Surfaces of Finite Total Curvature

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Introduction

Let $M$ be a connected, oriented, noncompact, complete surface in the 3-dimensional Euclidean space $E^3$. We denote the second fundamental form of $M$ by $B$, the mean curvature by $H$ and Gaussian curvature by $K$.

In this paper, we study the geometry of a surface $M$ which satisfies $\int_M |B| \, dM < \infty$, where $dM$ is the area element of $M$. For surfaces which satisfy $\int_M |B|^2 \, dM < \infty$, rather than $\int_M |B| \, dM < \infty$, extensive studies have been made. Since $|B|^2 = 4H^2 - 2K \geq 2|K|$, the condition $\int_M |B|^2 \, dM < \infty$ implies $\int_M |K| \, dM < \infty$. A theorem by Huber ([H]) says that an abstract noncompact complete 2-dimensional Riemannian manifold satisfying $\int_M |K| \, dM < \infty$ is conformally equivalent to a compact Riemann surface $\overline{M}$ punctured at a finite number of points $p_1, \ldots, p_k$. If $M$ is a minimal surface with $\int_M |B|^2 \, dM < \infty$ (or equivalently, $|\int_M K \, dM| < \infty$), Osserman showed that the Gauss map $G : M \to S^2$ is continuously extended to a map from $\overline{M}$ to $S^2$ ([O]). White ([W]) extended Osserman's theorem to surfaces with $\int_M |B|^2 \, dM < \infty$ and $K \leq 0$. The continuous extendability of the Gauss map cannot be derived without the condition $K \leq 0$, but Müller–Šverák ([M–S]) showed that if $M$ satisfies $\int_M |B|^2 \, dM < \infty$, then $M$ is properly immersed.

Let $M$ be a connected, oriented, noncompact, complete, properly immersed surface which satisfies a pointwise condition $|B(x)| \leq C|x|^{-1-\varepsilon}$ for $x$ in $M$, where $|x|$ is the Euclidean distance from the origin and $C$ and $\varepsilon$ are positive constants. In [E], the author showed that for $M$ satisfying this condition we have $\int_M |B|^2 \, dM < \infty$ and the Gauss map is continuously extended to a map from $\overline{M}$. Basic tools to study those surfaces are given by Kasue and Sugahara in [K] and [K–S]. In this paper we will study a surface $M$ which satisfies a stronger condition $|B(x)| \leq C|x|^{-2-\varepsilon}$ and show that we have $\int_M |B| \, dM < \infty$ (Theorem 1). We will also show that such a surface has a property that each end has an asymptotic plane (Theorem 2).
In general, of course, the condition $\int_M |B| dM < \infty$ does not give any pointwise estimate for $|B|$. Moreover, since $\int_M |B| dM < \infty$ does not necessarily imply $\int_M |K| dM < \infty$, Huber’s theorem cannot be applied for these surfaces and the geometry or the topology of $M$ near infinity can be very complicated in general. But if $M$ is a surface of revolution, we can say something about the geometry of the ends. In Theorem 3, we will show that if $M$ is a surface of revolution and satisfies $\int_M |B| dM < \infty$, then each end of $M$ has an asymptotic plane. We note that even when $M$ is a surface of revolution, $\int_M |B| dM < \infty$ does not imply that $|B|$ is uniformly bounded.

1. Surfaces whose second fundamental forms decay uniformly

Let $M$ be a connected, oriented, noncompact, complete, properly immersed surface in $E^3$. For $x$ in $M$, $|x|$ will denote the Euclidean distance to $x$ from the origin.

**Theorem 1.** If there exist positive constants $C$ and $\epsilon$ such that $|B(x)| \leq C|x|^{-2-\epsilon}$ for all $x$ in $M$, then we have $\int_M |B| dM < \infty$.

**Proof.** Since $M$ is properly immersed and the second fundamental form satisfies $|B(x)| \leq C|x|^{-2-\epsilon}$, $M(t) = \{x \in M : |x| \geq t\}$ is a union of a finite number of surfaces $M_1(t), \ldots, M_q(t)$, and for $\lambda = 1, \ldots, q$ and sufficiently large $t$, $M_\lambda(t)$ is diffeomorphic to $\partial M_\lambda(t) \times [t, \infty)$ ([K] Lemma 2). $\partial M_\lambda(t)$ is a closed curve whose length is less than $C_1 t$, where $C_1$ is a constant which does not depend on $t$ ([K–S] Lemma 6). Let $r : M \rightarrow \mathbb{R}$ be a function on $M$ which is defined by $r(x) = |x|$ for $x$ in $M$. Then there exists a positive constant $C_2$ such that $|\nabla r| \geq C_2$ if $r$ is sufficiently large ([K] Lemma 2). Hence, denoting the line element of $\partial M_\lambda(t)$ by $ds$, we have $dM \leq C_3 dr ds$ for some positive constant $C_3$. Now, if $R$ is sufficiently large, we have

$$\int_{M_\lambda(R)} |B| dM \leq C_3 \int_R^\infty \left( \int_{\partial M_\lambda(r)} |B| ds \right) dr \leq C_3 \int_R^\infty (Cr^{-2-\epsilon} \text{Length}(\partial M_\lambda(r))) dr \leq C_3 \int_R^\infty CC_1 r^{-1-\epsilon} dr < \infty.$$ 

This proves the theorem.
Theorem 2. If there exist positive constants $C$ and $\varepsilon$ such that $|B(x)| \leq C|x|^{-2-\varepsilon}$ for all $x$ in $M$, then each end of $M$ has an asymptotic plane, i.e., for each $\lambda = 1, \cdots, q$ there exists a plane $P_\lambda$ such that \[ \sup_{x \in M_\lambda(t)} \inf_{y \in P_\lambda} |x - y| \] tends to 0 as $t \to \infty$.

Proof. By Lemma 1.4 in [E], our assumption implies that the normal component of the position vector on each end of $M$ tends to a constant vector. To make it more precise, let $x$ be a point in $M_\lambda(t)$ and $N = N(x)$ be a unit normal vector of $M$ at $x$. Then, as $|x| \to \infty$, $N(x)$ converges to a constant unit vector $E_\lambda$ and $\langle x, N(x) \rangle$ converges to a constant $a_\lambda$. Let $P_\lambda$ be the plane defined by \( \{ y \in E^3 : \langle y, E_\lambda \rangle = a_\lambda \} \). First we note that \( \inf_{y \in P_\lambda} |x - y| = \langle x, E_\lambda \rangle - a_\lambda \) for $x$ in $M_\lambda$. By moving the origin of $E^3$ if necessary, we may assume that $a_\lambda \neq 0$. Since

\[
\lim_{|x| \to \infty} \langle x, a_\lambda E_\lambda \rangle = \lim_{|x| \to \infty} \langle x, \langle x, N \rangle N \rangle \\
= \lim_{|x| \to \infty} \langle x, N \rangle^2 \\
= a_\lambda^2,
\]

we see that $\langle x, E_\lambda \rangle - a_\lambda$ tends to 0 as $|x| \to \infty$. This shows that each end of $M$ has an asymptotic plane.

2. Surfaces of revolution

In this section, $M$ will be a surface of revolution in $E^3$ which is defined by $M = \{(x, y, z) = (r \cos \theta, r \sin \theta, f(r)) : r_0 \leq r < \infty, 0 \leq \theta < 2\pi \}$. The norm of the second fundamental form and the area element of $M$ are given as

\[
|B| = r^{-1}(1 + f'^2)^{-3/2}(r^2 f'^2 + f'^2(1 + f'^2)^2)^{1/2} \\
dM = r(1 + f'^2)^{1/2} dr d\theta.
\]

Theorem 3. Let $M$ be a surface of revolution given above. If $\int_M |B| dM < \infty$, then $\lim_{r \to \infty} f(r) = C$ for some constant $C$, i.e., each end of $M$ has an asymptotic plane.

Proof. Since

\[
\int_M |B| dM = \int_{r_0}^\infty dr \int_0^{2\pi} \frac{(r^2 f'^2 + f'^2(1 + f'^2)^2)^{1/2}}{1 + f'^2} d\theta
\]
$= 2\pi \int_{r_0}^{\infty} \frac{r^2 f''/2 + f'^2 (1 + f'^2)^2 / 2}{1 + f'^2} \, dr$
\[ \geq 2\pi \int_{r_0}^{\infty} |f'| \, dr, \]

the condition $\int_M |B| \, dM < \infty$ implies that $\lim_{r \to \infty} f(r) = C$ for some constant $C$.

The following example shows that $|B|$ may not be bounded on a surface of revolution with $\int_M |B| \, dM < \infty$.

**Example.** Let $f(r)$ be a function which has the following property:

$$f''(r) = \begin{cases} 
  n & \text{for } n + \frac{1}{n^4} \leq r \leq n + \frac{2}{n^4} \\
  0 & \text{for } n + \frac{3}{n^4} \leq r \leq n + 1 \\
  < n & \text{for } n + \frac{2}{n^4} < r < n + \frac{3}{n^4} \\
  \geq 0 & \text{for all } r
\end{cases}$$

Then for $n \leq r < n + 1$, we have

$$\sum_{k=1}^{n-1} \frac{1}{k^3} < f'(r) - f'(0) < \sum_{k=1}^{n} \frac{2}{k^3}.$$  

Hence, by a suitable choice for $f'(0)$, we can have $f'(r) = O(r^{-2})$. If $M$ is a surface of revolution obtained from $f(r)$, we have

$$\int_M |B| \, dM = 2\pi \int_{0}^{\infty} \frac{r^2 f''/2 + f'^2 (1 + f'^2)^2 / 2}{1 + f'^2} \, dr \leq 2\pi \int_{0}^{\infty} r|f''| \, dr + 2\pi \int_{0}^{\infty} |f'| \, dr < \infty.$$  

But $|B|$ is not bounded, since $|B| \approx n$ if $n + \frac{1}{n^4} \leq r \leq n + \frac{2}{n^4}$.

**References**


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