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<th>SPECTRAL GEOMETRY OF KAHLER HYPERSURFACES IN THE COMPLEX GRASSMANN MANIFOLD</th>
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<td>Author(s)</td>
<td>MIYATA, YOICHIRO</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1997), 995: 95-110</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1997-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/61215">http://hdl.handle.net/2433/61215</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
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<td>Textversion</td>
<td>publisher</td>
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Kyoto University
The spectral geometry of Kähler hypersurfaces in the complex Grassmann manifold

Yoichiro Miyata

§1. Introduction.

Let $M$ be a compact $C^\infty$-Riemannian manifold, $C^\infty(M)$ the space of all smooth functions on $M$, and $\Delta$ the Laplacian on $M$. Then $\Delta$ is a self-adjoint elliptic differential operator acting on $C^\infty(M)$, which has an infinite discrete sequence of eigenvalues: $\text{Spec}(M) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \uparrow \infty\}$. Let $V_k = V_k(M)$ be the eigenspace of $\Delta$ corresponding to the $k$-th eigenvalue $\lambda_k$. Then $V_k$ is finite-dimensional. We define an inner product $(\cdot, \cdot)_{L^2}$ on $C^\infty(M)$ by $(f, g)_{L^2} = \int_M fg \, dv_M$, where $dv_M$ denotes the volume element on $M$. Then $\sum_0^\infty V_t$ is dense in $C^\infty(M)$ and the decomposition is orthogonal with respect to the inner product $(\cdot, \cdot)_{L^2}$. Thus we have $C^\infty(M) = \sum_0^\infty V_t(M)$ (in $L^2$-sense). Since $M$ is compact, $V_0$ is the space of all constant functions which is 1-dimensional.

In this point of view, it is one of the simplest and the most interesting problems to estimate the first eigenvalue. In [10], A. Ros gave the following sharp upper bound for the first eigenvalue of Kähler submanifold of a complex projective space.

**Theorem 1.1.** Suppose that $M$ is a complex $m$-dimensional compact Kähler submanifold of the complex projective space $\mathbb{C}P^n$ of constant holomorphic sectional curvature $c$. Then the first eigenvalue $\lambda_1$ satisfies the following inequality:

$$\lambda_1 \leq c(m + 1)$$

The equality holds if and only if $M$ is congruent to the totally geodesic Kähler submanifold $\mathbb{C}P^m$ of $\mathbb{C}P^n$.

If $M$ is not totally geodesic, J-P. Bourguignon, P. Li and S. T. Yau in [1] gave the following more sharp estimate. (See also [7].)

**Theorem 1.2.** Suppose that $M$ is a complex $m$-dimensional compact Kähler submanifold of $\mathbb{C}P^n$, which is fully immersed and not totally geodesic. Then the first eigenvalue $\lambda_1$ satisfies the following inequality:

$$\lambda_1 \leq cm \frac{n + 1}{n}$$

It is unknown when the equality holds in this inequality.

Our purpose is to give the upper bound for the first eigenvalue of Kähler hypersurfaces of a complex Grassmann manifold.
Let denote by $G_r(\mathbb{C}^n)$ the complex Grassmann manifold of $r$-planes in $\mathbb{C}^n$, equipped with the Kähler metric of maximal holomorphic sectional curvature $c$. We obtain the following result which is a natural generalization of Theorem 1.1.

**Theorem A.** Suppose that $M$ is a compact connected Kähler hypersurface of $G_r(\mathbb{C}^n)$. Then the first eigenvalue $\lambda_1$ satisfies the following inequality:

$$\lambda_1 \leq c \left( n - \frac{n - 2}{r(n - r) - 1} \right)$$

The equality holds if and only if $r = 1, n$, and $M$ is congruent to the totally geodesic complex hypersurface $\mathbb{CP}^{n-2}$ of the complex projective space $\mathbb{CP}^{n-1}$.

The 2-plane Grassmann manifold $G_2(\mathbb{C}^n)$ admits the quaternionic Kähler structure $\mathfrak{J}$. For the normal bundle $T^\perp M$ of a Kähler hypersurface $M$ of $G_2(\mathbb{C}^n)$, $\mathfrak{J}T^\perp M$ is a vector bundle of real rank 6 over $M$ which is a subbundle of the tangent bundle of $G_2(\mathbb{C}^n)$. We consider a Kähler hypersurface $M$ of $G_2(\mathbb{C}^n)$ satisfying the property that $\mathfrak{J}T^\perp M$ is a subbundle of the tangent bundle $TM$ of $M$. In the section 4, we will introduce examples satisfying this property.

For a Kähler hypersurface of $G_2(\mathbb{C}^n)$ satisfying this property, we obtain the following upper bound of the first eigenvalue.

**Theorem B.** Suppose that $M$ is a compact connected Kähler hypersurface of $G_2(\mathbb{C}^n)$, $n \geq 4$. If $M$ satisfies the condition $\mathfrak{J}T^\perp M \subset TM$, then the following inequality holds:

$$\lambda_1 \leq c \left( n - \frac{n - 1}{2n - 5} \right)$$

The equality holds if and only if $n = 4$ and $M$ is congruent to the totally geodesic complex hypersurface $Q^3$ of the complex quadric $Q^4 = G_2(\mathbb{C}^4)$.

These two theorems are proved in the section 5. More detailed proofs of any our results are given in [8].

**Notations.** $M_{r,s}(\mathbb{C})$ denotes the set of all $r \times s$ matrices with entries in $\mathbb{C}$, and $M_r(\mathbb{C})$ stands for $M_{r,r}(\mathbb{C})$. $I_r$ and $O_r$ denote the identity $r$-matrix and the zero $r$-matrix.

§2. Preliminaries.

In this section, we discuss geometries of the complex $r$-plane Grassmann manifold and its first standard imbedding.

Let $M_r(\mathbb{C}^n)$ be the complex Stiefel manifold which is the set of all unitary $r$-systems of $\mathbb{C}^n$, i.e.,

$$M_r(\mathbb{C}^n) = \{ Z \in M_{n,r}(\mathbb{C}) | Z^*Z = I_r \} .$$

The complex $r$-plane Grassman manifold $G_r(\mathbb{C}^n)$ is defined by

$$G_r(\mathbb{C}^n) = M_r(\mathbb{C}^n)/U(r).$$
The origin \( o \) of \( G_r(\mathbb{C}^n) \) is defined by \( \pi(Z_0) \), where \( Z_0 = \begin{pmatrix} I_r \\ 0 \end{pmatrix} \) is a element of \( M_r(\mathbb{C}^n) \), and \( \pi: M_r(\mathbb{C}^n) \to G_r(\mathbb{C}^n) \) is the natural projection.

The left action of the unitary group \( \tilde{G} = SU(n) \) on \( G_r(\mathbb{C}^n) \) is transitive, and the isotropy subgroup at the origin \( o \) is

\[
\tilde{K} = S(U(r) \cdot U(n-r))
\]

so that \( G_r(\mathbb{C}^n) \) is identified with a homogeneous space \( \tilde{G}/\tilde{K} \).

Set \( \tilde{\mathfrak{g}} = \mathfrak{u}(n) \) and \( \tilde{\mathfrak{t}} = \mathbb{R} \oplus \epsilon \mathfrak{u}(r) \oplus \epsilon \mathfrak{u}(n-r) \), then \( \tilde{\mathfrak{g}} \) and \( \tilde{\mathfrak{t}} \) are the Lie algebra of \( \tilde{G} \) and \( \tilde{K} \), respectively. Define a linear subspace \( \tilde{\mathfrak{m}} \) of \( \tilde{\mathfrak{g}} \) by

\[
\tilde{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} \right\} \quad \xi \in M_{n-r}(\mathbb{C})
\]

then \( \tilde{\mathfrak{m}} \) is identified with the tangent space \( T_o(G_r(\mathbb{C}^n)) \). The \( \tilde{G} \)-invariant complex structure \( J \) of \( G_r(\mathbb{C}^n) \) and the \( \tilde{G} \)-invariant Kähler metric \( \tilde{g}_c \) of \( G_r(\mathbb{C}^n) \) of the maximal holomorphic sectional curvature \( c \) are given by

\[
J \begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1} \xi^* \\ \sqrt{-1} \xi & 0 \end{pmatrix},
\]

(2.1)

\[
\tilde{g}_c(X, Y) = -\frac{2}{c} \text{tr} XY, \quad X, Y \in \tilde{\mathfrak{m}}.
\]

In the case of \( r = 2 \), the complex 2-plane Grassmann manifold \( G_2(\mathbb{C}^n) \) admits another geometric structure named the quaternionic Kähler structure \( J \). \( J \) is a \( \tilde{G} \)-invariant subbundle of \( \text{End}(T(G_2(\mathbb{C}^n))) \) of rank 3, where \( \text{End}(T(G_2(\mathbb{C}^n))) \) is the \( \tilde{G} \)-invariant vector bundle of all linear endmorphisms of the tangent bundle \( T(G_2(\mathbb{C}^n)) \). Under the identification with \( T_o(G_r(\mathbb{C}^n)) \) and \( \tilde{\mathfrak{m}} \), the fiber \( J_o \) at the origin \( o \) is given by

\[
J_o = \left\{ J_\xi = \text{ad}(\xi) \mid \xi \in \tilde{\mathfrak{t}} \right\},
\]

where \( \tilde{\mathfrak{t}}_q \) is an ideal of \( \tilde{\mathfrak{t}} \) defined by

\[
\tilde{\mathfrak{t}}_q = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cong \mathfrak{su}(2).
\]
Choose a basis \( \{ \varepsilon_1, \varepsilon_2, \varepsilon_3 \} \) of \( \mathfrak{su}(2) \) satisfying \([\varepsilon_i, \varepsilon_{i+1}] = 2\varepsilon_{i+2}, \) (mod 3) . Set \( \tilde{\varepsilon}_i = \begin{pmatrix} \varepsilon_i \\ 0 \\ 0 \end{pmatrix} \) and \( J_i = J_{\varepsilon_i} \) for \( i = 1, 2, 3, \) then the basis \( \{ J_1, J_2, J_3 \} \) is a canonical basis of \( \mathfrak{z}_0, \) satisfying

\[
J_i^2 = -id_m \quad \text{for } i = 1, 2, 3,
\]

\[
J_1J_2 = -J_2J_1 = J_3, \quad J_2J_3 = -J_3J_2 = J_1, \quad J_3J_1 = -J_1J_3 = J_2,
\]

\[
\tilde{g}_{c_0}(J_iX, J_iY) = \tilde{g}_{c_0}(X, Y), \quad \text{for } X, Y \in \tilde{m} \text{ and } i = 1, 2, 3.
\]

There exists an element \( \tilde{\varepsilon}_C \) of the center of \( \mathfrak{k} \) such that \( J \) is given by \( J = ad(\tilde{\varepsilon}_C) \) on \( m. \) Therefore, \( J \) is commutative with \( \mathfrak{z}. \)

Let \( HM(n, \mathbb{C}) \) be the set of all Hermitian \((n, n)\)-matrices over \( \mathbb{C}, \) which can be identified with \( \mathbb{R}^{n^2}. \) For \( X, Y \in HM(n, \mathbb{C}), \) the natural inner product is given by

\[
(X, Y) = \frac{2}{c} trXY.
\]

\( GL(n, \mathbb{C}) \) acts on \( HM(n, \mathbb{C}) \) by \( X \mapsto BXB^*, \) \( B \in GL(n, \mathbb{C}), \) \( X \in HM(n, \mathbb{C}). \) Then the action of \( SU(n) \) leaves the inner product \((2.2)\) invariant.

The first standard imbedding \( \Psi \) of \( G_r(\mathbb{C}^n) \) is defined by

\[
\Psi(\pi(z)) = zZ^* \in HM(n, \mathbb{C}), \quad Z \in M_r(\mathbb{C}^n).
\]

\( \Psi \) is \( SU(n) \)-equivariant and the image \( N \) of \( G_r(\mathbb{C}^n) \) under \( \Psi \) is given as follows:

\[
N = \Psi(G_r(\mathbb{C}^n)) = \{ A \in HM(n, \mathbb{C}) \mid A^2 = A, \ trA = r \}.
\]

The tangent bundle \( TN \) and the normal bundle \( T^\perp N \) are given by

\[
T_{A_o}N = \{ X \in HM(n, \mathbb{C}) \midXA + AX = X \} \subset HM_0,
\]

\[
T_{A_o}^\perpN = \{ Z \in HM(n, \mathbb{C}) \midZA = ZX \}.
\]

In particular, at the origin \( A_o = \Psi(0) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \) we can obtain

\[
T_{A_o}N = \left\{ \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} \mid \xi \in M_{n-r,r}(\mathbb{C}) \right\},
\]

\[
T_{A_o}^\perpN = \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \mid Z_1 \in HM(r, \mathbb{C}), Z_2 \in HM(n-r, \mathbb{C}) \right\}.
\]

The complex structure \( J \) acts on \( T_{A_o}N \) as follows:

\[
J \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0 \end{pmatrix}.
\]
If $r = 2$, then the quaternionic Kähler structure $\mathfrak{J}$ acts on $T_{A_{o}}N$ as follows:

$$J_{\tilde{\epsilon}} \left( \begin{array}{cc} 0 & \xi^* \\
\xi & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & \epsilon \xi^* \\
-\xi \epsilon & 0 \end{array} \right), \quad \epsilon \in \text{su}(2).$$

Let $\tilde{\sigma}$ and $\tilde{H}$ denote the second fundamental form and the mean curvature vector of $\Psi$, respectively. Then, for $A \in N$ and $X, Y \in T_{A}N$, we can see

$$\tilde{\sigma}_{A}(X, Y) = (XY + YX)(I - 2A)$$

(2.8)

$$\tilde{H}_{A} = \frac{c}{2r(n - r)}(rI - nA)$$

(2.9)

and $\tilde{\sigma}$ satisfies the following:

$$\tilde{\sigma}_{A}(JX, JY) = \tilde{\sigma}_{A}(X, Y),$$

(2.10)

$$(\tilde{\sigma}_{A}(X, Y), A) = -(X, Y).$$

(2.11)

§3. Examples.

One of the most simple typical examples of submanifolds of $G_{r}(\mathbb{C}^{n})$ is a totally geodesic submanifold. B. Y. Chen and T. Nagano in [3, 4] determined maximal totally geodesic submanifolds of $G_{2}(\mathbb{C}^{n})$. For arbitrary $r$, I. Satake and S. Ihara in [11, 5] determined all (equivariant) holomorphic imbeddings of a symmetric domain into another symmetric domain. Taking a compact dual symmetric space if necessary, we obtain the complete list of maximal totally geodesic Kähler submanifolds of $G_{r}(\mathbb{C}^{n})$.

Since totally geodesic submanifolds of $G_{r}(\mathbb{C}^{n})$ are symmetric spaces, we can calculate the first eigenvalue of the Laplacian of $M$. (cf. [14])

**Theorem 3.1.** Let $M$ be a proper maximal totally geodesic Kähler submanifold of $G_{r}(\mathbb{C}^{n})$, and $\lambda_{1}$ the first eigenvalue of the Laplace-Beltrami operator with respect to the induced Kähler metric. Then, $M$ and $\lambda_{1}$ are one of the following (up to isomorphism).

1. $M_{1} = G_{r}(\mathbb{C}^{n-1}) \hookrightarrow G_{r}(\mathbb{C}^{n}), \quad 1 \leq r \leq n - 2, \quad \text{and} \quad \lambda_{1} = c(n - 1)$
2. $M_{2} = G_{r-1}(\mathbb{C}^{n-1}) \hookrightarrow G_{r}(\mathbb{C}^{n}), \quad 2 \leq r \leq n - 1, \quad \text{and} \quad \lambda_{1} = c(n - 1)$
3. $M_{3} = G_{r_{1}}(\mathbb{C}^{n_{1}}) \times G_{r_{2}}(\mathbb{C}^{n_{2}}) \hookrightarrow G_{r_{1} + r_{2}}(\mathbb{C}^{n_{1} + n_{2}}), \quad 1 \leq r_{i} \leq n_{i} - 1, \quad i = 1, 2,$
   \[ \text{and} \quad \lambda_{1} = c \min\{n_{1}, n_{2}\} \]
4. $M_{4} = M_{4,p} = Sp(p)/U(p) \hookrightarrow G_{p}(\mathbb{C}^{2p}), \quad p \geq 2, \quad \text{and} \quad \lambda_{1} = c(p + 1)$
5. $M_{5} = M_{5,p} = SO(2p)/U(p) \hookrightarrow G_{p}(\mathbb{C}^{2p}), \quad p \geq 4, \quad \text{and} \quad \lambda_{1} = c(p - 1)$
6. $M_{6,m} = \mathbb{C}P^{p} \hookrightarrow G_{r}(\mathbb{C}^{n})$: the complex projective space,
   \[ r = \left( \begin{array}{c} p \\
\din - 1 \end{array} \right), \quad n = \left( \begin{array}{c} p + 1 \\
m \end{array} \right), \quad 2 \leq m \leq p - 1, \quad \text{and} \quad \lambda_{1} = c(p + 1) \left( \begin{array}{c} p - 1 \\
m - 1 \end{array} \right)^{-1} \]
7. $M_{7} = Q^{3} \hookrightarrow Q^{4} = G_{2}(\mathbb{C}^{4})$: the complex quadric, \quad and \quad $\lambda_{1} = 3c$
8. $M_{8} = M_{8,2l} = Q^{2l} \hookrightarrow G_{r}(\mathbb{C}^{2r})$: the complex quadric, \quad $r = 2l - 1, \quad l \geq 3,$
   \[ \text{and} \quad \lambda_{1} = c\frac{2l}{2^{l-2}} \]
In above list, notice that $M_{4,2} = M_{7}$ and $M_{5,4} = M_{8,6}$.

Another one of the most simple typical examples of submanifolds of $G_r(\mathbb{C}^n)$ is a homogeneous Kähler hypersurface. K. Konno in [6] determined all Kähler C-spaces embedded as a hypersurface into a Kähler C-space with the second Betti number $b_2 = 1$.

**Theorem 3.2.** Let $M$ be a compact, simply connected homogeneous Kähler hypersurface of $G_r(\mathbb{C}^n)$, and $\lambda_1$ the first eigenvalue of the Laplace-Beltrami operator with respect to the induced Kähler metric. Then, $M$ and $\lambda_1$ are one of the following (up to isomorphism).

1. $M_9 = \mathbb{C}P^{n-2} \hookrightarrow \mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$ and $\lambda_1 = c(n-1)$
2. $M_{10} = Q^{n-2} \hookrightarrow \mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$ and $\lambda_1 = c(n-2)$
3. $M_7 = Q^3 \hookrightarrow Q^4 = G_2(\mathbb{C}^4)$ and $\lambda_1 = 3c$
4. $M_{11} = Sp(l)/U(2)Sp(l-2) \hookrightarrow G_1(\mathbb{C}^{2l})$: Kähler C-space of type $(C_l, \alpha_2)$, $l \geq 2$ and $\lambda_1 = c(2l-1)$

$M_9$ and $M_7$ are totally geodesic. $M_9$, $M_{10}$ and $M_7$ are symmetric spaces. If $l = 2$, then $M_{11}$ is congruent to $M_7$.

For each $l$ with $l > 2$, $M_{11}$ is not a symmetric space. Then, it is not easy to calculate the first eigenvalue $\lambda_1$ of $M_{11}$. We will calculate $\lambda_1$ of $M_{11}$ in the next section.

From these two theorems, we obtain the following proposition:

**Proposition 3.3.** Let $M$ be either a proper maximal totally geodesic Kähler submanifold of $G_r(\mathbb{C}^n)$ or a compact simply connected homogeneous Kähler hypersurface of $G_r(\mathbb{C}^n)$. Then, the first eigenvalue $\lambda_1$ of $M$ with respect to the induced Kähler metric satisfies the following inequality:

$$\lambda_1 \leq c(n-1).$$

Moreover, the equality holds if and only if $M$ is congruent to one of the follows:

$$M_1, \ M_2, \ M_{4,2} = M_7, \ M_9, \ M_{11}.$$  

§4. the homogeneous Kähler hypersurface $(C_l, \alpha_2)$.

In this section, we will consider the first eigenvalue of the Kähler C-space of type $(C_l, \alpha_r)$. For details, see [2] and [13].

The Kähler C-space of type $(C_l, \alpha_r)$ is a compact simply connected homogeneous Kähler manifold $M = G/K = Sp(l)/U(r) \cdot Sp(l-r)$, $1 \leq r \leq l$. Denote by $g$ and $\mathfrak{k}$ Lie algebras of $G$ and $K$, respectively, i.e.,

$$g = \mathfrak{sp}(l) = \left\{ \begin{pmatrix} A & -\overline{C} \\ C & A \end{pmatrix} \bigg| A, C \in M_l(\mathbb{C}), \ A^* = -A, \mathfrak{i}C = C \right\}.$$
\[ \mathfrak{t} = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A' & 0 & -C' \\ 0 & 0 & A & 0 \\ C' & 0 & 0 & A' \end{pmatrix} \right\} \begin{cases} A \in M_r(\mathbb{C}), \\ A', C' \in M_{l-r}(\mathbb{C}), \end{cases} \]

\[ A^* = -A, \quad A'^* = -A', \quad C' = C' \]

\[ = \mathfrak{u}(r) + \mathfrak{sp}(l-r). \]

\( \mathfrak{g} \) is a compact semisimple Lie algebra of type \( C_l \).

For \( x, y \in M_{l-r,r}(\mathbb{C}) \) and \( z \in M_r(\mathbb{C}) \) with \( ^t z = z \), define

\[ \eta(x, y, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ z & ty & 0 & -tx \\ y & 0 & 0 & 0 \end{pmatrix}. \]

Note that, if \( r = l \), then we ignore \( x \) and \( y \), and \( \eta(x, y, z) \) and \( \eta(0, 0, z) \) denote a matrix \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), \( z \in M_l(\mathbb{C}) \), \( ^t z = z \).

Let \( \mathfrak{m}, \mathfrak{m}^+ \) and \( \mathfrak{m}^- \) be subspaces of \( \mathfrak{g} \) defined by

\[ \mathfrak{m} = \{ \eta(x, y, z) - \eta(x, y, z)^* \}, \]

\[ \mathfrak{m}^+ = \{ \eta(x, y, z) \}, \]

\[ \mathfrak{m}^- = \{ \eta(x, y, z)^* \}, \]

so that \( \mathfrak{m}, \mathfrak{m}^+, \mathfrak{m}^- \) are \( K \)-invariant under the adjoint action, and \( \mathfrak{m} \) is identified with the tangent space \( T_0 M \) of \( M \) at the origin \( o = \{ K \} \). Moreover, the complexification \( \mathfrak{m}^\mathbb{C} \) of \( \mathfrak{m} \) is the direct sum \( \mathfrak{m}^\mathbb{C} = \mathfrak{m}^+ + \mathfrak{m}^- \), and \( \mathfrak{m}^\pm \) is the \( \pm \sqrt{-1} \)-eigenspace of the complex structure \( J \) of \( M \) at the origin \( o \).

For any positive real number \( a \), the Einstein-Kähler metric \( g(a) \) of \( M \) is given by

\[ g(a)(X, X) = 2a \text{tr}(x^* x + y^* y + \bar{z}z), \quad X = \eta(x, y, z) - \eta(x, y, z)^* \in \mathfrak{m}. \]

Relative to this metric, the scalar curvature \( \tau \) of \( M \) is given by

\[ \tau = \frac{2(2l - r + 1)}{a} \dim_{\mathbb{C}} M. \]

Y. Matsushima and M. Obata showed the following:

**Theorem 4.1 [9].** Let \( M \) be an \( n \)-dimensional compact Einstein Kähler manifold of positive scalar curvature \( \tau \). Then the first eigenvalue \( \lambda_1(M) \) of the Laplacian satisfies that

\[ \lambda_1(M) \geq \frac{\tau}{n}. \]

The equality holds if and only if \( M \) admits an one-parameter group of isometries (i.e., a non-trivial Killing vector field).

The natural inclusion \( \text{Sp}(l) \hookrightarrow \text{SU}(2l) \) defines an immersion \( \varphi \) of \( M \) into \( \tilde{M} = G_r(\mathbb{C}^{2l}) \cong \tilde{G}/\tilde{K} = \text{SU}(2l)/S(U(r) \cdot U(2l-r)) \) by

\[ \varphi(g \cdot K) = g \cdot \tilde{K}, \quad g \in G. \]
Under identification of $T_0\tilde{M}$ with $\tilde{m}$, the image of $X = \eta(x, y, z) - \eta(x, y, z)^* \in m$ is

$$\varphi_*(X) = \begin{pmatrix} 0 & -x^* & -\bar{z} & -y^* \\ x & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix},$$

so that we have

(4.2) \quad \tilde{g}_c(\varphi_*(X), \varphi_*(X)) = \frac{4}{c} \text{tr}(x^*x + y^*y + \bar{z}z).

Therefore, Theorem 4.1, (4.1) and (4.2) imply the following.

**Theorem 4.2.** For the Kähler C-space $M = \text{Sp}(l)/U(r) \cdot \text{Sp}(l-r)$ of type $(C_\iota, \alpha_r)$ equipped with the Kähler metric $g(\frac{2}{c})$, $M$ is immersed to $G_r(\mathbb{C}^{2l})$ by the Kähler immersion $\varphi$. The complex dimension, and the first eigenvalue $\lambda_1(M)$ of the Laplacian are given by

$$\dim_{\mathbb{C}} M = \frac{r(4l-3r+1)}{2}, \quad \lambda_1(M) = c(2l - r + 1).$$

In particular, if $r = 2$, then $M = \text{Sp}(l)/U(2) \cdot \text{Sp}(l-2)$ is a Kähler hypersurface of $G_2(\mathbb{C}^{2l})$, whose first eigenvalue $\lambda_1(M)$ of the Laplacian is given by

$$\lambda_1(M) = c(2l - 1).$$

For $z \in M_r(\mathbb{C})$, define an unit vector $\nu$ at the origin $o$ of $G_2(\mathbb{C}^{2l})$ by

$$\nu(z) = \begin{pmatrix} 0 & 0 & -z^* & 0 \\ 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \tilde{m}, \quad \frac{4}{c} \text{tr} z^*z = 1.$$

Then $\nu(z)$ is tangent to $M$ if and only if $z$ is symmetric.

The Kähler hypersurface $M = (C_\iota, \alpha_2)$ satisfies the following property relative to the quaternionic Kähler structure $\tilde{J}$ of $G_2(\mathbb{C}^{2l})$.

**Proposition 4.3.** The Kähler hypersurface $M = \text{Sp}(l)/U(2) \cdot \text{Sp}(l-2)$ of $G_2(\mathbb{C}^{2l})$ satisfies

(4.3) \quad \tilde{J} T^\perp M \subset TM \quad (\leftrightarrow \text{J} \xi \perp \text{J} \xi \text{ for any } \xi \in T^\perp M),

where $TM$ and $T^\perp M$ are the tangent bundle and the normal bundle of $M$, respectively.

**Proof.** Let $\nu_o$ be an unit normal vector of $M$ at $o$ defined by

$$\nu_o = \nu(z_o), \quad z_o = \frac{1}{2} \sqrt{\frac{c}{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
so that the normal space $T_o \perp M$ is given by

$$T_o \perp M = \mathbb{R}\{\nu_o, J\nu_o = \nu(\sqrt{-1}z_o)\}.$$ 

Then we see

$$\mathfrak{J}_o T_o \perp M = \mathbb{R}\{Ji\nu_o, J_iJ\nu_o, i = 1,2,3\} = \mathbb{R}\{\nu(Z_o\mathcal{E}i), \nu(\sqrt{-1}z_o\epsilon i), i = 1,2,3\},$$

where $J_1, J_2$ and $J_3$ are a canonical basis of $\mathfrak{J}_o$ defined in the section 2. It is easy to check that $z_o\epsilon_i$ and $\sqrt{-1}z_o\epsilon_i$ are symmetric, so that we obtain

$$\mathfrak{J}_o T_o \perp M \subset T_o M.$$ 

Since the quaternionic Kähler structure $\mathfrak{J}$ is $\tilde{G}$-invariant, and since the immersion $\varphi$ is $G$-equivariant, (4.3) holds at any point of $M$. \qed

If the ambient space is $G_2(\mathbb{C}^4)$, then the condition (4.3) determines a Kähler hypersurface as follows:

**Proposition 4.4.** Suppose that a Kähler hypersurface $M$ of $Q^4 = G_2(\mathbb{C}^4)$ satisfies the condition

$$\mathfrak{J} T \perp M \subset TM.$$ 

Then $M$ is totally geodesic. Moreover, if $M$ is compact, then $M$ is congruent to a complex quadric $Q^3 = Sp(2)/U(2)$.

**Proof.** Denote by $\tilde{\nabla}$ the Riemannian connection of $Q^4$, and denote by $\nabla, \sigma, A$ and $\nabla^\perp$, the Riemannian connection, the second fundamental form, the shape operator, and the normal connection of $M$, respectively. It is well-known that Gauss' formula and Weingarten's formula hold:

$$(4.4) \quad \tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad \tilde{\nabla}_X \xi = -A\xi X + \nabla^\perp_X \xi,$$

for $X, Y \in TM$ and $\xi \in T \perp M$. The metric condition implies

$$(4.5) \quad \tilde{g}_c(\sigma(X,Y), \xi) = \tilde{g}_c(A\xi X, Y).$$

Relative to the complex structure $J, \sigma$ and $A$ satisfy

$$(4.6) \quad \sigma(X, JY) = J\sigma(X,Y), \quad A\xi \circ J = -J \circ A\xi = -A J\xi.$$ 

For a local unit normal vector field $\xi$, we define local vector fields as follow: $e_i = J_i\xi, i = 1,2,3$, where $J_1, J_2$ and $J_3$ are a local canonical basis of $\mathfrak{J}$. Then,
under the assumption of this proposition, \{e_{1}, e_{2}, e_{3}, Je_{1}, Je_{2}, Je_{3}, \xi, J\xi\} is a local orthonormal frame field of \(Q^{4}\) such that \{e_{1}, e_{2}, e_{3}, Je_{1}, Je_{2}, Je_{3}\} is a tangent frame of \(M\). For \(X \in TM\), (4.4) implies
\[
\nabla_{X}e_{i} + \sigma(X, e_{i}) = \tilde{\nabla}xe_{i} = (\tilde{\nabla}XJ_{i})\xi + J_{i}(\tilde{\nabla}x\xi)
= (\tilde{\nabla}XJ_{i})\xi - J_{i}A\xi X + J_{i}(\nabla_{x}^{\perp}\xi)
\]
Since \(\mathfrak{J}\) is parallel with respect to the connection \(\tilde{\nabla}\), we have \(\tilde{\nabla}_{X}J_{i} \in \mathfrak{J}\), so that the normal component of (4.7) is
\[
\sigma(X, e_{i}) = -\tilde{g}_{c}(J_{i}A\epsilon X, \xi)\xi - \tilde{g}_{c}(J_{i}A\epsilon^{X}, J\xi)J\xi
= g_{c}(A\xi X, e_{i})\xi + g_{c}(A_{\xi}X, Je_{i})J\xi.
\]
From these two equations, we get (4.8)
\[
g_{c}(A_{\xi}X, Je_{i}) = 0.
\]
Instead of \(X\), applying to \(JX\), we have
\[
g_{c}(A_{\xi}X, e_{i}) = g_{c}(-A_{\xi}JX, Je_{i}) = 0.
\]
Therefore, we have \(A_{\xi} = 0\), or \(\sigma = 0\), so that \(M\) is totally geodesic. By B. Y. Chen and T. Nagano [3]'s results, if \(M\) is compact, \(M\) is congruent to a complex quadric \(Q^{3} = Sp(2)/U(2)\).

§5. proof of main theorems.

Let \(M\) be a compact connected Kähler hypersurface of \(G_{r}(\mathbb{C}^{n})\) immersed by a immersion \(\varphi\). It is well-known that every \(HM(n, \mathbb{C})\)-valued function \(F\) satisfies
\[
(\Delta F, \Delta F)_{L^{2}} - \lambda_{1}(\Delta F, F)_{L^{2}} \geq 0
\]
The equality holds if and only if \(F\) is a sum of eigenfunctions with respect to eigenvalues 0 and \(\lambda_{1}\). It is equivalent to that there exists a constant vector \(C \in HM(n, \mathbb{C})\) such that \(\Delta(F - C) = \lambda_{1}(F - C)\).

Denote by \(H\) the mean curvature vector of the isometric immersion \(\Phi = \Psi \circ \varphi\). Then, since \(M\) is minimal in \(G_{r}(\mathbb{C}^{n})\), (2.9) implies
\[
2(r(n-r)-1)H_{A} = 2r(n-r)\hat{H}_{A} - \hat{\sigma}_{A}(\xi, \xi) - \hat{\sigma}_{A}(J\xi, J\xi)
= c(rI - nA) - \hat{\sigma}_{A}(\xi, \xi) - \hat{\sigma}_{A}(J\xi, J\xi),
\]
where \(A\) is a position vector of \(\Phi(M)\) in \(HM(n, \mathbb{C})\), and \(\xi\) is a local unit normal vector field of \(\varphi\). Using (2.11) and (5.2), we get
\[
(H_{A}, A) = -1.
\]
\(HM(n, \mathbb{C})\)-valued function \(\Phi\) satisfies \(\Delta \Phi = -2(r(n-r)-1)H\), so that (5.1) and (5.3) imply the following. The equality condition dues to T. Takahashi's theorem in [12].
Lemma 5.1.

\begin{equation}
\frac{2(r(n-r)-1)}{1} \int_{M} (H_{A}, H_{A}) dv_{M} - \lambda_{1} \text{vol}(M) \geq 0.
\end{equation}

The equality holds if and only if $\Phi$ is a minimal immersion of $M$ into some round sphere in $HM(n, \mathbb{C})$, more precisely, there exists some positive constant $R$ and some constant vector $C \in HM(n, \mathbb{C})$ such that $H_{A}$ satisfies

\begin{equation}
H_{A} = \frac{1}{R^{2}} (C - A).
\end{equation}

Lemma 5.2. If the equality holds in (5.4), then $M$ is contained in a totally geodesic submanifold of $G_{r}(\mathbb{C}^{n})$ which is product of Grassmann manifolds, more precisely, there exist integers $k_{i}, r_{i}, i=1, \ldots , m$ such that

\begin{equation}
0 \leq r_{i} \leq k_{i}, \quad r_{1} \geq r_{2} \geq \cdots \geq r_{m},
\end{equation}

\begin{equation}
\sum_{i=1}^{m} r_{i} = r, \quad \sum_{i=1}^{m} k_{i} = n,
\end{equation}

\begin{equation}
M \subset G_{r_{1}}(\mathbb{C}^{k_{1}}) \times G_{r_{2}}(\mathbb{C}^{k_{2}}) \times \cdots \times G_{r_{m}}(\mathbb{C}^{k_{m}}) \subset G_{r}(\mathbb{C}^{n}).
\end{equation}

Notice that $G_{0}(\mathbb{C}^{k_{i}}) = G_{k_{i}}(\mathbb{C}^{k_{i}}) = \{\text{one point}\}$.

proof. Assume that this equality holds in (5.4).

Since $M$ is minimal in $G_{r}(\mathbb{C}^{n})$, $H$ is normal to $G_{r}(\mathbb{C}^{n})$. Then, from (2.4) and (5.5), we get

\begin{equation}
CA = AC,
\end{equation}

where $C$ is a constant vector in Lemma 5.1. Since $SU(n)$ acts on $G_{r}(\mathbb{C}^{n})$ transitively, without loss of generalization, we can assume that $C$ is a diagonal matrix as follows:

\begin{equation}
C = \begin{pmatrix}
c_{1}I_{k_{1}} & & 0 \\
c_{2}I_{k_{2}} & \ddots & \vdots \\
0 & \cdots & c_{m}I_{k_{m}}
\end{pmatrix}, \quad k_{i} > 0, \quad c_{i} \neq c_{j} (i \neq j).
\end{equation}

Notice that

\begin{equation}
n = k_{1} + k_{2} + \cdots + k_{m}.
\end{equation}

Define a linear subspace $L$ of $HM(n, \mathbb{C})$ by $L = \{ Z \in HM(n, \mathbb{C}) \mid ZC = CZ \}$, so that

\begin{equation}
L = \left\{ \begin{pmatrix}
Z_{1} & & 0 \\
& \ddots & \\
0 & \cdots & Z_{m}
\end{pmatrix} \mid Z_{i} \in M_{k_{i}}(\mathbb{C}) \right\}.
\end{equation}
From (5.7), $M$ is contained in $G_r(\mathbb{C}^n) \cap L$.

For each integer $r_i$ with $0 \leq r_i \leq k_i$, $\sum_{i=1}^{m} r_i = r$, let's define connected subsets of $G_r(\mathbb{C}^n)$ by

$$W_{r_1, \ldots, r_m} = \left\{ \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix} \mid A_i \in M_{k_i}(\mathbb{C}), A_i^2 = A_i, \text{ tr } A_i = r_i \right\}$$

So, $G_r(\mathbb{C}^n) \cap L$ is a disjoint union of all $W_{r_1, \ldots, r_m}$'s. Since $M$ is connected, $M$ is contained in suitable one of $W_{r_1, \ldots, r_m}$'s, saying $W_{r_1, \ldots, r_m}$. By the definition, we see

$$W_{r_1, \ldots, r_m} = G_{r_1}(\mathbb{C}^{k_1}) \times G_{r_2}(\mathbb{C}^{k_2}) \times \cdots \times G_{r_m}(\mathbb{C}^{k_m}).$$

Without loss of generalization, we can choose a diagonal matrix $C$ with respect to which the inequalities $r_1 \geq r_2 \geq \cdots \geq r_m$ hold. □

From (2.8), (2.10) and (5.2), we get

$$H_A = \frac{c}{2(r(n-r)-1)} \left\{ (rI-nA) - \frac{4}{c} (\Psi_\ast \xi)^2 (I-2A) \right\}.$$ (5.9)

Using (2.2) and (2.3), we see

$$(H_A, H_A) = \frac{c}{2(r(n-r)-1)^2} \left\{ nr(n-r) - 2 \text{ tr} \frac{4}{c} (\Psi_\ast \xi)^2 \left( I + \frac{n-2r}{r} A \right) \\
+ \text{ tr} \frac{16}{c^2} (\Psi_\ast \xi)^2 (I-2A)(\Psi_\ast \xi)^2 (I-2A) \right\}. $$ (5.10)

Since the immersion $\Psi$ is $\tilde{G}$-equivariant, for any $A \in \Phi(M)$, there exists a element $g_A \in \tilde{G}$ and a matrix $v_A \in M_{n-r,r}(\mathbb{C})$ satisfying $A_c = g_A A g_A^\ast$ and

$$\sqrt{\frac{c}{4}} \begin{pmatrix} 0 & v_A^\ast \\ v_A & 0 \end{pmatrix} = g_A (\Psi_\ast \xi) g_A^\ast. $$ (5.11)

Since the inner product $(,)$ is $\tilde{G}$-equivariant and $\xi$ is unit, we have $\text{ tr } v_A^\ast v_A = \text{ tr } v_A v_A^\ast = 1$. After translating by $g_A$, together with (5.11), (5.10) implies

$$(H_A, H_A) = \frac{c}{2(r(n-r)-1)^2} \{ n(r(n-r) - 2) + 2 \text{ tr } (v_A^\ast v_A v_A^\ast v_A) \}. $$ (5.12)
Lemma 5.3. (a) For $v \in M_{n-r,r}(\mathbb{C})$ with $tr v^*v = 1$, the following inequality holds

\[ tr v^*vv^*v \leq 1. \]

(b) Moreover, next three conditions are equivalent to each other.

1. The equality holds in (5.13)
2. The hermitian $r$-matrix $v^*v$ is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix}$.
3. The hermitian $(n-r)$-matrix $vv^*$ is similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}$.

(c) If the equality holds in (5.13), then there exists $R = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \in S(U(r) \cdot U(n-r))$ such that $v' = QvP^*$ satisfies

\[ v'^*v' = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \text{ and } v'v'^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}. \]

Proof. Lemma 5.3 follows from that both of hermitian matrices $v^*v$ and $vv^*$ are similar to diagonal matrices with non-negative eigenvalues.

Form (5.12) and Lemma 5.3, the following lemma is immediately obtained, which is used to prove Theorem A.

Lemma 5.4.

\[ (H_A, H_A) \leq \frac{c}{2(r(n-r)-1)} \left\{ n - \frac{n-2}{r(n-r)-1} \right\}. \]

The equality holds if and only if, for any $A \in \Phi(M)$, it is possible to choose $v_A$ satisfying

\[ v_A^*v_A = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \text{ and } v_Av_A^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}. \]

Proof of Theorem A. (5.4) and (5.14) imply

\[ \lambda_1 \leq c \left( n - \frac{n-2}{r(n-r)-1} \right). \]

Let's assume that this equality holds. Then, the equality conditions of Lemmas 5.1 and 5.4 hold.

Assume $m = 1$. Then, (5.5) and (5.9) imply

\[ \frac{1}{R^2} (c_1 I - A) = \frac{c}{2(r(n-r)-1)} \left\{ (rI - nA) - \frac{4}{c}(\Psi^*\xi)^2(I - 2A) \right\}. \]
After translating by \( g_A \), together with (5.11) and (5.15), we obtain

\[
\frac{1}{R^2} (c_1 - 1) I_r = \frac{c}{2(r(n-r) - 1)} \left\{ (r - n) I_r + \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \right\},
\]

\[
\frac{1}{R^2} c_1 I_{n-r} = \frac{c}{2(r(n-r) - 1)} \left\{ rI_{n-r} - \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix} \right\}.
\]

The first equation implies \( r = 1 \), and the second one implies \( n - r = 1 \). So, we have \( n = 2 \) and \( r = 1 \). This contradicts that \( \Lambda f \) is a complex hypersurface.

Since \( m \geq 2 \), from Lemma 5.2, \( M \) is contained in a proper totally geodesic submanifold of \( G_r(\mathbb{C}^n) \). On the other hand, \( M \) is of complex codimension 1 in \( G_r(\mathbb{C}^n) \). Consequently, either \( r = 1 \) or \( r = n - 1 \) occurs, and \( M \) is a totally geodesic complex hypersurface of a complex projective space \( \mathbb{C}P^{n-1} \cong G_1(\mathbb{C}^n) \cong G_{n-1}(\mathbb{C}) \). □

Proof of Theorem B. Let’s assume that \( M \) is a compact connected Kähler hypersurface of \( G_2(\mathbb{C}^n) \) satisfying the condition \( J \xi \perp \mathcal{J} \xi \). Since both of the complex structure and the quaternionic Kähler structure are \( \tilde{G} \)-invariant, we obtain, at the origin \( A_o \),

\[
J \begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix} \perp J_i \begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix}, \quad i = 1, 2, 3,
\]

where \( J_1, J_2 \) and \( J_3 \) are a canonical basis of \( \mathfrak{J}_o \) defined in the section 2. Set

\[
v_A = \begin{pmatrix} v_A' \\ v_A'' \end{pmatrix}, \quad v_A', v_A'' \in M_{n-2,1}(\mathbb{C}) \cong \mathbb{C}^{n-2}.
\]

Using (2.6) and (2.7), (5.16) implies that \( |v_A'| = |v_A''| \) and \( v_A' \perp v_A'' \). Combing them with \( tr v_A^* v_A = 1 \), we obtain \( |v_A'| = |v_A''| = \frac{1}{\sqrt{2}} \), so that

\[
v_A^* v_A = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Together with (5.17), (5.12) implies

\[
(H_A, H_A) = \frac{c}{2(2n - 5)} \left\{ n - \frac{n-1}{2n - 5} \right\}.
\]

Therefore, form Lemma 5.1, we obtain

\[
\lambda_1 \leq c \left( n - \frac{n-1}{2n - 5} \right).
\]

Let’s assume that this equality holds. Then, the equality conditions of Lemma 5.1 holds.
Computing dimensions of manifolds in (5.6), we have

\begin{equation}
2n - 5 \leq \sum_{i=1}^{m} r_i(k_i - r_i).
\end{equation}

From \( \sum_{i=1}^{m} r_i = 2 \) and \( r_1 \geq r_2 \geq \cdots \geq r_m \), the following two cases occur:

**Case I:** \( r_1 = r_2 = 1, \ r_3 = \cdots = r_m = 0 \),

**Case II:** \( r_1 = 2, \ r_2 = \cdots = r_m = 0 \).

In Case I, (5.18) implies \( 2n - 5 \leq k_1 + k_2 - 2 \leq n - 2 \), so \( n \leq 3 \). This is contradiction.

Therefore, Case II occurs. Then, (5.18) implies \( 2n - 5 \leq 2(k_1 - 2) \), so that we have \( n = k_1, \ m = 1, \ k_2 = \cdots = k_m = 0 \). (5.5) and (5.9) imply

\[
\frac{1}{R^2} (c_1 I - A) = \frac{c}{2(2n-5)} \left\{ (2I - nA) - \frac{4}{c} (\Psi_\ast \xi)^2 (I - 2A) \right\}.
\]

After translating by \( g_A \), together with (5.11) and (5.17), we obtain

\[
\frac{1}{R^2} (c_1 I - 1) = \frac{c}{2(2n-5)} \left\{ 2 - n + \frac{1}{2} \right\},
\]

\[
\frac{1}{R^2} c_1 I_n - 2 = \frac{c}{2(2n-5)} \left\{ 2I_n - 2 - v_A v_A^\ast \right\}.
\]

The second equation implies

\begin{equation}
v_A v_A^\ast = d I_{n-2}, \quad d = 2 - \frac{2(2n-5)}{c} \frac{c_1}{R^2}.
\end{equation}

From (5.17), we have

\[
d v_A = d I_{n-2} v_A = (v_A v_A^\ast) v_A = v_A (v_A^\ast v_A) = \frac{1}{2} v_A,
\]

so that \( d = \frac{1}{2} \). Consequently, taking traces of both sides of (5.19), we obtain \( n = 4 \).

Therefore, from Proposition 4.4, \( M \) is congruent to \( Q^3 \). \( \square \)

**REFERENCES**


