & -C' \\ 0 & 0 & A & 0 \\ C' & 0 & 0 & A' \end{pmatrix} \right\} | A \in M_{r}(\mathbb{C}), \\
A', C' \in M_{l-r}(\mathbb{C}), \\
A^* = -A, \ A'^* = -A', \ tC' = C' \}
= \mathfrak{u}(r) + \mathfrak{sp}(l - r).

\( \mathfrak{g} \) is a compact semisimple Lie algebra of type \( C_1 \).

For \( x, y \in M_{l-r,r}(\mathbb{C}) \) and \( z \in M_{r}(\mathbb{C}) \) with \( t_z = z \), define

\[
\eta(x, y, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ z & ty & 0 & -tx \\ y & 0 & 0 & 0 \end{pmatrix}.
\]

Note that, if \( r = l \), then we ignore \( x \) and \( y \), and \( \eta(x, y, z) \) and \( \eta(0,0,z) \) denote a matrix \( \begin{pmatrix} 0_l & 0_l \\ z & 0_l \end{pmatrix} \), \( z \in M_l(\mathbb{C}) \), \( t_z = z \).

Let \( \mathfrak{m}, \mathfrak{m}^+ \) and \( \mathfrak{m}^- \) be subspaces of \( \mathfrak{g} \) defined by

\[
\mathfrak{m} = \{ \eta(x, y, z) - \eta(x, y, z)^* \},
\mathfrak{m}^+ = \{ \eta(x, y, z) \},
\mathfrak{m}^- = \{ \eta(x, y, z)^* \},
\]

so that \( \mathfrak{m}, \mathfrak{m}^+ \) and \( \mathfrak{m}^- \) are \( K \)-invariant under the adjoint action, and \( \mathfrak{m} \) is identified with the tangent space \( T_o M \) of \( M \) at the origin \( o = \{K\} \). Moreover, the complexification \( \mathfrak{m}^\mathbb{C} \) of \( \mathfrak{m} \) is the direct sum \( \mathfrak{m}^\mathbb{C} = \mathfrak{m}^+ + \mathfrak{m}^- \), and \( \mathfrak{m}^\pm \) is the \( \pm \sqrt{-1} \)-eigenspace of the complex structure \( J \) of \( M \) at the origin \( o \).

For any positive real number \( a \), the Einstein-Kähler metric \( g(a) \) of \( M \) is given by

\[
g(a)(X, X) = 2a \text{tr}(x^* x + y^* y + \bar{z}z), \quad X = \eta(x, y, z) - \eta(x, y, z)^* \in \mathfrak{m}.
\]

Relative to this metric, the scalar curvature \( \tau \) of \( M \) is given by

\[
\tau = \frac{2(2l - r + 1)}{a} \dim_{\mathbb{C}} M.
\]

Y. Matsushima and M. Obata showed the following:

**Theorem 4.1** [9]. Let \( M \) be an \( n \)-dimensional compact Einstein Kähler manifold of positive scalar curvature \( \tau \). Then the first eigenvalue \( \lambda_1(M) \) of the Laplacian satisfies that

\[
\lambda_1(M) \geq \frac{\tau}{n}.
\]

The equality holds if and only if \( M \) admits an one-parameter group of isometries (i.e., a non-trivial Killing vector field).

The natural inclusion \( \text{Sp}(l) \hookrightarrow SU(2l) \) defines an immersion \( \varphi \) of \( M = G_r(\mathbb{C}^{2l}) = \tilde{G}/\tilde{K}_l = SU(2l)/S(U(r) \cdot U(2l - r)) \) by

\[
\varphi(g \cdot K) = g \cdot \tilde{K}, \quad g \in G.
\]
Under identification of $T_0 \tilde{M}$ with $\tilde{m}$, the image of $X = \eta(x, y, z) - \eta(x, y, z)^* \in m$ is

$$\varphi_*(X) = \begin{pmatrix} 0 & -x^* & -\bar{z} & -y^* \\ x & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix},$$
so that we have

$$\tilde{g}_c(\varphi_*(X), \varphi_*(X)) = \frac{4}{c} tr(x^*x + y^*y + \bar{z}z).$$

Therefore, Theorem 4.1, (4.1) and (4.2) imply the following.

**Theorem 4.2.** For the Kähler C-space $M = Sp(l)/U(r) \cdot Sp(l-r)$ of type $(C_1, \alpha_r)$ equipped with the Kähler metric $g(\frac{2}{c})$, $M$ is immersed to $G_r(\mathbb{C}^{2l})$ by the Kähler immersion $\varphi$. The complex dimension, and the first eigenvalue $\lambda_1(M)$ of the Laplacian are given by

$$\dim_{\mathbb{C}} M = \frac{r(4l-3r+1)}{2}, \quad \lambda_1(M) = c(2l-r+1).$$

In particular, if $r = 2$, then $M = Sp(l)/U(2) \cdot Sp(l-2)$ is a Kähler hypersurface of $G_2(\mathbb{C}^{2l})$, whose first eigenvalue $\lambda_1(M)$ of the Laplacian is given by

$$\lambda_1(M) = c(2l-1).$$

For $z \in M_r(\mathbb{C})$, define an unit vector $\nu$ at the origin $o$ of $G_2(\mathbb{C}^{2l})$ by

$$\nu(z) = \begin{pmatrix} 0 & 0 & -z^* & 0 \\ 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \tilde{m}, \quad \frac{4}{c} tr z^*z = 1.$$  

Then $\nu(z)$ is tangent to $M$ if and only if $z$ is symmetric.

The Kähler hypersurface $M = (C_1, \alpha_2)$ satisfies the following property relative to the quaternionic Kähler structure $\tilde{J}$ of $G_2(\mathbb{C}^{2l})$.

**Proposition 4.3.** The Kähler hypersurface $M = Sp(l)/U(2) \cdot Sp(l-2)$ of $G_2(\mathbb{C}^{2l})$ satisfies

$$\tilde{J} T^\perp M \subset TM \quad (\iff J\xi \perp \tilde{J}\xi \text{ for any } \xi \in T^\perp M),$$

where $TM$ and $T^\perp M$ are the tangent bundle and the normal bundle of $M$, respectively.

**Proof.** Let $\nu_o$ be an unit normal vector of $M$ at $o$ defined by

$$\nu_o = \nu(z_o), \quad z_o = \frac{1}{2} \sqrt{\frac{c}{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
so that the normal space $T_o^\perp M$ is given by

$$T_o^\perp M = \mathbb{R}\{\nu_o, J\nu_o = \nu(\sqrt{-1}z_o)\}.$$ 

Then we see

$$\tilde{\mathcal{J}}_o T_o^\perp M = \mathbb{R}\{J_i\nu_o, J_j\nu_o, \ i = 1, 2, 3\}$$

$$= \mathbb{R}\{\nu(z_o\epsilon_i), \nu(\sqrt{-1}z_o\epsilon_i), \ i = 1, 2, 3\},$$

where $J_1, J_2$ and $J_3$ are a canonical basis of $\mathfrak{J}_o$ defined in the section 2. It is easy to check that $z_o\epsilon_i$ and $\sqrt{-1}z_o\epsilon_i$ are symmetric, so that we obtain

$$\mathcal{J}_o T_o^\perp M \subset T_oM.$$

Since the quaternionic Kähler structure $\mathcal{J}$ is $\tilde{G}$-invariant, and since the immersion $\varphi$ is $G$-equivariant, (4.3) holds at any point of $M$. □

If the ambient space is $G_2(\mathbb{C}^4)$, then the condition (4.3) determines a Kähler hypersurface as follows:

**Proposition 4.4.** Suppose that a Kähler hypersurface $M$ of $Q^4 = G_2(\mathbb{C}^4)$ satisfies the condition

$$\mathcal{J} T^\perp M \subset TM.$$

Then $M$ is totally geodesic. Moreover, if $M$ is compact, then $M$ is congruent to a complex quadric $Q^3 = Sp(2)/U(2)$.

**Proof.** Denote by $\tilde{\nabla}$ the Riemannian connection of $Q^4$, and denote by $\nabla, \sigma, A$ and $\nabla^\perp$, the Riemannian connection, the second fundamental form, the shape operator, and the normal connection of $M$, respectively. It is well-known that Gauss’ formula and Weingarten’s formula hold:

$$\tilde{\nabla}_X Y = \nabla X Y + \sigma(X, Y),$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

for $X, Y \in TM$ and $\xi \in T^\perp M$. The metric condition implies

$$\tilde{g}_c(\sigma(X, Y), \xi) = \tilde{g}_c(A_\xi X, Y).$$

Relative to the complex structure $J, \sigma$ and $A$ satisfy

$$\sigma(X, JY) = J\sigma(X, Y), \ A_\xi \circ J = -J \circ A_\xi = -A_{J_\xi}.$$ 

For a local unit normal vector field $\xi$, we define local vector fields as follow: $e_i = J_i \xi, \ i = 1, 2, 3$, where $J_1, J_2$ and $J_3$ are a local canonical basis of $\mathcal{J}$. Then,
under the assumption of this proposition, \{e_1, e_2, e_3, Je_1, Je_2, Je_3, \xi, J\xi\} is a local orthonormal frame field of \( Q^4 \) such that \{e_1, e_2, e_3, Je_1, Je_2, Je_3 \} is a tangent frame of \( M \). For \( X \in TM \), (4.4) implies

\[
(4.7) \quad \nabla_X e_i + \sigma(X, e_i) = \tilde{\nabla}_X e_i = (\tilde{\nabla}_X J_i) \xi + J_i(\nabla_\xi e_i)
\]

Since \( J \) is parallel with respect to the connection \( \tilde{\nabla} \), we have \( \tilde{\nabla}_X J_i \in \mathfrak{J} \), so that the normal component of (4.7) is

\[
\sigma(X, e_i) = -\tilde{g}_c(J_i A \xi X, \xi) \xi - \tilde{g}_c(J_i A \xi X, J \xi) J \xi = g_c(A \xi X, e_i) \xi + g_c(A \xi X, Je_i) J \xi.
\]

From these two equations, we get

\[
(4.8) \quad g_c(A \xi X, Je_i) = 0.
\]

Instead of \( X \), applying to \( JX \), we have

\[
g_c(A \xi X, e_i) = g_c(-A \xi JX, Je_i) = 0.
\]

Therefore, we have \( A \xi = 0 \), or \( \sigma = 0 \), so that \( M \) is totally geodesic. By B. Y. Chen and T. Nagano [3]'s results, if \( M \) is compact, \( M \) is congruent to a complex quadric \( Q^3 = Sp(2)/U(2) \). \( \square \)

§5. proof of main theorems.

Let \( M \) be a compact connected Kähler hypersurface of \( G_r(\mathbb{C}^n) \) immersed by a immersion \( \varphi \). It is well-known that every \( HM(n, \mathbb{C}) \)-valued function \( F \) satisfies

\[
(5.1) \quad (\Delta F, F)_{L^2} - \lambda_1(\Delta F, F)_{L^2} \geq 0
\]

The equality holds if and only if \( F \) is a sum of eigenfunctions with respect to eigenvalues 0 and \( \lambda_1 \). It is equivalent to that there exists a constant vector \( C \in HM(n, \mathbb{C}) \) such that \( \Delta(F - C) = \lambda_1(F - C) \).

Denote by \( H \) the mean curvature vector of the isometric immersion \( \Phi = \Psi \circ \varphi \). Then, since \( M \) is minimal in \( G_r(\mathbb{C}^n) \), (2.9) implies

\[
(5.2) \quad 2(r(n-r) - 1)H_A = 2r(n-r)\hat{H}_A - \hat{\sigma}_A(\xi, \xi) - \hat{\sigma}_A(J \xi, J \xi)
\]

\[
= c(r I - n A) - \hat{\sigma}_A(\xi, \xi) - \hat{\sigma}_A(J \xi, J \xi),
\]

where \( A \) is a position vector of \( \Phi(M) \) in \( HM(n, \mathbb{C}) \), and \( \xi \) is a local unit normal vector field of \( \varphi \). Using (2.11) and (5.2), we get

\[
(5.3) \quad (H_A, A) = -1.
\]

\( HM(n, \mathbb{C}) \)-valued function \( \Phi \) satisfies \( \Delta \Phi = -2(r(n-r) - 1)H \), so that (5.1) and (5.3) imply the following. The equality condition dues to T. Takahashi’s theorem in [12].
Lemma 5.1.

\begin{equation}
2(r(n-r)-1) \int_{M} (H_{A}, H_{A}) \, dv_{M} - \lambda_{1} \text{vol}(M) \geqq 0.
\end{equation}

The equality holds if and only if $\Phi$ is a minimal immersion of $M$ into some round sphere in $HM(n, \mathbb{C})$, more precisely, there exists some positive constant $R$ and some constant vector $C \in HM(n, \mathbb{C})$ such that $H_{A}$ satisfies

\begin{equation}
H_{A} = \frac{1}{R^{2}} (C - A).
\end{equation}

Lemma 5.2. If the equality holds in (5.4), then $M$ is contained in a totally geodesic submanifold of $G_{r}(\mathbb{C}^{n})$ which is product of Grassmann manifolds, more precisely, there exist integers $k_{i}, r_{i}, i = 1, \cdots, m$ such that

\begin{align*}
0 & \leqq r_{i} \leqq k_{i}, \quad r_{1} \geqq r_{2} \geqq \cdots \geqq r_{m}, \\
\sum_{i=1}^{m} r_{i} &= r, \\
\sum_{i=1}^{m} k_{i} &= n,
\end{align*}

(5.6) \quad M \subset G_{r_{1}}(\mathbb{C}^{k_{1}}) \times G_{r_{2}}(\mathbb{C}^{k_{2}}) \times \cdots \times G_{r_{m}}(\mathbb{C}^{k_{m}}) \subset G_{r}(\mathbb{C}^{n}).

Notice that $G_{0}(\mathbb{C}^{k_{i}}) = G_{k_{i}}(\mathbb{C}^{k_{i}}) = \{\text{one point}\}$.

proof. Assume that this equality holds in (5.4).

Since $M$ is minimal in $G_{r}(\mathbb{C}^{n})$, $H$ is normal to $G_{r}(\mathbb{C}^{n})$. Then, from (2.4) and (5.5), we get

\begin{equation}
CA = AC,
\end{equation}

where $C$ is a constant vector in Lemma 5.1. Since $SU(n)$ acts on $G_{r}(\mathbb{C}^{n})$ transitively, without loss of generalization, we can assume that $C$ is a diagonal matrix as follows:

\begin{equation}
C = \begin{pmatrix}
c_{1}I_{k_{1}} & & 0 \\
& c_{2}I_{k_{2}} & & 0 \\
& & \ddots & 0 \\
0 & & & c_{m}I_{k_{m}}
\end{pmatrix}, \quad k_{i} > 0, \quad c_{i} \neq c_{j} (i \neq j).
\end{equation}

Notice that

\begin{equation}
n = k_{1} + k_{2} + \cdots + k_{m}.
\end{equation}

Define a linear subspace $L$ of $HM(n, \mathbb{C})$ by $L = \{Z \in HM(n, \mathbb{C}) \mid ZC = CZ\}$, so that

\begin{equation}
L = \left\{ \begin{pmatrix}
Z_{1} & & 0 \\
& Z_{2} & & 0 \\
& & \ddots & 0 \\
0 & & & Z_{m}
\end{pmatrix} \mid Z_{i} \in M_{k_{i}}(\mathbb{C}) \right\}.
\end{equation}
From (5.7), $M$ is contained in $G_r(\mathbb{C}^n) \cap L$.

For each integer $r_i$ with $0 \leq r_i \leq k_i$, $\sum_{i=1}^{m} r_i = r$, let's define connected subsets of $G_r(\mathbb{C}^n)$ by

$$W_{r_1, \ldots, r_m} = \left\{ \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \\ 0 & & & A_m \end{pmatrix} \right| A_i \in M_{k_i}(\mathbb{C}), \quad A_i^2 = A_i, \quad tr A_i = r_i \right\}.$$

So, $G_r(\mathbb{C}^n) \cap L$ is a disjoint union of all $W_{r_1, \ldots, r_m}$'s. Since $M$ is connected, $M$ is contained in suitable one of $W_{r_1, \ldots, r_m}$'s, saying $W_{r_1, \ldots, r_m}$. By the definition, we see

$$W_{r_1, \ldots, r_m} = G_{r_1}(\mathbb{C}^{k_1}) \times G_{r_2}(\mathbb{C}^{k_2}) \times \cdots \times G_{r_m}(\mathbb{C}^{k_m}).$$

Without loss of generalization, we can choose a diagonal matrix $C$ with respect to which the inequalities $r_1 \geq r_2 \geq \cdots \geq r_m$ hold. \square

From (2.8), (2.10) and (5.2), we get

$$H_A = \frac{c}{2(r(n-r)-1)} \left\{ (rI - nA) - \frac{4}{c} (\Psi_* \xi)^2 (I - 2A) \right\}.$$

Using (2.2) and (2.3), we see

$$H_A, H_A = \frac{c}{2(r(n-r)-1)^2} \left\{ 2tr \frac{4}{c} r (\Psi_* \xi)^2 \left( I + \frac{n-2r}{r} A \right) \\
+ tr \frac{16}{c^2} (\Psi_* \xi)^2 (I - 2A) (\Psi_* \xi)^2 (I - 2A) \right\}.$$

Since the immersion $\Psi$ is $\tilde{G}$-equivariant, for any $A \in \Phi(M)$, there exists a element $g_A \in \tilde{G}$ and a matrix $v_A \in M_{n-r,r}(\mathbb{C})$ satisfying $A_o = g_A A g_A^*$ and

$$\sqrt{\frac{c}{4}} = g_A (\Psi_* \xi) g_A^*.$$

Since the inner product $\langle \cdot, \cdot \rangle$ is $\tilde{G}$-equivariant and $\xi$ is unit, we have $tr v_A^* v_A = tr v_A v_A^* = 1$. After translating by $g_A$, together with (5.11), (5.10) implies

$$H_A, H_A = \frac{c}{2(r(n-r)-1)^2} \left\{ n(r(n-r) - 2) + 2tr (v_A^* v_A v_A^* v_A) \right\}.$$

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Lemma 5.3. (a) For $v \in M_{n-r,r}(\mathbb{C})$ with $tr v^* v = 1$, the following inequality holds

\begin{equation}
tr v^* vvv^* v \leq 1.
\end{equation}

(b) Moreover, next three conditions are equivalent to each other.

1. The equality holds in (5.13)
2. The hermitian $r$-matrix $v^* v$ is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix}$.
3. The hermitian $(n - r)$-matrix $vv^*$ is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}$.

(c) If the equality holds in (5.13), then there exists $R = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \in S(U(r) \cdot U(n-r))$ such that $v' = QvP^*$ satisfies

\begin{equation*}
v'^* v' = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \quad \text{and} \quad v' v'^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}.
\end{equation*}

Proof. Lemma 5.3 follows from that both of hermitian matrices $v^* v$ and $vv^*$ are similar to diagonal matrices with non-negative eigenvalues.

From (5.12) and Lemma 5.3, the following lemma is immediately obtained, which is used to prove Theorem A.

Lemma 5.4.

\begin{equation}
(H_A, H_A) \leq \frac{c}{2(r(n-r) - 1)} \left\{ n - \frac{n-2}{r(n-r)-1} \right\}.
\end{equation}

The equality holds if and only if, for any $A \in \Phi(M)$, it is possible to choose $v_A$ satisfying

\begin{equation}
v_A^* v_A = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \quad \text{and} \quad v_A v_A^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}.
\end{equation}

Proof of Theorem A. (5.4) and (5.14) imply

\begin{equation*}
\lambda_1 \leq c \left( n - \frac{n-2}{r(n-r)-1} \right).
\end{equation*}

Let’s assume that this equality holds. Then, the equality conditions of Lemmas 5.1 and 5.4 hold.

Assume $m = 1$. Then, (5.5) and (5.9) imply

\begin{equation*}
\frac{1}{R^2} (c_1 I - A) = \frac{c}{2(r(n-r)-1)} \left\{ (rI - nA) - \frac{4}{c} (\Psi \cdot \xi)^2 (I - 2A) \right\}.
\end{equation*}
After translating by $g_A$, together with (5.11) and (5.15), we obtain
\[
\frac{1}{R^2}(c_1 - 1)I_r = \frac{c}{2(r(n-r) - 1)} \left\{ \begin{pmatrix} r - n & I_r \\ 0 & 0_{r-1} \end{pmatrix} \right\},
\]
\[
\frac{1}{R^2}c_1I_{n-r} = \frac{c}{2(r(n-r) - 1)} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix} \right\}.
\]

The first equation implies $r = 1$, and the second one implies $n - r = 1$. So, we have $n = 2$ and $r = 1$. This contradicts that $M$ is a complex hypersurface.

Since $m \geq 2$, from Lemma 5.2, $M$ is contained in a proper totally geodesic submanifold of $G_r(\mathbb{C}^n)$. On the other hand, $M$ is of complex codimension 1 in $G_r(\mathbb{C}^n)$. Consequently, either $r = 1$ or $r = n - 1$ occurs, and $M$ is a totally geodesic complex hypersurface of a complex projective space $\mathbb{C}P^{n-1} \cong G_1(\mathbb{C}^n) \cong G_{n-1}(\mathbb{C})$. □

Proof of Theorem B. Let’s assume that $M$ is a compact connected Kähler hypersurface of $G_2(\mathbb{C}^n)$ satisfying the condition $J\xi \perp \bar{\zeta}$. Since both of the complex structure and the quaternionic Kähler structure are $G$-invariant, we obtain, at the origin $A_o$,
\[
(5.16)
\]
[math]
J \begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix} \perp J_i \begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix}, \quad i = 1, 2, 3,
\[/math]

where $J_1, J_2$ and $J_3$ are a canonical basis of $\mathcal{J}_o$ defined in the section 2. Set
\[
v_A = (v_A', v_A''), \quad v_A', v_A'' \in M_{n-2,1}(\mathbb{C}) \cong \mathbb{C}^{n-2}.
\]

Using (2.6) and (2.7), (5.16) implies that $|v_A'| = |v_A''|$ and $v_A' \perp v_A''$. Combing them with $tr v_A^*v_A = 1$, we obtain $|v_A'| = |v_A''| = \frac{1}{\sqrt{2}}$, so that
\[
(5.17)
\]
\[
v_A^*v_A = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Together with (5.17), (5.12) implies
\[
(H_A, H_A) = \frac{1}{2(2n-5)} \left\{ n - \frac{n-1}{2n-5} \right\}.
\]

Therefore, from Lemma 5.1, we obtain
\[
\lambda_1 \leq c\left( n - \frac{n-1}{2n-5} \right).
\]

Let’s assume that this equality holds. Then, the equality conditions of Lemma 5.1 holds.
Computing dimensions of manifolds in (5.6), we have

\begin{equation}
2n - 5 \leq \sum_{i=1}^{m} r_i (k_i - r_i).
\end{equation}

From \( \sum_{i=1}^{m} r_i = 2 \) and \( r_1 \geq r_2 \geq \cdots \geq r_m \), the following two cases occur:

Case I: \( r_1 = r_2 = 1, \ r_3 = \cdots = r_m = 0 \),

Case II: \( r_1 = 2, \ r_2 = \cdots = r_m = 0 \).

In Case I, (5.18) implies \( 2n - 5 \leq k_1 + k_2 - 2 \leq n - 2 \), so \( n \leq 3 \). This is contradiction.

Therefore, Case II occurs. Then, (5.18) implies \( 2n - 5 \leq 2(k_1 - 2) \), so that we have \( n = k_1, \ m = 1, \ k_2 = \cdots = k_m = 0 \). (5.5) and (5.9) imply

\[ \frac{1}{R^2} (c_1 I - A) = \frac{c}{2(2n - 5)} \left\{ (2I - nA) - \frac{4}{c} (\Psi_* \xi)^2 (I - 2A) \right\}. \]

After translating by \( g_A \), together with (5.11) and (5.17), we obtain

\[ \frac{1}{R^2} (c_1 - 1) = \frac{c}{2(2n - 5)} \left\{ 2 - n + \frac{1}{2} \right\}, \]

\[ \frac{1}{R^2} c_1 I_{n-2} = \frac{c}{2(2n - 5)} \left\{ 2I_{n-2} - v_A v_A^* \right\}. \]

The second equation implies

\begin{equation}
v_A v_A^* = d I_{n-2}, \quad d = 2 - \frac{2(2n - 5) c_1}{c R^2}.
\end{equation}

From (5.17), we have

\[ d v_A = d I_{n-2} v_A = (v_A v_A^*) v_A = v_A (v_A^* v_A) = \frac{1}{2} v_A, \]

so that \( d = \frac{1}{2} \). Consequently, taking traces of both sides of (5.19), we obtain \( n = 4 \).

Therefore, from Proposition 4.4, \( M \) is congruent to \( Q^3 \). \( \square \)

\textbf{References}


