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IN THE COMPLEX GRASSMANN MANIFOLD

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SPECTRAL GEOMETRY OF KÄHLER HYPERSURFACES
IN THE COMPLEX GRASSMANN MANIFOLD

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§1. Introduction.
Let $M$ be a compact $C^\infty$-Riemannian manifold, $C^\infty(M)$ the space of all smooth functions on $M$, and $\Delta$ the Laplacian on $M$. Then $\Delta$ is a self-adjoint elliptic differential operator acting on $C^\infty(M)$, which has an infinite discrete sequence of eigenvalues: $\text{Spec}(M) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \uparrow \infty\}$. Let $V_k = V_k(M)$ be the eigenspace of $\Delta$ corresponding to the $k$-th eigenvalue $\lambda_k$. Then $V_k$ is finite-dimensional. We define an inner product $(f, g)_{L^2} = \int_M fg \, dv_M$ on $C^\infty(M)$, where $dv_M$ denotes the volume element on $M$. Then $\sum_{k=0}^\infty V_k$ is dense in $C^\infty(M)$ and the decomposition is orthogonal with respect to the inner product $(\cdot, \cdot)_{L^2}$. Thus we have $C^\infty(M) = \sum_{k=0}^\infty V_k(M)$ (in $L^2$-sense).

Since $M$ is compact, $V_0$ is the space of all constant functions which is 1-dimensional.

In this point of view, it is one of the simplest and the most interesting problems to estimate the first eigenvalue. In [10], A. Ros gave the following sharp upper bound for the first eigenvalue of Kähler submanifold of a complex projective space.

**Theorem 1.1.** Suppose that $M$ is a complex $m$-dimensional compact Kähler submanifold of the complex projective space $\mathbb{C}P^n$ of constant holomorphic sectional curvature $c$. Then the first eigenvalue $\lambda_1$ satisfies the following inequality:

$$\lambda_1 \leq c(m+1)$$

The equality holds if and only if $M$ is congruent to the totally geodesic Kähler submanifold $\mathbb{C}P^m$ of $\mathbb{C}P^n$.

If $M$ is not totally geodesic, J-P. Bourguignon, P. Li and S. T. Yau in [1] gave the following more sharp estimate. (See also [7].)

**Theorem 1.2.** Suppose that $M$ is a complex $m$-dimensional compact Kähler submanifold of $\mathbb{C}P^n$, which is fully immersed and not totally geodesic. Then the first eigenvalue $\lambda_1$ satisfies the following inequality:

$$\lambda_1 \leq c\, m \, \frac{n+1}{n}$$

It is unknown when the equality holds in this inequality.

Our purpose is to give the upper bound for the first eigenvalue of Kähler hypersurfaces of a complex Grassmann manifold.
Let denote by $G_r(\mathbb{C}^n)$ the complex Grassmann manifold of $r$-planes in $\mathbb{C}^n$, equipped with the Kähler metric of maximal holomorphic sectional curvature $c$. We obtain the following result which is a natural generalization of Theorem 1.1.

**Theorem A.** Suppose that $M$ is a compact connected Kähler hypersurface of $G_r(\mathbb{C}^n)$. Then the first eigenvalue $\lambda_1$ satisfies the following inequality:

$$\lambda_1 \leq c \left( n - \frac{n-2}{r(n-r) - 1} \right)$$

The equality holds if and only if $r = 1, n$, and $M$ is congruent to the totally geodesic complex hypersurface $\mathbb{CP}^{n-2}$ of the complex projective space $\mathbb{CP}^{n-1}$.

The 2-plane Grassmann manifold $G_2(\mathbb{C}^n)$ admits the quaternionic Kähler structure $\mathfrak{J}$. For the normal bundle $T^\perp M$ of a Kähler hypersurface $M$ of $G_2(\mathbb{C}^n)$, $\mathfrak{J}T^\perp M$ is a vector bundle of real rank 6 over $M$ which is a subbundle of the tangent bundle of $G_2(\mathbb{C}^n)$. We consider a Kähler hypersurface $M$ of $G_2(\mathbb{C}^n)$ satisfying the property that $\mathfrak{J}T^\perp M$ is a subbundle of the tangent bundle $TM$ of $M$. In the section 4, we will introduce examples satisfying this property.

For a Kähler hypersurface of $G_2(\mathbb{C}^n)$ satisfying this property, we obtain the following upper bound of the first eigenvalue.

**Theorem B.** Suppose that $M$ is a compact connected Kähler hypersurface of $G_2(\mathbb{C}^n)$, $n \geq 4$. If $M$ satisfies the condition $\exists T^\perp M \subset TM$, then the following inequality holds:

$$\lambda_1 \leq c \left( n - \frac{n-1}{2n-5} \right)$$

The equality holds if and only if $n = 4$ and $M$ is congruent to the totally geodesic complex hypersurface $Q^3$ of the complex quadric $Q^4 = G_2(\mathbb{C}^4)$.

These two theorems are proved in the section 5. More detailed proofs of any our results are given in [8].

**Notations.** $M_{r,s}(\mathbb{C})$ denotes the set of all $r \times s$ matrices with entries in $\mathbb{C}$, and $M_r(\mathbb{C})$ stands for $M_{r,r}(\mathbb{C})$. $I_r$ and $O_r$ denote the identity $r$-matrix and the zero $r$-matrix.

§2. Preliminaries.

In this section, we discuss geometries of the complex $r$-plane Grassmann manifold and its first standard imbedding.

Let $M_r(\mathbb{C}^n)$ be the complex Stiefel manifold which is the set of all unitary $r$-systems of $\mathbb{C}^n$, i.e.,

$$M_r(\mathbb{C}^n) = \{ Z \in M_{n,r}(\mathbb{C}) | Z^*Z = I_r \}.$$ 

The complex $r$-plane Grassman manifold $G_r(\mathbb{C}^n)$ is defined by

$$G_r(\mathbb{C}^n) = M_r(\mathbb{C}^n)/U(r).$$
The origin \( o \) of \( G_r(\mathbb{C}^n) \) is defined by \( \pi(Z_0) \), where \( Z_0 = \begin{pmatrix} I_r \\ 0 \end{pmatrix} \) is a element of \( M_r(\mathbb{C}^n) \), and \( \pi: M_r(\mathbb{C}^n) \rightarrow G_r(\mathbb{C}^n) \) is the natural projection.

The left action of the unitary group \( \tilde{G} = SU(n) \) on \( G_r(\mathbb{C}^n) \) is transitive, and the isotropy subgroup at the origin \( o \) is

\[
\tilde{K} = S(U(r) \cdot U(n-r))
\]

so that \( G_r(\mathbb{C}^n) \) is identified with a homogeneous space \( \tilde{G}/\tilde{K} \)

Set \( \tilde{\mathfrak{g}} = su(n) \) and

\[
\hat{\mathfrak{e}} = \mathbb{R} \oplus su(r) \oplus su(n-r)
\]

then \( \tilde{\mathfrak{g}} \) and \( \hat{\mathfrak{e}} \) are the Lie algebra of \( \tilde{G} \) and \( \tilde{K} \), respectively. Define a linear subspace \( \hat{\mathfrak{m}} \) of \( \tilde{\mathfrak{g}} \) by

\[
\hat{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} \mid \xi \in M_{n-r,r}(\mathbb{C}) \right\},
\]

then \( \hat{\mathfrak{m}} \) is identified with the tangent space \( T_o(G_r(\mathbb{C}^n)) \). The \( \tilde{G} \)-invariant complex structure \( J \) of \( G_r(\mathbb{C}^n) \) and the \( \tilde{G} \)-invariant Kähler metric \( \tilde{g}_c \) of \( G_r(\mathbb{C}^n) \) of the maximal holomorphic sectional curvature \( c \) are given by

\[
J \begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1} \xi^* \\ -\sqrt{-1} \xi & 0 \end{pmatrix},
\]

(2.1)

\[
\tilde{g}_{c_o}(X, Y) = -\frac{2}{c} \text{tr} XY, \quad X, Y \in \hat{\mathfrak{m}}.
\]

In the case of \( r = 2 \), the complex 2-plane Grassmann manifold \( G_2(\mathbb{C}^n) \) admits another geometric structure named the quaternionic Kähler structure \( J \). \( J \) is a \( \tilde{G} \)-invariant subbundle of \( \text{End}(T(G_2(\mathbb{C}^n))) \) of rank 3, where \( \text{End}(T(G_2(\mathbb{C}^n))) \) is the \( \tilde{G} \)-invariant vector bundle of all linear endmorphisms of the tangent bundle \( T(G_2(\mathbb{C}^n)) \). Under the identification with \( T_o(G_r(\mathbb{C}^n)) \) and \( \hat{\mathfrak{m}} \), the fiber \( J_o \) at the origin \( o \) is given by

\[
J_o = \left\{ J_{\xi} = \text{ad}(\xi) \mid \xi \in \hat{\mathfrak{e}_q} \right\},
\]

where \( \hat{\mathfrak{e}_q} \) is an ideal of \( \hat{\mathfrak{e}} \) defined by

\[
\hat{\mathfrak{e}_q} = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & 0 \end{pmatrix} \mid u_1 \in su(2) \right\} \cong su(2).
Choose a basis \( \{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \} \) of \( \mathfrak{su}(2) \) satisfying \( [\varepsilon_i, \varepsilon_{i+1}] = 2\varepsilon_{i+2} \pmod{3} \). Set
\[
\tilde{\varepsilon}_i = \left( \begin{array}{cc} \varepsilon_i & 0 \\ 0 & 0 \end{array} \right)
\]
and \( J_i = J_{\tilde{\varepsilon}_i} \) for \( i = 1, 2, 3 \), then the basis \( \{ J_1, J_2, J_3 \} \) is a canonical basis of \( \mathfrak{J}_0 \), satisfying
\[
J_i^2 = -id_m \quad \text{for} \quad i = 1, 2, 3,
\]
\[
J_1J_2 = -J_2J_1 = J_3, \quad J_2J_3 = -J_3J_2 = J_1, \quad J_3J_1 = -J_1J_3 = J_2,
\]
\[
\tilde{g}_{c_{\mathcal{O}}}(J_iX, J_iY) = \tilde{g}_{c_{\mathcal{O}}}(X, Y), \quad \text{for} \ X, Y \in \tilde{m} \text{ and } i = 1, 2, 3.
\]

There exists an element \( \mathring{\varepsilon}_{\mathbb{C}} \) of the center of \( \mathfrak{g} \) such that \( J \) is given by \( J = ad(\mathring{\varepsilon}_{\mathbb{C}}) \) on \( m \). Therefore, \( J \) is commutable with \( \mathfrak{J} \).

Let \( HM(n, \mathbb{C}) \) be the set of all Hermitian \((n, n)\)-matrices over \( \mathbb{C} \), which can be identified with \( \mathbb{R}^{n^2} \). For \( X, Y \in HM(n, \mathbb{C}) \), the natural inner product is given by
\[
(2.2) \quad (X, Y) = \frac{2}{c} tr XY.
\]

\( GL(n, \mathbb{C}) \) acts on \( HM(n, \mathbb{C}) \) by \( X \mapsto BXB^*, B \in GL(n, \mathbb{C}), X \in HM(n, \mathbb{C}) \). Then the action of \( SU(n) \) leaves the inner product \((2.2)\) invariant.

The first standard imbedding \( \Psi \) of \( G_r(\mathbb{C}^n) \) is defined by
\[
\Psi(\pi(z)) = zZ^* \in HM(n, \mathbb{C}), \quad Z \in M_\Gamma(\mathbb{C}^n).
\]

\( \Psi \) is \( SU(n) \)-equivariant and the image \( N \) of \( G_r(\mathbb{C}^n) \) under \( \Psi \) is given as follows:
\[
(2.3) \quad N = \Psi(G_r(\mathbb{C}^n)) = \{ A \in HM(n, \mathbb{C}) \mid A^2 = A, \ trA = r \}.
\]

The tangent bundle \( TN \) and the normal bundle \( T^\perp N \) are given by
\[
(2.4) \quad T_{A_o}N = \{ X \in HM(n, \mathbb{C}) \mid AX +XA = X \} \subset HM_0,
\]
\[
T_{A_o}^\perp N = \{ Z \in HM(n, \mathbb{C}) \mid ZA = ZX \}.
\]

In particular, at the origin \( A_o = \Psi(0) = \left( \begin{array}{cc} I_r & 0 \\ 0 & 0 \end{array} \right) \), we can obtain
\[
(2.5) \quad T_{A_o}N = \left\{ \left( \begin{array}{cc} 0 & \xi^* \\ \xi & 0 \end{array} \right) \mid \xi \in M_{n-r, r}(\mathbb{C}) \right\},
\]
\[
T_{A_o}^\perp N = \left\{ \left( \begin{array}{cc} Z_1 & 0 \\ 0 & Z_2 \end{array} \right) \mid Z_1 \in HM(r, \mathbb{C}), Z_2 \in HM(n - r, \mathbb{C}) \right\}.
\]

The complex structure \( J \) acts on \( T_{A_o}N \) as follows:
\[
(2.6) \quad J \left( \begin{array}{cc} 0 & \xi^* \\ \xi & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & -\sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0 \end{array} \right).
\]
If \( r = 2 \), then the quaternionic Kähler structure \( \mathfrak{J} \) acts on \( T_{A_{\mathrm{o}}}N \) as follows:

\[
J_{\xi} \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \xi^* \\ -\xi \varepsilon & 0 \end{pmatrix}, \quad \varepsilon \in \text{su}(2).
\]

Let \( \tilde{\sigma} \) and \( \tilde{H} \) denote the second fundamental form and the mean curvature vector of \( \Psi \), respectively. Then, for \( A \in N \) and \( X, Y \in T_{A}N \), we can see

\[
\tilde{\sigma}_{A}(X, Y) = (XY + YX)(I - 2A)
\]

(2.8)

\[
\tilde{H}_{A} = \frac{c}{2r(n-r)}(rI - nA)
\]

and \( \tilde{\sigma} \) satisfies the following:

(2.10) \( \tilde{\sigma}_{A}(JX, JY) = \tilde{\sigma}_{A}(X, Y) \),

(2.11) \( (\tilde{\sigma}_{A}(X, Y), A) = -(X, Y) \).

§3. Examples.

One of the most simple typical examples of submanifolds of \( G_{r}(\mathbb{C}^{n}) \) is a totally geodesic submanifold. B. Y. Chen and T. Nagano in [3, 4] determined maximal totally geodesic submanifolds of \( G_{2}(\mathbb{C}^{n}) \). For arbitrary \( r \), I. Satake and S. Ihara in [11, 5] determined all (equivariant) holomorphic imbeddings of a symmetric domain into another symmetric domain. Taking a compact dual symmetric space if necessary, we obtain the complete list of maximal totally geodesic Kähler submanifolds of \( G_{r}(\mathbb{C}^{n}) \).

Since totally geodesic submanifolds of \( G_{r}(\mathbb{C}^{n}) \) are symmetric spaces, we can calculate the first eigenvalue of the Laplacian of \( M \). (cf. [14])

**Theorem 3.1.** Let \( M \) be a proper maximal totally geodesic Kähler submanifold of \( G_{r}(\mathbb{C}^{n}) \), and \( \lambda_{1} \) the first eigenvalue of the Laplace-Beltrami operator with respect to the induced Kähler metric. Then, \( M \) and \( \lambda_{1} \) are one of the following (up to isomorphism).

1. \( M_{1} = G_{r}(\mathbb{C}^{n-1}) \rightarrow G_{r}(\mathbb{C}^{n}), \quad 1 \leq r \leq n - 2, \quad \text{and} \quad \lambda_{1} = c(n - 1) \)
2. \( M_{2} = G_{r-1}(\mathbb{C}^{n-1}) \rightarrow G_{r}(\mathbb{C}^{n}), \quad 2 \leq r \leq n - 1, \quad \text{and} \quad \lambda_{1} = c(n - 1) \)
3. \( M_{3} = G_{r_{1}}(\mathbb{C}^{n_{1}}) \times G_{r_{2}}(\mathbb{C}^{n_{2}}) \rightarrow G_{1+r_{2}}(\mathbb{C}^{n_{1}+n_{2}}), \quad 1 \leq r_{i} \leq n_{i} - 1, \quad i = 1, 2, \quad \text{and} \quad \lambda_{1} = c \min\{n_{1}, n_{2}\} \)
4. \( M_{4} = M_{4,p} = Sp(p)/U(p) \rightarrow G_{p}(\mathbb{C}^{2p}), \quad p \geq 2, \quad \text{and} \quad \lambda_{1} = c(p + 1) \)
5. \( M_{5} = M_{5,p} = SO(2p)/U(p) \rightarrow G_{p}(\mathbb{C}^{2p}), \quad p \geq 4, \quad \text{and} \quad \lambda_{1} = c(p - 1) \)
6. \( M_{6,m} = \mathbb{C}P^{p} \rightarrow G_{r}(\mathbb{C}^{n}) : \text{the complex projective space}, \quad r = \left( \begin{array}{c} p \\ m-1 \end{array} \right), \quad n = \left( \begin{array}{c} p+1 \\ m \end{array} \right), \quad 2 \leq m \leq p-1, \quad \text{and} \quad \lambda_{1} = c(p+1) \left( \begin{array}{c} p-1 \\ m-1 \end{array} \right)^{-1} \)
7. \( M_{7} = Q^{3} \rightarrow Q^{4} = G_{2}(\mathbb{C}^{4}) : \text{the complex quadric}, \quad \text{and} \quad \lambda_{1} = 3c \)
8. \( M_{8} = M_{8,2l} = Q^{2l} \rightarrow G_{r}(\mathbb{C}^{2r}) : \text{the complex quadric}, \quad r = 2^{l-1}, \quad l \geq 3, \quad \text{and} \quad \lambda_{1} = c \frac{2^{l}}{2^{l-2}} \)
In above list, notice that $M_{4,2} = M_7$ and $M_{5,4} = M_{8,6}$.

Another one of the most simple typical examples of submanifolds of $G_r(C^n)$ is a homogeneous Kähler hypersurface. K. Konno in [6] determined all Kähler C-spaces embedded as a hypersurface into a Kähler C-space with the second Betti number $b_2 = 1$.

**Theorem 3.2.** Let $M$ be a compact, simply connected homogeneous Kähler hypersurface of $G_r(C^n)$, and $\lambda_1$ the first eigenvalue of the Laplace-Beltrami operator with respect to the induced Kähler metric. Then, $M$ and $\lambda_1$ are one of the following (up to isomorphism).

1. $M_9 = \mathbb{C}P^{n-2} \hookrightarrow \mathbb{C}P^{n-1} = G_1(C^n)$ and $\lambda_1 = c(n - 1)$
2. $M_{10} = Q^{n-2} \hookrightarrow \mathbb{C}P^{n-1} = G_1(C^n)$ and $\lambda_1 = c(n - 2)$
3. $M_7 = Q^3 \hookrightarrow Q^4 = G_2(C^4)$ and $\lambda_1 = 3c$
4. $M_{11} = Sp(l)/U(2)Sp(l - 2) \hookrightarrow G_l(C^{2l})$: Kähler C-space of type $(C_l, \alpha_2)$, $l \geq 2$ and $\lambda_1 = c(2l - 1)$

$M_9$ and $M_7$ are totally geodesic. $M_9$, $M_{10}$, and $M_7$ are symmetric spaces. If $l = 2$, then $M_{11}$ is congruent to $M_7$.

For each $l$ with $l > 2$, $M_{11}$ is not a symmetric space. Then, it is not easy to calculate the first eigenvalue $\lambda_1$ of $M_{11}$. We will calculate $\lambda_1$ of $M_{11}$ in the next section.

From these two theorems, we obtain the following proposition:

**Proposition 3.3.** Let $M$ be either a proper maximal totally geodesic Kähler submanifold of $G_r(C^n)$ or a compact simply connected homogeneous Kähler hypersurface of $G_r(C^n)$. Then, the first eigenvalue $\lambda_1$ of $M$ with respect to the induced Kähler metric satisfies the following inequality:

$$\lambda_1 \leq c(n - 1).$$

Moreover, the equality holds if and only if $M$ is congruent to one of the follows:

$$M_1, \ M_2, \ M_{4,2} = M_7, \ M_9, \ M_{11}.$$ 

**§4. the homogeneous Kähler hypersurface $(C_l, \alpha_2)$.**

In this section, we will consider the first eigenvalue of the Kähler C-space of type $(C_l, \alpha_r)$. For details, see [2] and [13].

The Kähler C-space of type $(C_l, \alpha_r)$ is a compact simply connected homogeneous Kähler manifold $M = G/K = Sp(l)/U(r) \cdot Sp(l - r)$, $1 \leq r \leq l$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ Lie algebras of $G$ and $K$, respectively, i.e.,

$$\mathfrak{g} = \mathfrak{sp}(l) = \left\{ \begin{pmatrix} A & -\overline{C} \\ C & A \end{pmatrix} \mid A, C \in M_l(C), \quad A^* = -A, \quad ^t\overline{C} = C \right\},$$
\[ \mathfrak{g} = \{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A' & 0 & -\overline{C'} \\ 0 & 0 & A & 0 \\ C' & 0 & 0 & \overline{A'} \end{pmatrix} \mid A \in M_r(\mathbb{C}), \quad A', C' \in M_{l-r}(\mathbb{C}), \quad A^* = -A, \quad A'^* = -A', \quad ^tC' = C' \} \]

\[ = \mathfrak{u}(r) + \mathfrak{sp}(l-r). \]

g is a compact semisimple Lie algebra of type $C_l$.

For $x, y \in M_{l-r,r}(\mathbb{C})$ and $z \in M_r(\mathbb{C})$ with $^t z = z$, define

\[ \eta(x, y, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ z & ty & 0 & -t^x \\ y & 0 & 0 & 0 \end{pmatrix}. \]

Note that, if $r = l$, then we ignore $x$ and $y$, and $\eta(x, y, z)$ and $\eta(0, 0, z)$ denote a matrix $\begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}$, $z \in M_l(\mathbb{C})$, $^t z = z$.

Let $\mathfrak{m}, \mathfrak{m}^+$ and $\mathfrak{m}^-$ be subspaces of $\mathfrak{g}$ defined by

\[ \mathfrak{m} = \{ \eta(x, y, z) - \eta(x, y, z)^* \}, \]
\[ \mathfrak{m}^+ = \{ \eta(x, y, z) \}, \]
\[ \mathfrak{m}^- = \{ \eta(x, y, z)^* \}, \]

so that $\mathfrak{m}, \mathfrak{m}^+$ and $\mathfrak{m}^-$ are $K$-invariant under the adjoint action, and $\mathfrak{m}$ is identified with the tangent space $T_o M$ of $M$ at the origin $o = \{ K \}$. Moreover, the complexification $\mathfrak{m}^\mathbb{C}$ of $\mathfrak{m}$ is the direct sum $\mathfrak{m}^\mathbb{C} = \mathfrak{m}^+ + \mathfrak{m}^-$, and $\mathfrak{m}^\pm$ is the $\pm \sqrt{-1}$-eigenspace of the complex structure $J$ of $M$ at the origin $o$.

For any positive real number $a$, the Einstein-Kähler metric $g(a)$ of $M$ is given by

\[ g(a)(X, X) = 2a \text{tr}(x^* x + y^* y + \overline{z}z), \quad X = \eta(x, y, z) - \eta(x, y, z)^* \in \mathfrak{m}. \]

Relative to this metric, the scalar curvature $\tau$ of $M$ is given by

\[ \tau = \frac{2(2l - r + 1)}{a} \dim_{\mathbb{C}} M. \]

Y. Matsushima and M. Obata showed the following:

**Theorem 4.1** [9]. Let $M$ be an $n$-dimensional compact Einstein Kähler manifold of positive scalar curvature $\tau$. Then the first eigenvalue $\lambda_1(M)$ of the Laplacian satisfies that

\[ \lambda_1(M) \geq \frac{\tau}{n}. \]

The equality holds if and only if $M$ admits an one-parameter group of isometries (i.e., a non-trivial Killing vector field).

The natural inclusion $\mathfrak{sp}(l) \hookrightarrow SU(2l)$ defines an immersion $\varphi$ of $M$ into $\tilde{M} = G_r(\mathbb{C}^{2l}) = \tilde{G}/\tilde{K} = SU(2l)/S(U(r) \cdot U(2l - r))$ by

\[ \varphi(g \cdot K) = g \cdot \tilde{K}, \quad g \in G. \]
Under identification of $T_o\tilde{M}$ with $\tilde{\mathfrak{m}}$, the image of $X = \eta(x, y, z) - \eta(x, y, z)^* \in \mathfrak{m}$ is
\[
\varphi_*(X) = \begin{pmatrix}
0 & -x^* & -\bar{z} & -y^* \\
x & 0 & 0 & 0 \\
z & 0 & 0 & 0 \\
y & 0 & 0 & 0
\end{pmatrix},
\]
so that we have
\[
(4.2) \quad \tilde{g}_c(\varphi_*(X), \varphi_*(X)) = \frac{4}{c} \text{tr}(x^*x + y^*y + \bar{z}z).
\]
Therefore, Theorem 4.1, (4.1) and (4.2) imply the following.

**Theorem 4.2.** For the Kähler C-space $M = Sp(l)/U(r) \cdot Sp(l-r)$ of type $(C_l, \alpha_r)$ equipped with the Kähler metric $g(\frac{2}{c})$, $M$ is immersed to $G_r(\mathbb{C}^{2l})$ by the Kähler immersion $\varphi$. The complex dimension, and the first eigenvalue $\lambda_1(M)$ of the Laplacian are given by
\[
\dim_{\mathbb{C}} M = \frac{r(4l - 3r + 1)}{2}, \quad \lambda_1(M) = c(2l - r + 1).
\]
In particular, if $r = 2$, then $M = Sp(l)/U(2) \cdot Sp(l - 2)$ is a Kähler hypersurface of $G_2(\mathbb{C}^{2l})$, whose first eigenvalue $\lambda_1(M)$ of the Laplacian is given by
\[
\lambda_1(M) = c(2l - 1).
\]
For $z \in M_r(\mathbb{C})$, define an unit vector $\nu$ at the origin $o$ of $G_2(\mathbb{C}^{2l})$ by
\[
\nu(z) = \begin{pmatrix}
0 & 0 & -z^* & 0 \\
0 & 0 & 0 & 0 \\
z & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \in \tilde{\mathfrak{m}}, \quad \frac{4}{c} \text{tr} z^*z = 1.
\]
Then $\nu(z)$ is tangent to $M$ if and only if $z$ is symmetric.

The Kähler hypersurface $M = (C_l, \alpha_2)$ satisfies the following property relative to the quaternionic Kähler structure $\tilde{J}$ of $G_2(\mathbb{C}^{2l})$.

**Proposition 4.3.** The Kähler hypersurface $M = Sp(l)/U(2) \cdot Sp(l - 2)$ of $G_2(\mathbb{C}^{2l})$ satisfies
\[
(4.3) \quad \tilde{J} T^\perp M \subset TM \quad (\iff J_\xi \perp \tilde{J}_\xi \text{ for any } \xi \in T^\perp M),
\]
where $TM$ and $T^\perp M$ are the tangent bundle and the normal bundle of $M$, respectively.

**Proof.** Let $\nu_o$ be an unit normal vector of $M$ at $o$ defined by
\[
\nu_o = \nu(z_o), \quad z_o = \frac{1}{2} \sqrt{\frac{c}{2}} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix},
\]
so that the normal space $T_o^\perp M$ is given by

$$T_o^\perp M = \mathbb{R}\{\nu_o, J\nu_o = \nu(\sqrt{-1}z_o)\}.$$ 

Then we see

$$\mathcal{J}_o T_o^\perp M = \mathbb{R}\{J_i\nu_o, J_iJ\nu_o, i = 1, 2, 3\} = \mathbb{R}\{\nu(z_0\varepsilon_i), \nu(\sqrt{-1}z_0\varepsilon_i), i = 1, 2, 3\},$$

where $J_1$, $J_2$ and $J_3$ are a canonical basis of $\mathfrak{J}_o$ defined in the section 2. It is easy to check that $z_0\varepsilon_i$ and $\sqrt{-1}z_0\varepsilon_i$ are symmetric, so that we obtain

$$\mathcal{J}_o T_o^\perp M \subset T_o M.$$ 

Since the quaternionic Kähler structure $\mathcal{J}$ is $\tilde{G}$-invariant, and since the immersion $\varphi$ is $G$-equivariant, (4.3) holds at any point of $M$. □

If the ambient space is $G_2(\mathbb{C}^4)$, then the condition (4.3) determines a Kähler hypersurface as follows:

**Proposition 4.4.** Suppose that a Kähler hypersurface $M$ of $Q^4 = G_2(\mathbb{C}^4)$ satisfies the condition

$$\mathcal{J} T^\perp M \subset TM.$$ 

Then $M$ is totally geodesic. Moreover, if $M$ is compact, then $M$ is congruent to a complex quadric $Q^3 = Sp(2)/U(2)$.

*Proof.* Denote by $\tilde{\nabla}$ the Riemannian connection of $Q^4$, and denote by $\nabla$, $\sigma$, $A$ and $\nabla^\perp$, the Riemannian connection, the second fundamental form, the shape operator, and the normal connection of $M$, respectively. It is well-known that Gauss’ formula and Weingarten’s formula hold:

$$\begin{align*}
\tilde{\nabla}_X Y &= \nabla_X Y + \sigma(X, Y), \\
\tilde{\nabla}_X \xi &= -A_\xi X + \nabla^\perp_X \xi,
\end{align*}$$

for $X, Y \in TM$ and $\xi \in T^\perp M$. The metric condition implies

$$\tilde{g}_c(\sigma(X, Y), \xi) = \tilde{g}_c(A_\xi X, Y).$$

Relative to the complex structure $J$, $\sigma$ and $A$ satisfy

$$\sigma(X, JY) = J\sigma(X, Y), \quad A_\xi \circ J = -J \circ A_\xi = -A_J \xi.$$ 

For a local unit normal vector field $\xi$, we define local vector fields as follow: $e_i = J_i \xi$, $i = 1, 2, 3$, where $J_1$, $J_2$ and $J_3$ are a local canonical basis of $\mathcal{J}$. Then,
under the assumption of this proposition, \( \{e_1, e_2, e_3, Je_1, Je_2, Je_3, \xi, J\xi\} \) is a local orthonormal frame field of \( Q^4 \) such that \( \{e_1, e_2, e_3, Je_1, Je_2, Je_3\} \) is a tangent frame of \( M \). For \( X \in TM \), (4.4) implies
\[
\nabla_X e_i + \sigma(X, e_i) = (\tilde{\nabla}_X J_i) \xi + J_i(\tilde{\nabla}_X \xi) = (\tilde{\nabla}_X J_i) \xi - J_i A \xi X + J_i(\nabla_{\xi}^\perp \xi)
\]
Since \( J \) is parallel with respect to the connection \( \tilde{\nabla} \), we have \( \tilde{\nabla}_X J_i \in J \), so that the normal component of (4.7) is
\[
\sigma(X, e_i) = -\tilde{g}_c(J_i A \xi X, \xi) \xi - \tilde{g}_c(J_i A \xi X, J \xi) J \xi = g_c(A \xi X, e_i) \xi + g_c(A \xi X, Je_i) J \xi.
\]
From these two equations, we get
\[
\nabla_{\xi}^\perp \xi = 0.
\]
Instead of \( X \), applying to \( JX \), we have
\[
g_c(A \xi X, e_i) = g_c(-A \xi JX, Je_i) = 0.
\]
Therefore, we have \( A = 0 \), or \( \sigma = 0 \), so that \( M \) is totally geodesic. By B. Y. Chen and T. Nagano [3]'s results, if \( M \) is compact, \( M \) is congruent to a complex quadric \( Q^3 = Sp(2)/U(2) \).

\section{proof of main theorems.}

Let \( M \) be a compact connected Kähler hypersurface of \( G_r(\mathbb{C}^n) \) immersed by a immersion \( \varphi \). It is well-known that every \( HM(n, \mathbb{C}) \)-valued function \( F \) satisfies
\[
(5.1) \quad (\Delta F, \Delta F)_{L^2} - \lambda_1(\Delta F, F)_{L^2} \geq 0
\]
The equality holds if and only if \( F \) is a sum of eigenfunctions with respect to eigenvalues 0 and \( \lambda_1 \). It is equivalent to that there exists a constant vector \( C \in HM(n, \mathbb{C}) \) such that \( \Delta(F - C) = \lambda_1(F - C) \).

Denote by \( H \) the mean curvature vector of the isometric immersion \( \Phi = \Psi \circ \varphi \). Then, since \( M \) is minimal in \( G_r(\mathbb{C}^n) \), (2.9) implies
\[
(5.2) \quad 2(r(n - r) - 1)HA = 2r(n - r)\tilde{H}_A - \tilde{\sigma}_A(\xi, \xi) - \tilde{\sigma}_A(J \xi, J \xi)
= c(rI - nA) - \tilde{\sigma}_A(\xi, \xi) - \tilde{\sigma}_A(J \xi, J \xi),
\]
where \( A \) is a position vector of \( \Phi(M) \) in \( HM(n, \mathbb{C}) \), and \( \xi \) is a local unit normal vector field of \( \varphi \). Using (2.11) and (5.2), we get
\[
(5.3) \quad (HA, A) = -1.
\]
\( HM(n, \mathbb{C}) \)-valued function \( \Phi \) satisfies \( \Delta \Phi = -2(r(n - r) - 1)H \), so that (5.1) and (5.3) imply the following. The equality condition dues to T. Takahashi's theorem in [12].
Lemma 5.1.

\[(5.4) \quad 2(r(n-r) - 1) \int_M (H_A, H_A) dv_M - \lambda_1 vol(M) \geq 0.\]

The equality holds if and only if $\Phi$ is a minimal immersion of $M$ into some round sphere in $HM(n, \mathbb{C})$, more precisely, there exists some positive constant $R$ and some constant vector $C \in HM(n, \mathbb{C})$ such that $H_A$ satisfies

\[(5.5) \quad H_A = \frac{1}{R^2} (C - A).\]

Lemma 5.2. If the equality holds in (5.4), then $M$ is contained in a totally geodesic submanifold of $G_r(\mathbb{C}^n)$ which is product of Grassmann manifolds, more precisely, there exist integers $k_i, r_i, i=1, \cdots, m$ such that

\[
0 \leq r_i \leq k_i, \quad r_1 \geq r_2 \geq \cdots \geq r_m, \\
\sum_{i=1}^{m} r_i = r, \quad \sum_{i=1}^{m} k_i = n,
\]

\[(5.6) \quad M \subset G_{r_1}(\mathbb{C}^{k_1}) \times G_{r_2}(\mathbb{C}^{k_2}) \times \cdots \times G_{r_m}(\mathbb{C}^{k_m}) \subset G_r(\mathbb{C}^n).\]

Notice that $G_0(\mathbb{C}^{k_i}) = G_{k_i}(\mathbb{C}) = \{\text{one point}\}$.

**proof.** Assume that this equality holds in (5.4).

Since $M$ is minimal in $G_r(\mathbb{C}^n)$, $H$ is normal to $G_r(\mathbb{C}^n)$. Then, from (2.4) and (5.5), we get

\[(5.7) \quad CA = AC,\]

where $C$ is a constant vector in Lemma 5.1. Since $SU(n)$ acts on $G_r(\mathbb{C}^n)$ transitively, without loss of generalization, we can assume that $C$ is a diagonal matrix as follows:

\[
(5.8) \quad C = \begin{pmatrix} c_1 I_{k_1} & 0 \\ c_2 I_{k_2} & \cdots \\ \vdots & \ddots \\ 0 & \cdots & c_m I_{k_m} \end{pmatrix}, \quad k_i > 0, \quad c_i \neq c_j (i \neq j).
\]

Notice that

\[n = k_1 + k_2 + \cdots + k_m.\]

Define a linear subspace $L$ of $HM(n, \mathbb{C})$ by

\[
L = \left\{ Z \in HM(n, \mathbb{C}) \mid ZC = CZ \right\},
\]

so that

\[
L = \left\{ \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_m \end{pmatrix} \mid Z_i \in M_{k_i}(\mathbb{C}) \right\}.
\]
From (5.7), $M$ is contained in $G_r(\mathbb{C}^n) \cap L$.

For each integer $r_i$ with $0 \leq r_i \leq k_i$, $\sum_{i=1}^{m} r_i = r$, let's define connected subsets of $G_r(\mathbb{C}^n)$ by

$$W_{r_1, \ldots, r_m} = \left\{ \begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & A_m \end{pmatrix} | A_i \in M_{k_i}(\mathbb{C}), A_i^2 = A_i, \quad tr A_i = r_i \right\}.$$ 

So, $G_r(\mathbb{C}^n) \cap L$ is a disjoint union of all $W_{r_1, \ldots, r_m}$'s. Since $M$ is connected, $M$ is contained in suitable one of $W_{r_1, \ldots, r_m}$'s, saying $W_{r_1, \ldots, r_m}$. By the definition, we see

$$W_{r_1, \ldots, r_m} = G_{r_1}(\mathbb{C}^{k_1}) \times G_{r_2}(\mathbb{C}^{k_2}) \times \cdots \times G_{r_m}(\mathbb{C}^{k_m}).$$

Without loss of generalization, we can choose a diagonal matrix $C$ with respect to which the inequalities $r_1 \geq r_2 \geq \cdots \geq r_m$ hold. □

From (2.8), (2.10) and (5.2), we get

$$(5.9) \quad H_A = \frac{c}{2(r(n-r)-1)} \left\{ (rI-nA) - \frac{4}{c}(\Psi_* \xi)^2(I-2A) \right\}. $$

Using (2.2) and (2.3), we see

$$(5.10) \quad (H_A, H_A) = \frac{c}{2(r(n-r)-1)^2} \left\{ nr(n-r) - 2tr \frac{4}{c} (\Psi_* \xi)^2 \left(I + \frac{n-2r}{r} A \right) \\
+ tr \frac{16}{c^2} (\Psi_* \xi)^2(I-2A)(\Psi_* \xi)^2(I-2A) \right\}. $$

Since the immersion $\Psi$ is $\tilde{G}$-equivariant, for any $A \in \Phi(M)$, there exists a element $g_A \in \tilde{G}$ and a matrix $v_A \in M_{n-r, r}(\mathbb{C})$ satisfying $A_o = g_A A g_A^*$ and

$$(5.11) \quad \sqrt{\frac{c}{4}} \begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix} = g_A (\Psi_* \xi) g_A^*.$$ 

Since the inner product $(,)$ is $\tilde{G}$-equivariant and $\xi$ is unit, we have $tr v_A^* v_A = tr v_A v_A^* = 1$. After translating by $g_A$, together with (5.11), (5.10) implies

$$(5.12) \quad (H_A, H_A) = \frac{c}{2(r(n-r)-1)^2} \left\{ n(r(n-r) - 2) + 2tr (v_A^* v_A v_A^* v_A) \right\}.$$
Lemma 5.3. (a) For $v \in M_{n-r,r}(\mathbb{C})$ with $\text{tr} \, v^*v = 1$, the following inequality holds

\begin{equation}
\text{tr} \, v^*vv^*v \leq 1.
\end{equation}

(b) Moreover, next three conditions are equivalent to each other.

1. The equality holds in (5.13)
2. The hermitian $r$-matrix $v^*v$ is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix}$.
3. The hermitian $(n-r)$-matrix $vv^*$ is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}$.

(c) If the equality holds in (5.13), then there exists $R = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \in S(U(r) \cdot U(n-r))$ such that $v' = QvP^*$ satisfies $v'^*v' = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}$.

Proof. Lemma 5.3 follows from that both of hermitian matrices $v^*v$ and $vv^*$ are similar to diagonal matrices with non-negative eigenvalues.

Form (5.12) and Lemma 5.3, the following lemma is immediately obtained, which is used to prove Theorem A.

Lemma 5.4.

\begin{equation}
(H_A, H_A) \leq \frac{c}{2(r(n-r)-1)} \left\{ n - \frac{n-2}{r(n-r)-1} \right\}.
\end{equation}

The equality holds if and only if, for any $A \in \Phi(M)$, it is possible to choose $v_A$ satisfying

\begin{equation}
v_A^*v_A = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \quad \text{and} \quad v_Av_A^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}.
\end{equation}

proof of Theorem A. (5.4) and (5.14) imply

$$\lambda_1 \leq c \left( n - \frac{n-2}{r(n-r)-1} \right).$$

Let's assume that this equality holds. Then, the equality conditions of Lemmas 5.1 and 5.4 hold.

Assume $m = 1$. Then, (5.5) and (5.9) imply

$$\frac{1}{R^2} (c_1 I - A) = \frac{c}{2(r(n-r)-1)} \left\{ (rI - nA) - \frac{4}{c} (\Psi^*\xi)^2 (I - 2A) \right\}.$$
After translating by $g_{A}$, together with (5.11) and (5.15), we obtain

$$\frac{1}{R^{2}}(c_{1} - 1)I_{r} = \frac{c}{2(r(n-r) - 1)} \left\{ (r - n)I_{r} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \right\},$$

$$\frac{1}{R^{2}}c_{1}I_{n-r} = \frac{c}{2(r(n-r) - 1)} \left\{ rI_{n-r} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix} \right\}.$$

The first equation implies $r = 1$, and the second one implies $n - r = 1$. So, we have $n = 2$ and $r = 1$. This contradicts that $\Lambda f$ is a complex hypersurface.

Since $m \geq 2$, from Lemma 5.2, $M$ is contained in a proper totally geodesic submanifold of $G_{r}(\mathbb{C}^{n})$. On the other hand, $M$ is of complex codimension 1 in $G_{r}(\mathbb{C}^{n})$. Consequently, either $r = 1$ or $r = n - 1$ occurs, and $M$ is a totally geodesic complex hypersurface of a complex projective space $\mathbb{C}P^{n-1} \cong G_{1}(\mathbb{C}^{n}) \cong G_{n-1}(\mathbb{C})$. □

**Proof of Theorem B.** Let’s assume that $M$ is a compact connected Kähler hypersurface of $G_{2}(\mathbb{C}^{n})$ satisfying the condition $J \xi \perp \exists \xi$. Since both of the complex structure and the quaternionic Kähler structure are $\tilde{G}$-invariant, we obtain, at the origin $A_{o}$,

$$J \left( \begin{array}{c} 0 \\ v_{A}^{*} \\ 0 \end{array} \right) \perp J_{i} \left( \begin{array}{c} 0 \\ v_{A}^{*} \\ 0 \end{array} \right), \quad i = 1, 2, 3,$$

where $J_{1}$, $J_{2}$ and $J_{3}$ are a canonical basis of $\exists_{o}$ defined in the section 2. Set

$$v_{A} = (v_{A}', v_{A}''), \quad v_{A}', v_{A}'' \in M_{n-2,1}(\mathbb{C}) \cong \mathbb{C}^{n-2}.$$

Using (2.6) and (2.7), (5.16) implies that $|v_{A}'| = |v_{A}''|$ and $v_{A}' \perp v_{A}''$. Combing them with $tr v_{A}'^{*}v_{A} = 1$, we obtain $|v_{A}'| = |v_{A}''| = \frac{1}{\sqrt{2}}$, so that

$$v_{A}'^{*}v_{A} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Together with (5.17), (5.12) implies

$$(H_{A}, H_{A}) = \frac{c}{2(2n - 5)} \left\{ n - \frac{n-1}{2n-5} \right\}.$$

Therefore, form Lemma 5.1, we obtain

$$\lambda_{1} \leq c \left( n - \frac{n-1}{2n-5} \right).$$

Let’s assume that this equality holds. Then, the equality conditions of Lemma 5.1 holds.
Computing dimensions of manifolds in (5.6), we have

\[(5.18) \quad 2n - 5 \leq \sum_{i=1}^{m} r_i(k_i - r_i). \]

From \( \sum_{i=1}^{m} r_i = 2 \) and \( r_1 \geq r_2 \geq \cdots \geq r_m \), the following two cases occur:

Case I: \( r_1 = r_2 = 1, \quad r_3 = \cdots = r_m = 0 \),

Case II: \( r_1 = 2, \quad r_2 = \cdots = r_m = 0 \).

In Case I, (5.18) implies \( 2n - 5 \leq k_1 + k_2 - 2 \leq n - 2 \), so \( n \leq 3 \). This is contradiction.

Therefore, Case II occurs. Then, (5.18) implies \( 2n - 5 \leq 2(k_1 - 2) \), so that we have \( n = k_1, \quad m = 1, \quad k_2 = \cdots = k_m = 0 \). (5.19) and (5.17) imply

\[ \frac{1}{R^2} \left( c_1 I - A \right) = \frac{c}{2(2n-5)} \left\{ (2I - nA) - \frac{4}{c}(\Psi_\epsilon \epsilon)'(I - 2A) \right\}. \]

After translating by \( g_A \), together with (5.11) and (5.17), we obtain

\[ \frac{1}{R^2} (c_1 - 1) = \frac{c}{2(2n-5)} \left\{ 2 - n + \frac{1}{2} \right\}, \]

\[ \frac{1}{R^2} c_1 I_{n-2} = \frac{c}{2(2n-5)} \left\{ 2I_{n-2} - v_A v_A^* \right\}. \]

The second equation implies

\[(5.19) \quad v_A v_A^* = dI_{n-2}, \quad d = 2 - \frac{2(2n-5)}{c} \frac{c_1}{R^2}. \]

From (5.17), we have

\[ d v_A = dI_{n-2} v_A = (v_A v_A^*) v_A = v_A (v_A^* v_A) = \frac{1}{2} v_A, \]

so that \( d = \frac{1}{2} \). Consequently, taking traces of both sides of (5.19), we obtain \( n = 4 \).

Therefore, from Proposition 4.4, \( M \) is congruent to \( Q^3 \). \( \square \)

**REFERENCES**


