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TWO DIMENSIONAL CONFORMALLY INVARIANT GEOMETRIC VARIATIONAL PROBLEMS

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1. Prescribed mean curvature functional

In this article, we summerize some results on the existence of an extremal for some two dimensional conformally invariant functional which is closely related to surfaces of constant mean curvature. We interpret the results from the differential geometric point of view and formulate a variational problem for the further investigation. We start with the formulation.

Let Σ be a closed Riemann surface with positive genus g and N a closed hyperbolic 3-manifold, i.e. N is a quotient space $\mathbb{H}^3/\Gamma$ of hyperbolic 3-space $\mathbb{H}^3$ by torsion free cocompact Kleinnian group Γ.

We fix a free homotopy class γ of maps of Σ to N and fix a map $u_0 \in \gamma$. For any map $u \in \gamma$, we define a volume functional $V(\cdot, u_0)$ as follows.

$$V(u, u_0) := \iint_{\Sigma\times[0,1]} f^* vol_N \tag{1.1}$$

where f is a homotopy between u and u_0 and vol_N denotes the volume form of N. $V(u, u_0)$ does not depend on the choice of homotopy of f.

For $\Omega \subset \Sigma$ and $V \in \mathbb{R}$, set

$$D(u,\Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2 dV$$
$$\mathcal{V}_\gamma(V) := \{u \in \gamma : V(u, u_0) = V\}.$$

First, we work in the prescribed mean curvature formulation. For $u : \Sigma \to N$, we define

$$I_H(u,\Sigma) := D(u,\Sigma) + 2HV(u,u_0).$$

We consider the following minimizing problem.

(i) For a given $H \in \mathbb{R}$, find a minimizer $(u, \Sigma) \in \gamma \times \mathcal{M}_g$ of the functional $I_H(u, \Sigma)$ where $\mathcal{M}_g$ denotes the moduli space of closed surfaces with genus g.

Any solution for problem (i) satisfies equations (1.2)-(1.3) below for a *prescribed* H.

$$trace(\nabla du) = 2H\nabla_1 u \times \nabla_2 u, \tag{1.2}$$

$$|\nabla_1 u|^2 - |\nabla_2 u|^2 = \langle \nabla_1 u, \nabla_2 u\rangle = 0 \tag{1.3}$$

where all derivatives are taken with respect to the hyperbolic metric on Σ induced by the conformal structure, ∇_i denotes the derivative with respect to a local orthonormal frame of and cross product $\times$ denotes the tensor defined by

$$vol_p(X, Y, Z) = \langle X, Y \times Z\rangle.$$

Hence, the solution is a conformal (branched) parametrization of a surface of constant mean curvature H. The first existence result of an extremal goes as follows.

Theorem 1.1 (Toda). *Suppose γ induces an injective action on the fundamental groups. If $|H| < 1$, there exists a minimizer for problem (i).*

We shall briefly sketch the procedure of the proof.

(1) We consider the minimizing problem for functional I_H for the source surface with a fixed conformal structure. With a help of Eells-Sampson's heat equation technic, we obtain the existence result below.

Theorem 1.2 (Toda). *For any free homotopy class γ, any fixed complex structure and $|H| < 1$, there exists a minimizer for I_H.*

(2)Consider the variation of complex structures. The incompressibility assumption on γ excludes the degeneration of a minimizing sequence of complex structures.

2. Area minimizing with a volume constraint

In the previous section, we obtained the existence of a minimizer for a prescribed mean curvature formulation. To relax the strong requirement in the stability of a minimizer, we consider a constrained problem in this section. The problem can be formulated as follows.

(ii) For a given real number V, find a minimizer $(u, \Sigma) \in \gamma \times \mathcal{M}_g$ of the Dirichlet integral $D(u, \Sigma)$ in $\mathcal{V}_\gamma(V_0)$ where $\mathcal{M}_g$ denotes the moduli space of closed surfaces with genus g.

Of course, (ii) is a formulation to find an area minimizing surface under a volume constraint. Any solution for problem (ii) satisfies equations (1.2)-(1.3) for some constant H. In this case, H appears as the Lagrange multiplier. For this problem, the following existence theorem holds.

Theorem 2.1(Toda). *Suppose γ induces an injective action on the fundamental group. Then, there exists a minimizer for minimizing problem (ii). Moreover, any minimizer is an immersion.*

To prove the theorem, there are two essential steps. The first step is to prove the optimal energy loss estimate and the second is to construct a energy comparison map. This strategy for proof was invented by Wente for surfaces in $R3$. The hardest part in our hyperbolic case is the estimate in the first step. In contrast to the case of $\mathbb{R}^3$, we can not directly use the expansions of the volume functional for the estimate. Moreover, we don't have the optimal isoperimetric inequality in hyperbolic *manifolds* in general. To overcome the difficulty, we localize the energy loss carefully and obtain the estimate by lifting singularities to universal cover $\mathbb{H}^3$ where we have the optimal isoperimetric inequality obtained by Schmidt. The second step is done by the "sphere attaching lemma" which is developed by Wente. Since he proved the lemma by the local arguement, almost the same arguement works for our problem.

A non-existence result obtained by Theorem 1.2 compliments Theorem 1.1.

Corollary. *Suppose γ induces an injective action on the fundamental groups. Any minimizer for problem (ii) satisfies (1.2) for $|H| < 1$. Especially, the bound $|H| < 1$ in Theorem 1.1 is optimal.*

This corollary is proved by investigations of the dependence of the minimizing area on a given volume constraint.

3. A problem in the classical differential geometry

The immersions obtained in Theorem 1.1 and Theorem 2.1 can be developed to Kleinian periodic immersions of $\mathbb{H}^2$ into $\mathbb{H}^3$. Thus, our theorems can also be intepreted as existence results for periodic surface in hyperbolic space form $\mathbb{H}^3$.

In classical differential geometry, it is known that according to $H < 1, > 1, = 1$, the situation is completely different and among those, $H < 1$ is the least investigated.

On the other hand, because of the complexity of the period condition, to my knowledge, the following problem is still open.

Problem 1. *Does there exist a surface of constant mean curvature 1 with a non-elementary Kleinnian period in $\mathbb{H}^3$?*

The corollary to Theorem 2.1 supports the following conjecture,

Conjecture. *There exists no surface of constant mean curvature 1 with a quasi-Fuchsian period in $\mathbb{H}^3$.*

If this conjecture holds true, the situation must depend on the algebraic or geometric property of the period. Thus, Problem 1 should be considered more in detail; if it is affirmative, when can one construct the surface?

4. A variational problem with a group action

One way to study Problem 1 is to consider the limit as $H \uparrow 1$. Taking Theorem 2.1 and Conjecture above into account, we have to find a surface with constant mean curvature < 1 which has a non-Fuchsian period. This is a less investigated subject both in differential geometric and in variational context. We give here only a formulation.

We start with the definition of the Teichmüller space. Let M be a closed surface with genus g. By $\mathcal{T}_g$, we denote the Teichmüller space with base surface M. An element of $\mathcal{T}_g$ is represented by a pair (Σ, S) where Σ is a Riemann surface and S is a homeomorphism of M to Σ. Two pairs (Σ, S) and (Σ', S') denote the same element of $\mathcal{T}_g$ if and only if $S \circ S'^{-1}$ is homotopic to a holomorphic map.

We define the functional for $f \in \gamma$ and $p = (\Sigma, S)$ as follows,

$$I_H(f, p) := I_H(f \circ S^{-1}, \Sigma). \tag{4.1}$$

This definition is independent of choice of representative (Σ, S). If we choose as f the unique solution in Theorem 1.2 for each Σ, it induces smooth function Φ of $\mathcal{T}_g$. It is classically known that for any critical value, the solution for equation (1.2)-(1.3) coresponds. So, the problem reduces to study critical points of Φ.

Set

$$K := ker \left\{ f \circ S^{-1} : \pi_1(M, p) \to \pi_1(N, f \circ S^{-1}(p)) \right\}.$$

Since the kernel is a normal subgroup, K is independent of choice of $f \in \gamma$. We define a subgroup of the mapping class group by

$$G := \{ g \in Out(\pi_1(M)); g(K) = K \}$$

By observing the geometric effect of the action, we can see that Φ is a G-invariant function.

Thus, our problem reduces to this G-invariant variational problem. Hopefully, it is expected that the interplay between the algebraic property of G, which contains the topological information of γ, and critical point theory (or Morse theory) describes the situation. But this optimistic forecast will be carried over only after the investigation of possible degenerations of complex structures.