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HOMOLOGICAL ASPECTS OF EQUIVARIANT MODULES
Matijevic-Roberts and Buchsbaum-Rim*

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Abstract

Homological aspects of equivariant modules and sheaves are discussed. Construction of Ext_A and Tor^A in the category of (G,A)-modules is the main subject. As an application, Matijevic-Roberts type theorem on homological properties such as Cohen-Macaulay property for graded noetherian rings are generalized to a theorem on group action. As another application, Cohen-Macaulay approximation and Ringel's Δ-good approximation on reductive groups (quasi-hereditary algebras) are unified, and resolutions of Buchsbaum-Rim type are studied, utilizing the notion of tilting modules.

1 Introduction

The purpose of these notes is to present some results on homological aspects on equivariant modules and sheaves.

Let R be a commutative ring, G an affine R-group scheme, and A a noetherian commutative G-algebra, namely, R-algebra with a G-action. We say that M is a (G,A)-module [52] if M is both a G-module and an A-module, and the A-action $A \otimes M \to M$ is a G-homomorphism. Equivariant module theory is nothing but a study of (G,A)-modules. A (G,A)-module is sometimes called a G-equivariant A-module. It seems that there has been no coherent treatment of this object over arbitrary base R and general G, except for Seshadri's paper [52]. However, the work presented here is deeply related to the theory of relative Hopf modules over a field (see, e.g., [43]), some works in representation theory over a field, and the study of graded modules over graded noetherian rings by Goto-Watanabe [23, 24].

This paper consists of four parts.

The first part (sections 2–7) is devoted to give G-equivariant structures to $\mathrm{Ext}^i_A(M,N)$ and $\mathrm{Tor}_i^A(M,N)$ for (G,A)-modules M and N, under appropriate assumptions. If G is a split torus $\mathbb{G}_m^n$, then a G-algebra A is nothing

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but a $\mathbb{Z}^n$-graded R-algebra (see Example 3.4), and a (G, A)-module is nothing but a graded A-module. The grading of $\mathrm{Ext}^i_A(M, N)$ and $\mathrm{Tor}^A_i(M, N)$ agree with those used in Goto-Watanabe papers [23, 24]. What we try to do here is to generalize homological theories on graded rings and modules to G-equivariant theory for more general groups. This part contains some considerable survey materials (mainly for ring theorists) on basics on Hopf algebras and representation theory of algebraic groups over arbitrary base.

In section 2, we review basics on comodules over R-flat coalgebras over arbitrary base. Restriction and induction will be important in the third part. In section 3, assuming that R and A are noetherian and G is R-flat, we give a definition of the (G, A)-module $\mathrm{Ext}^i_A(M, N)$, where M is an A-finite (G, A)-module, and N a (G, A)-module. Graded modules over $\mathbb{Z}^n$-graded noetherian rings are good examples. Note that if M is not A-finite, then $\mathrm{Ext}^i_A(M, N)$ can not be graded in an appropriate way, even if both M and N are graded. In section 4, we give a definition of $\mathrm{Tor}^A_i(M, N)$ for (G, A)-modules M and N, under the assumption R noetherian, G an affine flat group scheme over R which is *ind-finite projective* (IFP for short, see Definition 4.1), and A a commutative G-algebra. In section 5, we introduce hyperalgebras (algebras of distributions) of an algebraic group, and give a sufficient condition for the theory of rational modules goes well, which should be compared with the results in [33]. We also give a sufficient condition for an affine group scheme being IFP. Section 6 is an attempt to extend the category of (G, A)-modules to a module category in an appropriate way. The category of G-modules and (G, A)-modules do not have enough projective objects in general. Although they have projective limits if A and R are noetherian and G is R-flat, the projective limits in these categories do not agree with the projective limits as R-modules in general. These are sometimes inconvenient. For example, even if A is a noetherian G-algebra which is local, the completion $\hat{A}$ is not a G-algebra in general, although there is a canonical way to define G-action when G is a finite flat group. We treat another case, the case G is flat, infinitesimally flat, affine, of finite type with connected geometric fibers. The hyperalgebra and its smash product with A is used to extend the category of (G, A)-modules. The group $\mathrm{Ext}^i_A(M, N)$ for non-finite M lies in the extended category. The constructions up to section 5 will be generalized to quasi-coherent sheaves over a scheme, in section 7.

The second part (sections 8–9) is a first application of the constructions in the first part. Matijevic-Roberts type theorems on Cohen-Macaulay, Gorenstein and regular properties in [24] and for complete intersection property in [15] for graded rings are generalized to equivariant context.

In section 8, we study the stability of subschemes under the action of an algebraic group. In section 9, we discuss Matijevic-Roberts type problem, using the results in section 8. The notion of G-linearlization by Mumford-Fogarty [44], and normal flatness play important roles there.

The third part (sections 10–13) is a generalization to the arbitrary base of the theory of good filtrations on representations of reductive algebraic groups over a field. Although the representations of (split) reductive groups are strongly related to the theory of quasi-hereditary algebras, and the notion of quasi-hereditary algebras is generalized to the notion over arbitrary base by Cline-Parshall-Scott [16], some considerable part of the theory over fileds has not been generalized to the theory over arbitrary base. For example, construction of tilting modules over arbitrary base is new here. Section 10 is a survey on the Auslander-Buchweitz theory, which contains an important application to ring theory, Cohen-Macaulay approximations. The Auslander-Buchweitz theory is convenient to describe Ringel's theorem and its generalization discussed here. Section 11 is a survey on split reductive groups. In this section, we remark that any split reductive group is IFP, and remove the embeddability assumption in Seshadri's theorem [52, Theorem 2]. In section 12, we extend the definitions and related results on good filtrations over a field to those over arbitrary base. The notion 'good' is divided into two different ones, good and u-good. U-good is an abbreviation for universally good, which means the goodness which is stable under base change. In section 13, we generalize Ringel's theorem to a theorem over arbitrary base.

The fourth part (sections 14–15) is an application of the first and the third part to commutative ring theory again. The notion of tilting modules, and the (G, A)-structure of higher Ext_A-groups play important roles here.

Existence of a G-equivariant resolution of determinantal ideals of certain type, which generalizes Buchsbaum-Rim resolution for maximal minors in a natural way, is discussed in section 14. In section 15, Ringel's approximation and Cohen-Macaulay approximation are unified in some sense. The determinantal rings give an example.

General notation

Throughout these notes, R denotes a commutative ring. The symbols $\otimes$ and Hom denote $\otimes_R$ and Hom_R, respectively. For a commutative ring A, the prime spectrum of A is denoted by $\operatorname{Spec} A$. For $\mathfrak{p} \in \operatorname{Spec} A$, the symbol $\kappa(\mathfrak{p})$ stands for the field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$. All schemes are assumed to be separated. For an R-scheme X and Y, we denote the set of $\operatorname{Spec} R$-morphisms from Y to X by $X(Y)$. Moreover, $X \times Y$ stands for $X \times_R Y$. If $Y = \operatorname{Spec} R'$ is affine, then, $X(Y)$ and $X \times Y$ may also be denoted by $X(R')$ and $X \otimes R'$, respectively. Action of an algebraic group scheme G over R on an R-scheme X is a right R-action, unless otherwise specified. Modules (over a non-commutative ring or a group) are left modules unless otherwise specified, and ${}_A\mathbb{M}$ denotes the category of A-modules for a ring A.

2 Comodules over flat coalgebras over commutative rings

For the theory of Hopf algebras over a field, see [53, 43].

Let R be a commutative ring. An R-algebra is nothing but a monoid in the symmetric category [36] of R-modules. An R-coalgebra is a comonoid in the symmetric category of R-modules by definition. Namely, an R-module C, together with R-linear maps

$$R \xleftarrow{\varepsilon_C} C \xrightarrow{\Delta_C} C \otimes C$$

satisfying the commutativity $(1_C \otimes \Delta_C)\Delta_C = (\Delta_C \otimes 1_C)\Delta_C$, $\rho_C(1_C \otimes \varepsilon_C)\Delta_C = 1_C = \lambda_C(\varepsilon_C \otimes 1_C)\Delta_C$, where $\rho_C : C \otimes R \to C$ is given by $c \otimes r \mapsto rc$, and $\lambda_C : R \otimes C \to C$ is given by $r \otimes c \mapsto rc$. A coalgebra map $C \to C'$ is a map of comonoids by definition. Namely, an R-linear map $f : C \to C'$ is said to be a coalgebra map if $\varepsilon_C = \varepsilon_{C'} f$ and $(f \otimes f)\Delta_C = \Delta_{C'} f$ hold.

A (right) C-comodule is a (right) coaction of C. Namely, an R-module M, endowed with an R-linear map $\omega_M : M \to M \otimes C$ satisfying $\rho_M(1_M \otimes \varepsilon_C)\omega_M = 1_M$ and $(1_M \otimes \Delta_C)\omega_M = (\omega_M \otimes 1_C)\omega_M$. A (right) C-comodule map is an R-linear map $f : M \to M'$ satisfying $(f \otimes 1_C)\omega_M = \omega_{M'} f$. Left C-comodules and comodule maps are defined similarly. Unless otherwise specified, a C-comodule mean a right C-comodule. The category of C-comodules is denoted by $\mathbb{M}^C$. The category $\mathbb{M}^C$ is an additive category with cokernels and arbitrary inductive limits, as cokernels and inductive limits are compatible with the functor $? \otimes C$. If C is R-flat, then $\mathbb{M}^C$ is abelian.

General theory of R-flat coalgebras and comodules goes like a dual of the theory of algebras and modules. Let C be an R-flat coalgebra. It is obvious that C is a C-comodule (in fact, a (C, C)-bicomodule, which is appropriately defined) with $\omega_C = \Delta_C$. For an R-module V and a C-comodule M, $V \otimes M$ is a C-comodule with $\omega_{V \otimes M} := 1_V \otimes \omega_M$. If I is an injective R-module, then $I \otimes C$ is an injective C-comodule (i.e., an injective object in $\mathbb{M}^C$). For a C-comodule M, the map $\omega_M : M \to M \otimes C$ is a C-comodule map, where $\omega_{M \otimes C} := 1_M \otimes \Delta_C$. Combining these, it is easy to verify that $\mathbb{M}^C$ has enough injectives, and any injective C-comodule is a direct summand of an injective comodule of the form $I \otimes C$, where I is an injective R-module.

The dual algebra C^* is $\operatorname{Hom}(C, R)$. It is an R-algebra with the product defined by

$$\langle c^* d^*, c \rangle := \sum_{(c)} \langle c^*, c_1 \rangle \langle d^*, c_2 \rangle$$

for $c^*, d^* \in C^*$ and $c \in C$, where $\sum_{(c)} c_1 \otimes c_2 = \Delta_C(c)$. A C-comodule M is a C^*-module. The C^*-action is defined by

$$c^* m := \sum_{(m)} \langle c^*, m_1 \rangle m_0,$$

for $c^* \in C^*$ and $m \in M$, where $\sum_{(m)} m_0 \otimes m_1 = \omega_M(m)$. A C-comodule map is a C^*-linear map, and we obtain a functor $\Phi : \mathbb{M}^C \to {}_{C^*}\mathbb{M}$. It is an equivalence, if C is R-finite projective.

Let M be a C-comodule, and N a left C-comodule. The *cotensor product* of M and N, denoted by $M \boxtimes^C N$ is the kernel of the map

$$M \otimes N \xrightarrow{\omega_M \otimes 1_N - 1_M \otimes \omega_N} M \otimes C \otimes N.$$

Note that $? \boxtimes^C N$ and $M \boxtimes^C ?$ are left exact, and we have

$$R^i(? \boxtimes^C N)(M) \cong R^i(M \boxtimes^C ?)(N)$$

in a natural way for $i \geq 0$, which we denote by $\mathrm{Cotor}^i_C(M, N)$.

Let $f : D \to C$ be a coalgebra map of R-flat coalgebras. For $M \in \mathbb{M}^D$, M is a C-comodule by $\omega^C_M := (1_M \otimes f)\omega^D_M$. A D-comodule map is a C-comodule map, and we obtain a functor $\mathbb{M}^D \to \mathbb{M}^C$, which we denote by res^D_C and call the *restriction functor.* The restriction for left comodules are defined similarly.

The restriction functor has a right adjoint. For $N \in \mathbb{M}^C$, the tensor product $N \otimes D$ is a D-comodule, and $N \boxtimes^C (\mathrm{res}^D_C D) \subset N \otimes D$ is a D-subcomodule. We denote $N \boxtimes^C (\mathrm{res}^D_C D)$ by $\mathrm{ind}^D_C N$. It is easy to check that ind^D_C gives a functor $\mathbb{M}^C \to \mathbb{M}^D$, and is right adjoint to res^D_C, and hence is left exact. Note also that ind^D_C preserves injectives, as it has an exact left adjoint.

Let M be an R-module and N an R-submodule of M. We say that N is a *pure* R-submodule of M if for any R-module W, the canonical map $W \otimes N \to W \otimes M$ is injective. Note that N is R-pure if the inclusion $N \hookrightarrow M$ R-splits or M/N is R-flat.

Definition 2.1 Let C be an R-coalgebra. We say that D is an R-*subcoalgebra* of C when D is an R-pure submodule of C, and $\Delta_C(D) \subset D \otimes D$.

It is obvious that an R-subcoalgebra D is an R-coalgebra so that the inclusion map $D \hookrightarrow C$ is an R-coalgebra map. If D is an R-subcoalgebra of C, then the unit of the adjunction $M \to (\mathrm{ind}^D_C \circ \mathrm{res}^D_C)(M)$ is an isomorphism. In fact, $\mathrm{ind}^D_C(N)$ is identified with the R-submodule

$$\{n \in N \mid \omega_N(n) \in N \otimes D\}$$

of N. This shows that res^D_C is fully faithful (and obviously exact, as $\mathrm{res}^D_C(N) = N$ as an R-module). Thus, a D-comodule is identified with a C-comodule M such that $\omega^C_M(M) \subset M \otimes D$.

We say that an R-coalgebra, which is also an R-algebra, B is an R-*bialgebra* if both $\varepsilon_B : B \to R$ and $\Delta_B : B \to B \otimes B$ are R-algebra maps.

An *antipode* of an R-bialgebra B is an R-linear map $S : B \to B$ satisfying $m_B(1_B \otimes S)\Delta_B = 1_B = m_B(S \otimes 1_B)\Delta_B$. An antipode of an R-bialgebra is unique, if it exists. If an R-bialgebra B has an antipode, then it is called an *R-Hopf algebra.*

The R-algebra R is an R-Hopf algebra in an obvious way.

Let B be an R-bialgebra. It is easy to see that the unit map $u : R \to B$ ($u(1) = 1_B$) is an R-bialgebra map. Hence, any R-module is a B-comodule via restriction. For an R-module M, the B-comodule $M = \mathrm{res}^R_B(M)$ is called the trivial B-comodule M. For a B-comodule M, we define the *invariance* of M, denoted by M^B, as

$$\{m \in M \mid \omega_M(m) = m \otimes 1_B\}.$$

The R-submodule M^B of M is identified with $\mathrm{Hom}_B(R, M)$, where Hom_B means $\mathrm{Hom}_{\mathbb{M}^B}$. The higher B-cohomology of M, denoted by $H^i(B, M)$ is $\mathrm{Ext}^i_B(R, M)$ by definition.

3 (G, A)-modules and Ext_A-groups

Let G be an affine R-group scheme. In other words, the coordinate ring H, sometimes denoted by $R[G]$, of G is a commutative R-Hopf algebra. A G-module is an H-comodule by definition. A G-linear map of G-modules is an H-comodule map by definition.

Alternative definition of a G-module is given as follows [33].

Let $\mathcal{C}$ be a category. An R-functor (with valued in $\mathcal{C}$) is a covariant functor from the category of commutative R-algebras (to $\mathcal{C}$). An affine R-group scheme G is a representable R-functor with valued in the category of abstract groups (by Yoneda's lemma). A G-module is an R-module M together with a natural transformation $G \to GL(M)$ between R-functors with valued in the category of abstract groups, where the functor $GL(M)$ is given by $GL(M)(A) := \mathrm{End}_A(M \otimes A)^\times$ for any commutative R-algebra A (if M is R-finite projective, then $GL(M)$ is also representable, but is not in general). For the correspondence between this definition and the definition via comodules, see [33].

Thus, a G-module M is nothing but the functorial A-linear action of the group of A-valued points $G(A)$ on $M \otimes A$, for any commutative R-algebra A. We express this situation with a set-theoretic (naive, and sometimes dangerous) description: For $g \in G$ and $m \in M$, $gm \in M$ is given (and satisfies certain axioms of linear action).

The category of G-modules (and G-linear maps) is not necessarily abelian. However, if G is R-flat, then the category of G-modules, denoted by ${}_G\mathbb{M}$, is abelian.

From now on, until the end of these notes, G denotes a flat affine R-group scheme.

Let V and W be G-modules. Then, $V \otimes W$ is a G-module. The G-action is given by $g(v \otimes w) := gv \otimes gw$. If V is of finite presentation as an R-module moreover, then $\mathrm{Hom}(V, W)$ is a G-module by $(gf)(v) := g(f(g^{-1}v))$ for $f \in \mathrm{Hom}(V, W)$, $g \in G$ and $v \in V$, in a naive sense.

Let A be a commutative G-algebra, namely, a commutative R-algebra on which G-acts as R-algebra automorphisms. This is equivalent to say that A is a commutative R-algebra, which is also a G-module, and the product $A \otimes A \rightarrow A$ is G-linear. We say that M is a (G, A)-module when M is both a G-module and an A-module, and the A-action $A \otimes M \rightarrow M$ is G-linear. Hence, A is a (G, A)-module in a natural way. In the language of Hopf algebras, G-algebras and H-comodule algebras are the same, and (G, A)-modules and (H, A)-Hopf modules are the same. An R-linear map $f : M \rightarrow N$ between (G, A)-modules is called a (G, A)-linear map when f is both A-linear and G-linear. We obtain an abelian category of (G, A)-modules, which is denoted by ${}_{G,A}\mathbb{M}$. Note that R is a trivial G-algebra, and a (G, R)-module is nothing but a G-module.

Lemma 3.1 *The category ${}_{G,A}\mathbb{M}$ has exact filtered inductive limits, and the forgetful functor from ${}_{G,A}\mathbb{M}$ to the category of abelian groups preserves arbitrary inductive limits.*

Let M and N be (G, A)-modules, and V a G-module. Then, the G-module $M \otimes V$ is a (G, A)-module by the A-action $a(m \otimes v) := am \otimes v$. The tensor product $M \otimes_A N$ is uniquely a (G, A)-module so that the canonical projection $M \otimes N \rightarrow M \otimes_A N$ is (G, A)-linear.

We denote the full subcategory of ${}_{G,A}\mathbb{M}$ consisting of all (G, A)-modules which is of finite presentation as an A-module by ${}_{G,A}\mathbb{M}_f$. ${}_{G,R}\mathbb{M}_f$ is denoted by ${}_{G}\mathbb{M}_f$. If $M \in {}_{G,A}\mathbb{M}_f$ and $N \in {}_{G,A}\mathbb{M}$, then the A-module $\mathrm{Hom}_A(M, N)$ is a (G, A)-module by $(gf)(m) := g(f(g^{-1}m))$.

The following is a slight generalization of [52, Proposition 3].

Lemma 3.2 (Local noetherian property) *Let $M \in {}_{G,A}\mathbb{M}$. Then, for any A-finite A-submodule M_0 of M, there is a sequence of A-submodules $M_0 \subset N \subset L$ of M with N being a (G, A)-submodule of M and L is A-finite. In particular, ${}_{G,A}\mathbb{M}$ is Grothendieck, and has injective hulls. If A or R is noetherian, then we can take them so that $N = L$. If A is noetherian, then ${}_{G,A}\mathbb{M}$ is locally noetherian, and hence has arbitrary projective limits.*

Note that the forgetful functor ${}_{G,A}\mathbb{M} \rightarrow {}_{A}\mathbb{M}$ does not even preserves infinite direct products, in general.

From now on, we assume that both R and A are noetherian. The first theorem of these notes is:

Theorem 3.3 *Let $M \in {}_{G,A}\mathbb{M}$, and I an injective (G, A)-module. Then, we have $\mathrm{Ext}^i_A(M, I) = 0$ for $i > 0$.*

This does not mean that an injective (G, A)-module I is A-injective, as the class of A-modules which admit (G, A)-module structures is much smaller than the class of all A-modules. In fact, an injective (G, A)-module is not A-injective in general even if R is an algebraically closed field and $G = \mathbb{G}_m$ [23].

Let $M \in {}_{G,A}\mathbb{M}_f$ and $N \in {}_{G,A}\mathbb{M}$. By the theorem, it follows that the underlying A-module of the (G, A)-module $R^i \operatorname{Hom}_A(M, ?)(N)$ is $\operatorname{Ext}^i_A(M, N)$, since a (G, A)-injective resolution of N is $\operatorname{Hom}_A(M, ?)$-acyclic *in the category of A-modules.* Thus, we have a canonical G-equivariant structure of $\operatorname{Ext}^i_A(M, N)$. In particular, if J is a G-ideal (a (G, A)-submodule) of A, then the local cohomology $H^i_J(N) = \varinjlim \operatorname{Ext}^i_A(A/J^n, N)$ has a canonical (G, A)-module structure.

Example 3.4 Let n be a positive integer, and G the n-fold split torus $\mathbb{G}_m^n$, where $\mathbb{G}_m := GL_1$. Then, a G-module is nothing but a $\mathbb{Z}^n$-graded R-module.

For $V \in {}_G\mathbb{M}$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, we define

$$V_\lambda := \{v \in V \mid t \cdot v = t^\lambda v \quad \text{for any } t = (t_1, \ldots, t_n) \in G\}, \tag{1}$$

where $t^\lambda := t^{\lambda_1} \cdots t^{\lambda_n} \in R^\times$, and the product $t^\lambda v$ is a scalar multiple. It is easy to verify that $V = \bigoplus_\lambda V_\lambda$, and an R-linear map $f : V \to W$ between G-modules is G-linear if and only if $f(V_\lambda) \subset W_\lambda$ for any $\lambda \in \mathbb{Z}^n$.

Conversely, if $V = \bigoplus_\lambda V_\lambda$ is a $\mathbb{Z}^n$-graded R-module, then letting (1) as the definition of the G-action, V is a G-module.

A G-algebra is nothing but a $\mathbb{Z}^n$-graded R-algebra, and a (G, A)-module is nothing but a graded A-module. The (G, A)-structure of $M \otimes_A N$ and $\operatorname{Ext}^i_A(M, N)$ are those discussed in [23] and [24].

Related to the example, we introduce some important notion in representation theory. For an affine R-group scheme G, we denote the set of isomorphism classes of rank-one R-free G-modules by $X(G)$, and call it the *character group* of G. It is an abelian group with $\otimes$ as the group-law. Note that $X(G)$ is identified with a subset of $R[G]^\times$ by

$$X(G) = \operatorname{Hom}_{\text{alg-grp}}(G, \mathbb{G}_m) \subset \operatorname{Hom}_{R\text{-sch}}(G, \operatorname{Spec} R[t, t^{-1}]) = R[G]^\times.$$

With this identification, $X(G)$ is a subgroup of $R[G]^\times$. However, it is common to express the group-law of $X(G)$ *additively.* For a split torus $T = \mathbb{G}_m^n$, the number n is called the *rank* of T. The example above shows that $X(T) \cong \mathbb{Z}^n$ as additive groups. An element of $X(T)$ is sometimes called a *weight* of T.

Unfortunately, $\operatorname{Hom}_A(M, N)$ does not allow any G-module structure in general, if M is not A-finite. However, if G is good enough, $\operatorname{Ext}^i_A(M, N)$ admits certain 'extended' equivariant structure (extending the category of (G, A)-modules), see section 6.

An R-module can be viewed as a G-module with a trivial G-action. For a G-module M, we denote the invariance $M^{R[G]}$ by M^G and call it the submodule of (absolute) invariants. The module $\mathrm{Ext}^i_G(R, M)$ is denoted by $H^i(G, M)$, and called the ith cohomology of M. For $M \in {}_{G,A}\mathbb{M}_f$ and $N \in {}_{G,A}\mathbb{M}$, we have that $\mathrm{Hom}_A(M, N)^G = \mathrm{Hom}_{G,A}(M, N)$, where $\mathrm{Hom}_{G,A}$ denotes the set of morphisms in ${}_{G,A}\mathbb{M}$. Similarly, $\mathrm{Ext}^i_{{}_{G,A}\mathbb{M}}(M, N)$ is denoted by $\mathrm{Ext}^i_{G,A}(M, N)$.

Proposition 3.5 *Let $A \to B$ be a map of noetherian G-algebras, $V \in {}_{G,A}\mathbb{M}$, $M \in {}_{G,B}\mathbb{M}$, and $N \in {}_{G,B}\mathbb{M}$. If M is A-flat and B-finite, then there is a spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}^p_{G,A}(V, \mathrm{Ext}^q_B(M, N)) \Rightarrow \mathrm{Ext}^{p+q}_{G,B}(M \otimes V, N).$$

If V is A-finite projective, then there are isomorphisms

$$\mathrm{Ext}^i_{G,B}(M \otimes V, N) \cong \mathrm{Ext}^i_{G,B}(M, \mathrm{Hom}_A(V, N)) \cong \mathrm{Ext}^i_{G,B}(M, N \otimes V^\star),$$

where $V^\star := \mathrm{Hom}_A(V, A)$.

4 IFP groups and Tor^A-groups

Let R be a noetherian commutative ring, G a flat affine R-group scheme, and A a commutative G-algebra.

The definition of the (G, A)-structure of Tor^A is a little bit more difficult. Consider the following condition.

($*$) For any R-finite G-module M, there exists some surjective G-linear map $P \to M$ with P being R-finite projective.

It was questioned by Seshadri that if any reductive R-group scheme satisfies this condition. We will see later that this is true for *split* reductive groups.

If the condition ($*$) holds, then for any (non-R-finite) G-module M, there exists some surjective G-linear map $P \to M$ with P being R-projective, by Lemma 3.2 (applied for the case $A = R$). It follows that if M is a (G, A)-module (resp. A-finite (G, A)-module), then there is a (G, A)-epimorphism $P \to M$ with P being A-projective (resp. A-finite projective). We take a G-submodule (resp. R-finite G-submodule) M_0 of M which A-generates M, and take a surjective G-linear map $P_0 \to M_0$ with P_0 being R-projective (resp. R-finite projective). Then, it is easy to see that the composite map

$$P = A \otimes P_0 \to A \otimes M_0 \to A \otimes M \to M$$

is (G, A)-linear and surjective.

Assume $(*)$ holds and let N be a (G, A)-module. We have a derived functor

$$\underline{L}(? \otimes_A N) : D^-({}_{G,A}\mathbb{M}) \to D^-({}_{G,A}\mathbb{M})$$

by [25, Corollary 5.3.β]. For $M, N \in {}_{G,A}\mathbb{M}$, we have $L_i(? \otimes_A N)(M) \cong L_i(M \otimes_A ?)(N)$ in a natural way, which we denote by $\operatorname{Tor}_i^A(M, N)$. The underlying A-module is the same as the usual one, because it is calculated by an A-projective (G, A)-resolution.

Similarly, if $(*)$ is satisfied, A is noetherian, and $N \in {}_{G,A}\mathbb{M}$, then, the derived functor

$$\underline{R}\operatorname{Hom}_A(?, N) : D^-({}_{G,A}\mathbb{M}_f) \to D^+({}_{G,A}\mathbb{M})$$

is induced, and $R^i \operatorname{Hom}_A(?, N)(M) \cong \operatorname{Ext}_A^i(M, N)$ in a natural way for $M \in {}_{G,A}\mathbb{M}_f$, where $\operatorname{Ext}_A^i(M, N)$ is the one defined in the last section.

Definition 4.1 We say that an R-coalgebra C is ind-finite projective (IFP, for short) if any R-finite R-submodule M of C is contained in some R-finite projective R-subcoalgebra D of C. We say that G is IFP if the coordinate ring H is IFP as an R-coalgebra.

Note that an IFP R-coalgebra C is R-flat. A base change of an IFP coalgebra is IFP.

Lemma 4.2 *Let C be an IFP R-coalgebra. If M is an R-finite C-comodule, then there exists some surjective C-comodule map $P \to M$ such that P is R-finite projective. In particular, if G is IFP, then the condition $(*)$ is satisfied.*

A base change of an IFP group is IFP. A sufficient condition for IFP will be given in the next section.

5 Universal density

Definition 5.1 Let R be a field, V an R-space, and W an R-subspace of V^*. We say that W is *dense* in V^* when the canonical map $V \to W^*$ ($v \mapsto (w \mapsto wv)$) is injective. Let R be a general commutative ring, and V and W be R-modules. We say that an R-module map $f : V \to W^*$ is *universally dense* when for any R-module U, the canonical R-linear map

$$\theta_U : U \otimes V \to \operatorname{Hom}(W, U) \qquad u \otimes v \mapsto (w \mapsto \langle v, w \rangle u)$$

is injective.

If R is a field and W is a dense subspace of V^*, then the inclusion $W \hookrightarrow V^*$ is universally dense. In the definition above, the canonical pairing which corresponds to f by the canonical isomorphism

$$\mathrm{Hom}(W, V^*) \cong \mathrm{Hom}(W \otimes V, R)$$

is denoted by $\langle -, - \rangle$. For any commutative R-algebra R', the pairing $R' \otimes \langle -, - \rangle$ yields an R'-linear map $f' : W' \to \mathrm{Hom}_{R'}(V', R')$. Note that the map θ_U is natural with respect to U.

Lemma 5.2 *Let R be a general commutative ring, and $f : W \to V^*$ be an R-linear map. Consider the following conditions.*

1 *f is universally dense.*

2 *V is R-flat, and for any $\mathfrak{p} \in \mathrm{Spec}\, R$, the image of the canonical map*

$$f(\mathfrak{p}) : W \otimes \kappa(\mathfrak{p}) \to \mathrm{Hom}_{\kappa(\mathfrak{p})}(V \otimes \kappa(\mathfrak{p}), \kappa(\mathfrak{p}))$$

is a dense subspace of $\mathrm{Hom}_{\kappa(\mathfrak{p})}(V \otimes \kappa(\mathfrak{p}), \kappa(\mathfrak{p}))$.

It holds in general that **1** *implies* **2**. *If R is noetherian, then* **2** *implies* **1**.

If f is universally dense, then the canonical map $\theta_R : V \to W^*$ is injective and R-pure.

Let C be an R-coalgebra, and $A \to C^*$ be a universally dense R-algebra map (in particular, C is R-flat).

A C-comodule M is a C^*-module, hence is an A-module. We denote the canonical exact functor $\mathbb{M}^C \to {}_A\mathbb{M}$ by Φ. As an R-module, we have $\Phi(M) = M$.

Let V be an A-module, and we denote the action $A \otimes V \to V$ by a_V. We denote the canonical isomorphism

$$\mathrm{Hom}(A \otimes V, V) \cong \mathrm{Hom}(V, \mathrm{Hom}(A, V))$$

by ρ_V. By assumption, the map

$$\theta_V : V \otimes C \to \mathrm{Hom}(A, V)$$

is injective.

We denote the pull-back of $\mathrm{Im}(\theta_V)$ by $\rho_V a_V : V \to \mathrm{Hom}(A, V)$ by V_{rat}, and we call V_{rat} the *rational part* of V.

Lemma 5.3 *V_{rat} is an A-submodule of V. Moreover, the composite map $V_{\mathrm{rat}} \hookrightarrow V \to \mathrm{Hom}(A, V)$ factors through*

$$V_{\mathrm{rat}} \otimes C \hookrightarrow V \otimes C \xrightarrow{\theta_V} \mathrm{Hom}(A, V),$$

the map $\omega_{V_{\mathrm{rat}}} : V_{\mathrm{rat}} \to V_{\mathrm{rat}} \otimes C$ is obtained, and V_{rat} is a C-comodule with the structure map $\omega_{V_{\mathrm{rat}}}$. With respect to this C-comodule structure, the A-module structure of $\Phi(V_{\mathrm{rat}})$ agrees with that of V_{rat} as an A-submodule of V.

Lemma 5.4 *Let* $V \in \mathbb{M}^C$, $W \in {}_A\mathbb{M}$ *and* $f \in \operatorname{Hom}_A(\Phi V, W)$. *Then, we have* $f(V) \subset W_{\mathrm{rat}}$. *Thus, we have an isomorphism*

$$\operatorname{Hom}_A(\Phi V, W) \cong \operatorname{Hom}_C(V, W_{\mathrm{rat}}).$$

In particular, if $W' \in {}_A\mathbb{M}$ *and* $g \in \operatorname{Hom}_A(W', W)$, *then we have* $g(W'_{\mathrm{rat}}) \subset W_{\mathrm{rat}}$. *Letting* $g_{\mathrm{rat}} := g|_{W'_{\mathrm{rat}}} \in \operatorname{Hom}_C(W'_{\mathrm{rat}}, W_{\mathrm{rat}})$, *we have a functor* $(?)_{\mathrm{rat}} : {}_A\mathbb{M} \to \mathbb{M}^C$. *The functor* $(?)_{\mathrm{rat}}$ *is a right adjoint of* Φ.

Corollary 5.5 *The functor* $(?)_{\mathrm{rat}}$ *is left exact, and preserves injective objects. The functor* Φ *is fully faithful and exact.*

In what follows, R is noetherian, and $G = \operatorname{Spec} H$ is a flat affine R-group scheme of finite type. We denote the defining ideal of the unit element $\{e\} \hookrightarrow G$ by I. In other words, I is the kernel of the counit map $\varepsilon_H : H \to R$.

Definition 5.6 We call

$$\operatorname{Hy} G = U := \varinjlim (H/I^n)^*$$

the *hyperalgebra* or the *algebra of distributions* of G.

Note that U is an R-subalgebra of H^*. In fact, we have $1_{H^*} = \varepsilon_H \in (H/I)^* \subset U$, and $(H/I^n)^* \cdot (H/I^m) \subset (H/I^{n+m-1})^*$ for any $m, n \geq 1$.

The following definition is in [33].

Definition 5.7 We say that G is infinitesimally flat when H/I^n is R-finite projective for any $n \geq 1$.

Assume that G is infinitesimally flat. Then $U = \operatorname{Hy} G$ is R-projective. Moreover, as we have $(H/I^n)^*(I^n \cdot H + H \cdot I^n) = 0$ and $(H/I^n \otimes H/I^n)^* \cong (H/I^n)^* \otimes (H/I^n)^*$, we have a canonical map $\Delta_U : U \to U \otimes U$ given by

$$\langle \Delta_U(u), a \otimes b \rangle := \langle u, ab \rangle \qquad (u \in U,\ a, b \in H).$$

It is straightforward to check that U is an R-Hopf algebra with these structures in this case.

The following is well-known [7, VIII.1.3].

Lemma 5.8 *If* G *is* R*-smooth, then* G *is infinitesimally flat.*

The following is also known as a special case of a theorem proved by Raynaud [47].

Theorem 5.9 *If* G *is* R*-smooth with connected geometric fibers, then* H *is* R*-projective.*

The following is a slight generalization of Theorem 5.9, and gives a sufficient condition for universal density of U.

Theorem 5.10 *Assume that G is infinitesimally flat with connected fibers (i.e., $G \otimes \kappa(\mathfrak{p})$ is connected for any $\mathfrak{p} \in \operatorname{Spec} R$). Then $\operatorname{Hy} G \to H^*$ is universally dense, and H is R-projective.*

Here we sketch an outline of the proof. Note that if G is infinitesimally flat, then $(\operatorname{Hy} G)^*$ is nothing but the I-adic completion of H. Hence, if R is a field, the universal density in problem is nothing but the assertion $\bigcap_{i>0} I^i = 0$, which follows from the fact G is irreducible and Cohen-Macaulay (so any zerodivisor in H is nilpotent). The universal density in the general case follows from Lemma 5.2. A slight modification of [47, Corollary 1] shows that $(\operatorname{Hy} G)^*$ is R-Mittag-Leffler, hence so is H, as H is an R-pure submodule of $(\operatorname{Hy} G)^*$ by universal density. As H is R-countable and Mittag-Leffler, it is R-projective.

Assume that G is infinitesimally flat with connected fibers. We say that a U-module V is rational if $V = V_{\mathrm{rat}}$. By Corollary 5.5, the full subcategory of rational U-modules is identified with ${}_G\mathbb{M}$ via the fully faithful exact functor Φ.

The following is proved in the same line of [52, Proposition 4].

Corollary 5.11 *If one of the following holds, then G is IFP.*

1 $\operatorname{gl.dim} R = 0$

2 $\operatorname{gl.dim} R = 1$, *and G is infinitesimally flat with connected fibers.*

The structure of $\operatorname{Tor}_i^A(M, N)$ for $G = \mathbb{G}_m^n$ given in [24] agrees with our definition for the IFP group G.

6 Smash products and (G, A)-modules

Let R be a noetherian commutative ring, and $G = \operatorname{Spec} H$ a flat, infinitesimally flat, affine R-group scheme with connected fibers. We set $U := \operatorname{Hy} G$.

Let A be a commutative noetherian G-algebra. As A is a U-module algebra [43], we can define the *smash product* [43] $\Gamma := A\#U$. As an R-module, $A\#U = A \otimes U$, and the product of $A\#U$ is given by

$$(a \otimes u)(b \otimes v) := \sum_{(u)} a \cdot (u_1 b) \otimes u_2 v \qquad (a, b \in A,\ u, v \in U).$$

With this product, $\Gamma = A\#U$ is an associative R-algebra. A Γ-module is an A-module U-module via the R-algebra maps $A \hookrightarrow \Gamma$ and $U \hookrightarrow \Gamma$ given by $a \mapsto a \otimes 1$ and $u \mapsto 1 \otimes u$, respectively (note that $u 1_A = (\varepsilon_U u) 1_A$ for $u \in U$).

For a Γ-module M, the A-module G-module M_{rat} is a (G, A)-module, and we have a functor $(?)_{\mathrm{rat}} : {}_{\Gamma}\mathbb{M} \to {}_{G,A}\mathbb{M}$. Conversely, for a (G, A)-module M, the A-module U-module $M = \Phi M$ gives a Γ-module M defined by

$$(a \otimes u)m := a(um) \qquad (a \in A,\ u \in U,\ m \in M).$$

Hence, $\Phi : {}_{G,A}\mathbb{M} \to {}_{\Gamma}\mathbb{M}$ is left adjoint to $(?)_{\mathrm{rat}}$, and we identify (G, A)-modules and rational Γ-modules via Φ.

Let $M, N \in {}_{\Gamma}\mathbb{M}$ and $V \in {}_{U}\mathbb{M}$. As in the case of (G, A)-modules, $M \otimes V$ is a Γ-module, and $M \otimes_A N$ is a quotient Γ-module of $M \otimes N$. Moreover, $\mathrm{Hom}(M, V)$ is a Γ-module, and $\mathrm{Hom}_A(M, N)$ is a Γ-submodule of $\mathrm{Hom}(M, N)$, without assuming that M is A-finite. Another advantage of Γ-modules compared with (G, A)-modules is, the category of Γ-modules ${}_{\Gamma}\mathbb{M}$ has enough projectives. Moreover, the projective limit in ${}_{\Gamma}\mathbb{M}$ is compatible with the forgetful functor ${}_{\Gamma}\mathbb{M} \to {}_{A}\mathbb{M}$.

If $M, N \in {}_{G,A}\mathbb{M}$ and $V \in {}_{G}\mathbb{M}$, then $\Phi(M \otimes V) \cong \Phi M \otimes \Phi V$. In particular, we have $\Phi(M \otimes_A N) \cong \Phi M \otimes_A \Phi N$. If M is A-finite moreover, then we have $\Phi \operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_A(\Phi M, \Phi N)$.

Lemma 6.1 *A Γ-projective (resp. Γ-injective) module is A-projective (resp.. A-injective).*

Let $M, N \in {}_{\Gamma}\mathbb{M}$. By the lemma, we have an isomorphism of Γ-modules

$$L_i(? \otimes_A N)(M) \cong L_i(M \otimes_A ?)(N),$$

and both are isomorphic to $\mathrm{Tor}_i^A(M, N)$ as A-modules, so we have a canonical Γ-module structure of $\mathrm{Tor}_i^A(M, N)$. If G is IFP, and both M and N are rational, then $\mathrm{Tor}_i^A(M, N)$ is rational for $i \geq 0$, and this structure coincide with the one we defined before. Similarly, we have an isomorphism

$$R^i \operatorname{Hom}_A(M, ?)(N) \cong R^i \operatorname{Hom}_A(?, N)(M),$$

and both Γ-modules are isomorphic to $\mathrm{Ext}_A^i(M, N)$ as A-modules, so we have a canonical Γ-module structure of $\mathrm{Ext}_A^i(M, N)$. If both M and N are rational and M is A-finite, then $\mathrm{Ext}_A^i(M, N)$ is rational, and this (G, A)-module agrees with the one we have defined before.

Any R-module is a U-module via the restriction through the R-algebra map $\varepsilon_U : U \to R$. The R-submodule $\mathrm{Hom}_U(R, V)$ of a U-module V is called the invariance of V, and denoted by V^U. For a rational U-module V, we have $V^G = V^U$. Note also that $\mathrm{Hom}_A(M, N)^U = \mathrm{Hom}_{\Gamma}(M, N)$.

Lemma 6.2 *Let $M \in {}_{\Gamma}\mathbb{M}$, $N \in {}_{\Gamma}\mathbb{M}$ and $V \in {}_{U}\mathbb{M}$. Then, the canonical isomorphism*

$$\mathrm{Hom}_A(M \otimes V, N) \cong \mathrm{Hom}(V, \mathrm{Hom}_A(M, N))$$

is a Γ-isomorphism. In particular, the isomorphism

$$\mathrm{Hom}_\Gamma(M \otimes V, N) \cong \mathrm{Hom}_U(V, \mathrm{Hom}_A(M, N))$$

is induced, taking the U-invariance.

By the lemma, if I is an injective Γ-module and M is an R-flat Γ-module, then $\mathrm{Hom}_A(M, I)$ is U-injective. So a spectral sequence analogous to Proposition 3.5 is obtained.

7 Linearlizable quasi-coherent $(G, \mathcal{O}_X)$-modules, $\underline{\mathrm{Ext}}_{\mathcal{O}_X}$, and $\underline{\mathrm{Tor}}^{\mathcal{O}_X}$

In this section, R denotes a noetherian commutative ring, and $G = \operatorname{Spec} H$ denotes a flat affine R-group scheme of finite type. X denotes a locally noetherian G-action (an R-scheme together with an R-morphism $a_X : X \times G \to X$ which satisfies the usual axiom of the right action). We denote the product of G by μ_G.

The following definition is essentially due to Mumford-Fogarty [44].

Definition 7.1 A G-linearlized $\mathcal{O}_X$-module is an $\mathcal{O}_X$-module M together with an $\mathcal{O}_{X\times G}$-isomorphism $\phi : a_X^* M \to p_1^* M$, satisfying the condition: The isomorphism

$$[a_X \circ (a_X \times 1_G)]^* M = [a_X \circ (1_X \times \mu_G)]^* M \xrightarrow{(1_X \times \mu_G)^* \phi} [p_1 \circ (1_X \times \mu_G)]^* M$$

agrees with the composite map

$$[a_X \circ (a_X \times 1_G)]^* M \xrightarrow{(a_X \times 1_G)^* \phi} [p_1 \circ (a_X \times 1_G)]^* M = [a_X \circ p_{12}]^* M \xrightarrow{p_{12}^* \phi}$$
$$[p_1 \circ p_{12}]^* M = [p_1 \circ (1_X \times \mu_G)]^* M.$$

We give another definition, which will appear to be equivalent for quasi-coherent $\mathcal{O}_X$-modules, of G-linearlization.

First, assume that $X = \operatorname{Spec} A$ is affine. Then, an A-module M corresponds to a quasi-coherent $\mathcal{O}_X$-module $\tilde{M}$. Moreover, $M \otimes H$ equipped with the $A \otimes H$-module structure by $(a \otimes h)(m \otimes h') := am \otimes hh'$ corresponds to $p^* \tilde{M}$, and $M \otimes H$ equipped with the $A \otimes H$-module structure by $(a \otimes h)(m \otimes h') := \sum_{(a)} a_{(0)} m \otimes (S a_{(1)}) hh'$ corresponds to $a^* \tilde{M}$, where S denotes the antipode of H.

A (G, A)-module M yields a canonical map $\Box : M \otimes H \to M \otimes H$ defined by $m \otimes h \mapsto \sum_{(m)} m_{(0)} \otimes m_{(1)} h$, and the corresponding map $\phi := \tilde{\Box}$ satisfies the commutativity. Thus, a (G, A)-module M gives a G-linearlized quasi-coherent $\mathcal{O}_X$-module $(\tilde{M}, \phi)$.

Conversely, a G-linearlized quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ gives an A-module $M := \Gamma(X, \mathcal{M})$, and the map $\omega : M \to M \otimes H$ derived from the map

$$\mathcal{M} \to a_* a^* \mathcal{M} \xrightarrow{a_*\phi} a_* p^* \mathcal{M}$$

gives the (G, A)-module structure of M, and this correspondence gives an equivalence.

This correspondence is extended to the case X is not necessarily affine in a natural way.

We define $\mathcal{G}_X$ to be the category of noetherian (G, X)-schemes of finite type. Namely, an object in $\mathcal{G}_X$ is a quasi-compact G-scheme Y together with a G-morphism $Y \to X$ locally of finite type. Note that $Y \in \mathcal{G}_X$ implies Y is noetherian and of finite type over X. A $\mathcal{G}_X$-morphism $Y \to Y'$ is a G-morphism X-morphism. $\mathcal{G}_X^{\mathrm{fl}}$ denotes the full subcategory of $\mathcal{G}_X$ of flat X-schemes, affine over $\operatorname{Spec} \mathbb{Z}$.

For a category $\mathcal{C}$, a $\mathcal{C}$-valued (G, X)-functor (resp. $(G, X)^{\mathrm{fl}}$-functor) is a contravariant functor from $\mathcal{G}_X$ (resp. $\mathcal{G}_X^{\mathrm{fl}}$) to $\mathcal{C}$ by definition. Note that the category $\mathcal{G}_X$ is skeletally small, so the category of (G, X)-functors with valued in any $\mathcal{C}$ is a category with small hom sets.

Definition 7.2 Assume that $\mathcal{C}$ has finite projective limits. A $\mathcal{C}$-valued (G, X)-functor F is called a (G, X)-faisceau if it satisfies the conditions:

1 If $Y = Y_1 \coprod \cdots \coprod Y_r \in \mathcal{G}_X$ is a finite disjoint union of (G, X)-schemes, then the canonical maps $F(Y) \to F(Y_i)$ yield an isomorphism $F(Y) \cong \prod_i F(Y_i)$.

2 If $Y \to Y'$ is a morphism in $\mathcal{G}_X$ which is faithfully flat, then the map $F(Y') \to F(Y)$ is a difference kernel of two maps $F(f_1)$ and $F(f_2)$, where $f_i : Y' \times_Y Y' \to Y'$ is the ith projection.

$(G, X)^{\mathrm{fl}}$-faisceau is defined similarly.

The functor $Y \mapsto \Gamma(Y, \mathcal{O}_Y)$ is a faisceau of R-algebras, which we denote by $\mathcal{O}_X$, by abuse of notation. Moreover, $\mathcal{O}_X$ is a faisceau of G-algebras. Let $Y \in \mathcal{G}_X$. The action $a_Y : Y \times G \to Y$ yields a G-algebra map

$$\omega : \mathcal{O}_X(Y) = \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y \times G, \mathcal{O}_{G \times Y}) = \mathcal{O}_X(Y) \otimes H$$

by [26, Proposition 9.3]. It is easy to see that ω defines a G-module structure of $\mathcal{O}_X(Y)$, and $\mathcal{O}_X(Y)$ is a G-algebra in fact.

An $\mathcal{O}_X$-module M is said to be *quasi-coherent*, by abuse of terminology, if for any $A \to B$ such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is a morphism in $\mathcal{G}_X$, the canonical B-module map $B \otimes_A M(A) \to M(B)$ is an isomorphism. M is called *coherent* if it is quasi-coherent and $M(A)$ is A-finite for any A with

$\operatorname{Spec} A \in \mathcal{G}_X$. A quasi-coherent $\mathcal{O}_X$-module is a faisceau as a (G,X)-functor [40, Corollary 1.6].

For a quasi-coherent sheaf (for the usual Zariski topology) M on X, we define a (G,X)-functor of $\mathcal{O}_X$-modules $W(M)$ by $W(M)(Y) = \Gamma(Y, f^*M)$ for $Y \in \mathcal{G}_X$, where $f : Y \to X$ is the structure map. It is easy to check that $W(M)$ is a quasi-coherent faisceau. These definitions are done for $(G,X)^{\mathrm{fl}}$-functors similarly.

Now we give an alternative definition of $(G,\mathcal{O}_X)$-modules.

Definition 7.3 A $(G,\mathcal{O}_X)$-module (faisceau) M is a collection of data:

1 M is an $\mathcal{O}_X$-module (faisceau).

2 For $Y \in \mathcal{G}_X$, $M(Y)$ is a $(G,\mathcal{O}_X(Y))$-module.

3 For each $Y \to Y'$, the canonical map $M(Y') \to M(Y)$ is a G-linear map.

A $(G,\mathcal{O}_X)^{\mathrm{fl}}$-module (faisceau) is defined similarly.

Let M be a quasi-coherent $(G,\mathcal{O}_X)$-module. Then, M is a quasi-coherent $(G,\mathcal{O}_X)^{\mathrm{fl}}$-module by restriction.

Let M be a quasi-coherent $(G,\mathcal{O}_X)^{\mathrm{fl}}$-module. We show that a G-linearlized quasi-coherent $\mathcal{O}_X$-module corresponds in a natural way.

First, note that X is covered by a G-stable open subschemes (for the definition of G-stability of subschemes, see the next section). Let V be a quasi-compact open subset of X. Then, the image VG of the action $V \times G \to X$ $((v,g) \mapsto vg)$ is a quasi-compact open subset of X, and is G-stable. So, if (V_i) is an open covering of X, then (V_iG) is the desired one.

Hence, we may assume that X is quasi-compact without loss of generality. Let (V_i) be a finite covering of X with each V_i affine. We set $Y := (\coprod_i V_i) \times G$, and we regard Y to be a principal G-action. Then, the action $Y \to X$ preserves G-action, and we have that $Y \in \mathcal{G}_X^{\mathrm{fl}}$.

Note that to give a quasi-coherent $\mathcal{O}_X$-module is the same as to give a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{M}$ and an isomorphism $\psi : \pi_1^*\mathcal{M} \to \pi_2^*\mathcal{M}$ which satisfies the cocycle condition $\pi_{31}^*\psi = \pi_{32}^*\psi \circ \pi_{21}^*\psi$, where $\pi_i : Y \times_X Y \to Y$ is the ith projection, and $\pi_{ji} : Y \times_X \times_X Y \to Y \times_X Y$ is the map given by $(y_1, y_2, y_3) \mapsto (y_i, y_j)$, see [40, p.19].

The map ψ is defined as the restriction map $\mathcal{M}(\tau)$, where $\tau : Y \times_X Y \to Y \times_X Y$ is given by $\tau(y_1, y_2) = (y_2, y_1)$, as τ is a morphism in $\mathcal{G}_X^{\mathrm{fl}}$. Thus, $\mathcal{M}$ gives a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}_0$.

As $\psi : \mathcal{M}(\pi_1) \to \mathcal{M}(\pi_2)$ is G-linear by the definition of $(G,\mathcal{O}_X)^{\mathrm{fl}}$-module, it is easy to verify that $\phi : a^*\mathcal{M} \to p^*\mathcal{M}$ induces $\phi_0 : a^*\mathcal{M}_0 \to p^*\mathcal{M}_0$, and this defines a G-linearlization of $\mathcal{M}_0$.

Finally, if M is a quasi-coherent G-linearlized $\mathcal{O}_X$-module, then f^*M is a G-linearlized $\mathcal{O}_Y$-module for any $f : Y \to X$ with $Y \in \mathcal{G}_X$. It is easy to see that $f^*M \to a_*a^*f^*M \xrightarrow{a_*\phi} a_*p^*f^*M$ gives the coaction $W(M)(Y) \to W(M)(Y) \otimes H$, which is natural on Y. From the linearlizable sheaf condition, we can check that $W(M)$ is a quasi-coherent $(G, \mathcal{O}_X)$-module.

Thus, a quasi-coherent $(G, \mathcal{O}_X)$-module, a quasi-coherent $(G, \mathcal{O}_X)^{\mathrm{fl}}$-module, and a G-linearlized quasi-coherent $\mathcal{O}_X$-module are one and the same thing by the correspondence above.

Let M be a coherent $(G, \mathcal{O}_X)$-module, and N a quasi-coherent $(G, \mathcal{O}_X)$-module. Then, we define the $(G, \mathcal{O}_X)^{\mathrm{fl}}$-module $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(M, N)$ by

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(M, N)(A) := \mathrm{Ext}^i_A(M(A), N(A)).$$

It is easy to see that $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(M, N)$ is quasi-coherent, and can be grasped as a $(G, \mathcal{O}_X)$-module.

Assume that G is IFP. Then, we can define a G-linearlized structure of $\underline{\mathrm{Tor}}_i^{\mathcal{O}_X}(M, N)$ for quasi-coherent $(G, \mathcal{O}_X)$-modules M and N, in a similar way.

8 Stability of subschemes

In this section, R, G, and X are the same as in the last section. Moreover, we assume that G is R-smooth with connected geometric fibers.

Note that if Y is an irreducible (resp. reduced, integral) locally noetherian R-scheme, then so is $Y \times G$.

We denote the action (resp. the first projection) $X \times G \to X$ by $a = a_X$ (resp. $p = p_X = p_1$). The isomorphism $X \times G \to X \times G$ given by $(x, g) \mapsto (xg, g)$ is denoted by h_X. Note that it holds $p_X \circ h_X = a_X$.

For a subscheme Y of X, we say that Y is *G-stable* when the action $Y \times G \to X$ ($(y, g) \mapsto yg$) factors through $Y \hookrightarrow X$. If Y is G-stable, then Y has a unique G-action such that $Y \hookrightarrow X$ is a G-morphism. We say that $x \in X$ is G-stable when $\overline{\{x\}}$, as an integral subscheme of X, is G-stable.

Lemma 8.1 *The following hold:*

1 *Let Y be a closed subscheme of X, and Y^* denotes the closure of the image of the action $Y \times G \to X$. Then, Y^* is the smallest G-stable closed subscheme containing Y. If Y is irreducible (resp. reduced), then so is Y^*.*

2 *For a reduced closed subscheme Y of X, the following are equivalent.*

a *Y is G-stable*

b *Any irreducible component (maximal integral closed subscheme) of Y is G-stable*

c *$X - Y$ is G-stable.*

3 *When $X = X_0 \times_R G$ is a principal G-bundle, then any G-stable open set of X is of the form $V \times_R G$, where V is an open set of X_0.*

4 *If $\varphi : X \to X'$ is a G-morphism of locally of finite type between locally noetherian R-schemes with G-actions, then the flat locus* $\mathrm{Flat}(\varphi)$ *is a G-stable open subset of X.*

5 *If the Cohen-Macaulay (resp. Gorenstein, local complete intersection, regular) locus is an open set of X, then it is G-stable.*

6 *If M is a G-linearlized coherent $\mathcal{O}_X$-module, then, for any $r \geq 0$, the set $\{x \in X \mid M_x$ is $\mathcal{O}_{X,x}$-free of rank $r\}$ is a G-stable open set of X. In particular,* $\mathrm{Supp}\, M$ *(as a reduced closed subscheme) is a G-stable closed subscheme of X.*

For $x \in X$, we denote the generic point of $\overline{\{x\}}^*$ by x^*. x is G-stable if and only if $x = x^*$.

Corollary 8.2 *Assume that the Cohen-Macaulay (resp. Gorenstein, local complete intersection, regular) locus U of X is an open subset of X (e.g., X is excellent). If U contains all of the G-stable points of X, then $U = X$.*

Proof. Assume the contrary. Then, $Y := X - U$ with the reduced closed subscheme structure is a non-empty G-stable closed subscheme of X. It follows that the generic point η of any irreducible component of Y is G-stable, but $\eta \notin U$. This is a contradiction. □

This corollary remains true without assuming that X is excellent, as will be seen in the next section.

9 First application — Matijevic-Roberts type theorem

As in the last section, R is a noetherian commutative ring, G a smooth affine R-group scheme with connected geometric fibers, X a locally noetherian G-action. The following is the first application of our construction.

Theorem 9.1 *Let $x \in X$ and $\mathcal{M}$ a coherent $(G, \mathcal{O}_X)$-module. Then, the following hold:*

1 *If $\mathcal{O}_{X,x^*}$ is a regular local ring (resp. a complete intersection), then so is $\mathcal{O}_{X,x}$. In particular, if X is regular (resp. a complete intersection) at any stable point, then X is regular (resp. a local complete intersection).*

2 *If $\mathcal{M}_{x^*}$ is Gorenstein (resp. Cohen-Macaulay, free), then so is $\mathcal{M}_x$. In particular, if $\mathcal{M}$ is Gorenstein (resp. Cohen-Macaulay, free) at any stable point, then $\mathcal{M}$ is Gorenstein (resp. Cohen-Macaulay, locally free).*

In [39], J. Matijevic and P. Roberts proved the following theorem, proving Nagata's conjecture [45] affirmatively.

Theorem 9.2 (Matijevic-Roberts) *Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a commutative $\mathbb{Z}$-graded noetherian ring. If $A_\mathfrak{p}$ is Cohen-Macaulay for every graded prime ideal $\mathfrak{p}$ of A, then A is Cohen-Macaulay.*

Nagata's conjecture was proved also by Hochster and Ratliff [30].

After a while, Matijevic-Roberts theorem was generalized to the theorem on Gorenstein and regular properties and any finitely generated module M, see J. Matijevic [38], and Y. Aoyama and S. Goto [3].

Moreover, these generalized versions were extended to the case of $\mathbb{Z}^n$-graded rings by S. Goto and K.-i. Watanabe [24]:

Theorem 9.3 *Let A be a $\mathbb{Z}^n$-graded commutative noetherian ring, and M a finitely generated graded A-module. Let $\mathfrak{p} \in \operatorname{Spec} A$, and we denote by $\mathfrak{p}^*$ the maximal graded (prime) ideal contained in $\mathfrak{p}$. Then, the following holds:*

1 *If $A_{\mathfrak{p}^*}$ is a regular local ring, then so is $A_\mathfrak{p}$.*

2 *If $M_{\mathfrak{p}^*}$ is Cohen-Macaulay (resp. Gorenstein), then so is $M_\mathfrak{p}$.*

A similar result on the complete intersection property was obtained by Cavaliere and Niesi [15] after a while.

Theorem 9.1 is yet another generalization of these theorems, see Example 3.4. The rest of this section is devoted to describe how Theorem 9.1 is proved as an application of G-linearlization.

Lemma 9.4 *Let Y be a closed integral subscheme of X with the generic point η. Then, the following hold:*

1 *$\mathcal{O}_{Y^*,\eta}$ is a regular local ring*

2 *Let $\mathcal{M}$ be a coherent $(G, \mathcal{O}_{Y^*})$-module. Then, $\mathcal{M}_\eta$ is $\mathcal{O}_{Y^*,\eta}$-free. If $\mathcal{N}$ is a coherent $(G, \mathcal{O}_X)$-module moreover, then $\operatorname{Ext}^i_{\mathcal{O}_{X,\eta}}(\mathcal{M}_\eta, \mathcal{N}_\eta)$ is also $\mathcal{O}_{Y^*,\eta}$-free for any $i \geq 0$.*

Proof. Consider the morphism $\varphi : Y \times G \to Y^*$ defined by $(y, g) \mapsto yg$. It is clear that φ is a G-morphism, and $\operatorname{Flat}(\varphi)$ is a G-stable open subset of $Y \times G$. By Lemma 8.1, **3**, $\operatorname{Flat}(\varphi)$ is of the form $F \times G$ for some open subset

F of Y. As φ is dominating and both $Y \times G$ and Y^* are integral, we have that $\eta \in F$. Hence, the composite morphism

$$\psi : \operatorname{Spec} \kappa(\eta) \times G \to Y \times G \xrightarrow{\varphi} Y^*$$

is flat, as the first arrow is obviously flat, and the second one is flat at the image of the first. As $\operatorname{Spec} \kappa(\eta) \times G$ is $\kappa(\eta)$-smooth and the unit element $\operatorname{Spec} \kappa(\eta) \times \{e\}$ is mapped to $\eta \in Y^*$ by ψ, the local ring $\mathcal{O}_{Y^*,\eta}$ is a regular local ring, and **1** is proved.

We prove **2**. By Lemma 8.1, **6**, the free locus U of $\mathcal{M}$ is a G-stable open subset of Y^*, and is non-empty, as it contains the generic point of Y^*. If we have $Y \subset Y^* - U$, then we have $Y^* \subset Y^* - U$, as $Y^* - U$ with the reduced structure is G-stable. This is a contradiction, and we have $\eta \in U$. This proves that $\mathcal{M}_\eta$ is $\mathcal{O}_{Y^*,\eta}$-free. Applying this observation to the coherent $(G, \mathcal{O}_{Y^*})$-module $\underline{\operatorname{Ext}}^i_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$, we are done. □

Now the proof of Theorem 9.1 is reduced to a purely ring-theoretic argument. Let $x \in X$ as in the theorem, and we set $Y := \overline{\{x\}}$. We set $B := \mathcal{O}_{X,x}$, $B/P = \mathcal{O}_{Y^*,x}$, and $M := \mathcal{M}_x$. We know the following:

1 B/P is a regular local ring (Lemma 9.4, **1**).

2 B is normally flat along P, as the defining ideal sheaf $\mathcal{P}$ of Y^* is a coherent $(G, \mathcal{O}_X)$-module and we have $(\mathcal{P}^n/\mathcal{P}^{n+1})_x = P^n/P^{n+1}$ (see Lemma 9.4, **2**).

3 Similarly, M is normally flat along P (i.e., $\operatorname{Gr}_P M$ is B/P-flat).

4 $\operatorname{Ext}^i_B(B/P, B/P)$ is B/P-free for $i \geq 0$.

5 $\operatorname{Ext}^i_B(B/P, M)$ is B/P-free for $i \geq 0$.

4 B_P is a regular local ring (resp. complete intersection) by the assumption of the theorem.

6 M_P is Cohen-Macaulay (resp. Gorenstein, free) by assumption.

The second part of Theorem 9.1 is reduced to the following:

Lemma 9.5 *Let $(B, \mathfrak{n})$ be a noetherian local ring, M a finite B-module. Let P be a prime ideal of B. Assume that M is normally flat along P. Then, the following hold:*

1 *If $M \neq 0$, then we have* $\dim M = \dim M_P + \dim B/P$.

2 *Assume that B is normally flat along P. Then, M is B-free if and only if M_P is B_P-free.*

If $\operatorname{Ext}^i_B(B/P, M)$ *is* B/P*-free for* $i \geq 0$ *moreover, then, the following hold:*

3 *We have* $\operatorname{depth} M = \operatorname{depth} M_P + \operatorname{depth} B/P$. *In particular,* M *is Cohen-Macaulay if and only if so are* M_P *and* B/P. *If* M *is Cohen-Macaulay, then*

$$\operatorname{type} M = \operatorname{type} M_P \cdot \operatorname{type} B/P,$$

where type *denotes the Cohen-Macaulay type. In particular,* M *is Gorenstein if and only if so are* M_P *and* B/P.

4 *Assume that* B/P *is Gorenstein. Then, we have* $\mu_B^{i+\dim B/P}(M) = \mu^i_{B_P}(M_P)$, *where* μ^i *denotes the Bass number.*

The first part of Theorem 9.1 is reduced to the following:

Lemma 9.6 *Let* $(B, \mathfrak{n})$ *be a local ring,* P *a prime ideal of* B. *Assume that* B *is normally flat along* P, *and* B/P *is regular. Then the following hold:*

1 B *is regular if and only if* B_P *is regular.*

2 *If* $\operatorname{Ext}^i_B(B/P, B/P)$ *is* B/P*-free for* $i \geq 0$ *moreover, then* B *is a complete intersection if and only if* B_P *is a complete intersection.*

These lemmas above are already proved or essentially proved and used in [24], except that the complete intersection property was not treated. Important theorems on normal flatness are proved in [49], which includes some part of these lemmas.

10 Auslander-Buchweitz theory

In [5], an important theory on abelian categories, containing Cohen-Macaulay approximations as an important example, was developed.

For a category $\mathcal{C}$, the set of objects in $\mathcal{C}$ is also denoted by $\mathcal{C}$. A subset $\mathcal{X}$ of $\mathcal{C}$ and the full subcategory $\mathcal{X}$ of $\mathcal{C}$ will be sometimes identified. For a category $\mathcal{C}$, both the set of null objects in $\mathcal{C}$ and one of the null objects in $\mathcal{C}$ will be denoted by the same symbol 0, which will not cause a confusion. For a functor $F : \mathcal{C} \to \mathcal{C}'$ and $\mathcal{S} \subset \mathcal{C}$, we denote the set of objects in $\mathcal{C}'$ which is isomorphic to $F(S)$ for some $S \in \mathcal{S}$ by $F(\mathcal{S})$.

Let $\mathcal{A}$ be an abelian category. A morphism $p : M \to N$ in $\mathcal{A}$ is said to be *right minimal* if for any $\varphi \in \operatorname{End}_{\mathcal{A}}(M)$, we have that $p\varphi = p$ implies that φ is an isomorphism. Left minimality is the dual notion of the right minimality. That is to say, a morphism in $\mathcal{A}$ is said to be left minimal, if it is right minimal as a morphism in the opposite category $\mathcal{A}^{\mathrm{op}}$ of $\mathcal{A}$.

Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$. A morphism $X \to M$ in $\mathcal{A}$ is called a *right* $\mathcal{X}$*-approximation* if $X \in \mathcal{X}$, and for any $X' \in \mathcal{X}$ and any $g \in \mathcal{A}(X', M)$, there exists some $h \in \mathcal{A}(X', X)$ such that $fh = g$. Left approximation is the dual notion of right approximation. A right (resp. left) minimal right (resp. left) $\mathcal{X}$-approximation is called a right (resp. left) minimal $\mathcal{X}$-approximation, for short. A right minimal $\mathcal{X}$-approximation of M is unique up to isomorphisms in $\mathcal{A}/M$, if it exists.

We say that $\mathcal{B} \subset \mathcal{A}$ has right (resp. left) (minimal) $\mathcal{X}$-approximations if any $M \in \mathcal{B}$ has a right (resp. left) (minimal) $\mathcal{X}$-approximation.

Lemma 10.1 *If*

$$0 \to Y \xrightarrow{i} X \xrightarrow{p} M \to 0$$

is an exact sequence in $\mathcal{A}$ and $\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{X}, Y) = 0$, then we have that p is a right $\mathcal{X}$-approximation of M.

Proof. For $X' \in \mathcal{X}$, we have that

$$\mathcal{A}(X', X) \to \mathcal{A}(X', M) \to \operatorname{Ext}^1_{\mathcal{A}}(X', Y) = 0$$

is exact. Hence, for any $p' \in \mathcal{A}(X', M)$, there is $h \in \mathcal{A}(X', X)$ such that $ph = p'$. $\square$

We say that a ring E is *semiperfect* if $E/\operatorname{rad} E$ is semisimple, and if for any idempotent $\bar{e} \in E/\operatorname{rad} E$, there exists some idempotent $e \in E$ which specializes to $\bar{e}$. E is semiperfect if and only if any finitely generated E-module admits a projective cover (essential epimorphism from a projective E-module). A local ring is semiperfect. A module-finite algebra over a Henselian local ring is semiperfect. The following is a slight modification of [41, Theorem 3.4].

Proposition 10.2 *Let $\mathcal{A}$ be an abelian category, and*

$$0 \to Y \xrightarrow{i} X \xrightarrow{p} M$$

an exact sequence in $\mathcal{A}$. Consider the following two conditions:

1 *p is right minimal*

2 *The image of i does not contain any non-zero direct summand of X.*

*We have **1**⇒**2** in general. If $\operatorname{End}_{\mathcal{A}} X$ is semiperfect, then there exists some decomposition $X = X_0 \oplus X_1$ such that $X_0 \subset \operatorname{Im} i$ and that $X_1 \to M$ is right minimal. In particular, we have **2**⇒**1**.*

Corollary 10.3 *Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$ closed under direct summands. Assume that any object in $\mathcal{X}$ has a semiperfect endomorphism ring. If $M \in \mathcal{A}$ has a right (resp. left) $\mathcal{X}$-approximation, then M has a unique (up to isomorphisms in $\mathcal{A}/M$ (resp. in $M/\mathcal{A}$)) right (resp. left) minimal $\mathcal{X}$-approximation.*

Let $\mathcal{X}$ be a full subcategory of $\mathcal{A}$. We define some full subcategories related to $\mathcal{X}$ as follows:

add $\mathcal{X}$: Consists of $M \in \mathcal{A}$ which is isomorphic to a direct summand of a finite direct sum of objects in $\mathcal{X}$. Note that $0 \in \operatorname{add} \mathcal{X} \supset \mathcal{X}$, and $\operatorname{add} \mathcal{X}$ is closed under isomorphisms, finite direct sums, and direct summands. If $\mathcal{X}$ is closed under extensions, then so is $\operatorname{add} \mathcal{X}$.

$\mathcal{F}(\mathcal{X})$: Consists of $M \in \mathcal{A}$ such that there exists some $r \geq 0$ and a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that M_i/M_{i-1} is isomorphic to an object in $\mathcal{X}$ for $i = 1, 2, \ldots, r$. Note that $0 \in \mathcal{F}(\mathcal{X}) \supset \mathcal{X}$, and $\mathcal{F}(\mathcal{X})$ is closed under isomorphisms and extensions. Note also that $\mathcal{F}(\mathcal{X})$ is not closed under direct summands in general even if so is $\mathcal{X}$.

$\hat{\mathcal{X}}$: Consists of $M \in \mathcal{A}$ such that there exists some resolution of finite length

$$0 \to X_h \to \cdots \to X_1 \to X_0 \to M \to 0$$

with $X_i \in \mathcal{X}$. For $M \in \hat{\mathcal{X}}$, the smallest integer among the lengths h of $\mathcal{X}$-resolutions of M is called the $\mathcal{X}$-resolution dimension, and is denoted by $\mathcal{X}\text{-resol.dim}\, M$. For $M \notin \hat{\mathcal{X}}$, we define $\mathcal{X}\text{-resol.dim}\, M = \infty$.

$\mathcal{X}^{\perp}$: For $M \in \mathcal{A}$, we set

$$\mathcal{X}\text{-inj.dim}\, M := \sup(\{i \geq 0 \mid \operatorname{Ext}^i_{\mathcal{A}}(\mathcal{X}, M) \neq 0\} \cup \{0\}),$$

and call $\mathcal{X}\text{-inj.dim}\, M$ the $\mathcal{X}$-injective dimension of M. We say that M is $\mathcal{X}$-injective if $\mathcal{X}\text{-inj.dim}\, M = 0$. The full subcategory of $\mathcal{X}$-injective objects is denoted by $\mathcal{X}^{\perp}$. Similarly, $\mathcal{X}$-projective dimension, denoted by $\mathcal{X}\text{-proj.dim}\, M$, and $\mathcal{X}$-projective objects are defined. The category of $\mathcal{X}$-projective objects is denoted by ${}^{\perp}\mathcal{X}$. Note that $\mathcal{X}^{\perp}$ is closed under isomorphisms, extensions, direct summands, and monocokernels. Similarly, ${}^{\perp}\mathcal{X}$ is closed under isomorphisms, extensions, direct summands, and epikernels.

We say that $\omega \subset \mathcal{X}$ is a cogenerator of $\mathcal{X}$ if for any $X \in \mathcal{X}$, there exists some exact sequence

$$0 \to X \to T \to X' \to 0$$

such that $T \in \omega$ and $X' \in X$. If we have $\omega \subset \mathcal{X}^{\perp}$ moreover, then we say that ω is an *injective cogenerator* of $\mathcal{X}$.

The following is a digest of a series of theorems due to M. Auslander and R. O. Buchweitz [5]. Some of them are taken from [6] in the context of representations of algebras.

Theorem 10.4 (Auslander-Buchweitz) *Let $\mathcal{A}$ be an abelian category, and $\mathcal{X}$, $\mathcal{Y}$ and ω full subcategories of $\mathcal{A}$. Assume that the following conditions are satisfied:*

AB 1 *$\mathcal{X}$ is closed under extensions, direct summands and epikernels.*

AB 2 *$\mathcal{Y} \subset \hat{\mathcal{X}}$, and $\mathcal{Y}$ is closed under extensions, direct summands and monocokernels.*

AB 3 *It holds $\omega = \mathcal{X} \cap \mathcal{Y}$, and ω is an injective cogenerator of $\mathcal{X}$.*

Then, the following hold:

1 $\hat{\omega} = \mathcal{Y}$

2 *If $\omega' \subset \mathcal{X}$ is an injective cogenerator of $\mathcal{X}$, then we have* $\operatorname{add} \omega' = \omega$.

3 *If we have $M \in \hat{\mathcal{X}}$, then the following hold:*

i *($\mathcal{X}$-approximation) There is an exact sequence in $\mathcal{A}$*

$$0 \to Y \to X \xrightarrow{p} M \to 0 \tag{2}$$

such that $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

ii *($\mathcal{Y}$-hull) There is an exact sequence in $\mathcal{A}$*

$$0 \to M \xrightarrow{\iota} Y' \to X' \to 0 \tag{3}$$

such that $X' \in \mathcal{X}$ and $Y' \in \mathcal{Y}$.

4 *For $M \in \hat{\mathcal{X}}$, the following are equivalent:*

i $M \in \mathcal{X}$ **ii** $M \in {}^{\perp}\mathcal{Y}$ **ii'** $\operatorname{Ext}^1_{\mathcal{A}}(M, \mathcal{Y}) = 0$ **iii** $M \in {}^{\perp}\omega$.

In particular, p in the exact sequence in (2) is a right $\mathcal{X}$-approximation of M.

5 *For $N \in \hat{\mathcal{X}}$, the following are equivalent.*

i $N \in \mathcal{Y}$ **ii** $N \in \mathcal{X}^{\perp}$ **ii'** $\operatorname{Ext}^1_{\mathcal{A}}(\mathcal{X}, N) = 0$.

In particular, ι in (3) is a left $\mathcal{Y}$-approximation of M.

6 *For $M \in \hat{\mathcal{X}}$, we have that* $\mathcal{X}$-resol.dim(M) = $\mathcal{Y}$-proj.dim(M) = ω-proj.dim(M).

7 *For $Y \in \mathcal{Y}$, we have* ω-resol.dim(Y) = $\mathcal{X}$-resol.dim(Y).

8 *If* $0 \to M_1 \to M_2 \to M_3 \to 0$ *is an exact sequence in* $\mathcal{A}$ *and two of* M_1, M_2 *and* M_3 *belong to* $\hat{\mathcal{X}}$, *then all of them belong to* $\hat{\mathcal{X}}$.

We call the exact sequence (2) the $\mathcal{X}$*-approximation* of M. It is said to be *minimal* when p in (2) is right minimal. We call the exact sequence (3) the $\mathcal{Y}$*-hull* of M. It is said to be *minimal* when ι in (3) is left minimal.

Definition 10.5 If the conditions **AB 1–3** are satisfied, then the triple $(\mathcal{X}, \mathcal{Y}, \omega)$ is called a *weak Auslander-Buchweitz context* (weak AB context, for short) in the abelian category $\mathcal{A}$. If it holds $\hat{\mathcal{X}} = \mathcal{A}$ moreover, then it is called an *Auslander-Buchweitz context* (AB context, for short) in $\mathcal{A}$.

If $(\mathcal{X}, \mathcal{Y}, \omega)$ is an AB context in $\mathcal{A}$, then any of $\mathcal{X}$, $\mathcal{Y}$ and ω determine others.

We exhibit some examples of weak AB contexts in commutative ring theory. Let R be a noetherian commutative ring, and $\mathcal{A}_R = {}_R\mathbb{M}_f$ denote the category of finitely generated R-modules.

Projective modules We define

$$\mathcal{X}_R^{\text{pro}} = \omega_R^{\text{pro}} := \operatorname{add} R = {}^{\perp}\mathcal{A}_R = \{R\text{-finite projective modules}\},$$

and

$$\mathcal{Y}_R^{\text{pro}} := \hat{\mathcal{X}}_R^{\text{pro}} = \{R\text{-finite modules of finite projective dimension}\}.$$

Then, $(\mathcal{X}_R^{\text{pro}}, \mathcal{Y}_R^{\text{pro}}, \omega_R^{\text{pro}})$ is a weak AB context with $\hat{\mathcal{X}} = \mathcal{Y}_R$. It is an AB context if and only if R is regular.

Modules of Gorenstein dimension 0 Let $M \in \mathcal{A}_R$. We say that M is of Gorenstein dimension 0 if M is reflexive (i.e., the canonical map $M \to M^{**}$ is an isomorphism), and $M, M^* \in {}^{\perp}\omega_R^{\text{pro}}$, see [4]. We set

$$\mathcal{X}_R^{\text{gor}} := \{R\text{-finite modules of Gorenstein dimension } 0\}.$$

Then, we have that $(\mathcal{X}_R^{\text{gor}}, \mathcal{Y}_R^{\text{pro}}, \omega_R^{\text{pro}})$ is a weak AB context, which is essentially proved by Auslander and Bridger [4]. This weak AB context is an AB context if and only if R is Gorenstein.

Cohen-Macaulay approximation We say that an R-module K is pointwise dualizing, if K is R-finite, $\operatorname{Ext}_R^i(K, K) = 0$ for $i > 0$, the canonical map $R \to \operatorname{Hom}_R(K, K)$ is isomorphic, and for any prime ideal $\mathfrak{p}$ of R, $K_{\mathfrak{p}}$ is of finite injective dimension. We say that an R-module M is maximal Cohen-Macaulay (MCM, for short), if M is R-finite, and for any $\mathfrak{p} \in \operatorname{Spec} R$, it holds $\operatorname{depth} M_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$. We assume that a pointwise dualizing module K_R exists. This implies that R is Cohen-Macaulay [51]. Now we set:

$$\begin{aligned} \mathcal{X}_R^{\mathrm{cm}} &:= \{\text{MCM } R\text{-modules}\}, \\ \mathcal{Y}_R^{\mathrm{cm}} &:= \{Y \in \mathcal{A}_R \mid \operatorname{inj.dim}_{R_\mathfrak{p}} Y_\mathfrak{p} < \infty \quad (\mathfrak{p} \in \operatorname{Spec} R)\}, \\ \omega_R^{\mathrm{cm}} &:= \operatorname{add} K_R. \end{aligned}$$

As a matter of fact, $(\mathcal{X}_R^{\mathrm{cm}}, \mathcal{Y}_R^{\mathrm{cm}}, \omega_R^{\mathrm{cm}})$ is an AB-context [5]. It follows that ω_R^{cm} is independent of the choice of K_R (see [25]). An $\mathcal{X}_R^{\mathrm{cm}}$-approximation is called a *Cohen-Macaulay approximation.*

11 Split reductive group

Soon after the works of Auslander-Buchweitz [5] and Auslander-Reiten [6], another important example of Auslander-Buchweitz context was found in representation theory by Ringel. His result was proved as a theorem on quasi-hereditary algebras [48], and applied to the representations of reductive groups by Donkin [20].

In this section, we briefly review the theory of split reductive groups.

Let R be a noetherian commutative ring.

Definition 11.1 An R-complex

$$\mathbb{F} : 0 \to F^0 \xrightarrow{\delta^0} F^1 \xrightarrow{\delta^1} F^2 \xrightarrow{\delta^2} \cdots$$

is said to be *universally acyclic* if $H^i(\mathbb{F} \otimes M) = 0$ $(i > 0)$, and the canonical map

$$\rho_M : H^0(\mathbb{F}) \otimes M \to H^0(\mathbb{F} \otimes M)$$

is an isomorphism for any R-module M.

Lemma 11.2 *Assume that* $\operatorname{gl.dim} R \leq 1$. *If* $\mathbb{F}$ *is an* R*-flat complex with* F^0 *being* R*-projective, then the following are equivalent.*

1 $\mathbb{F}$ *is universally acyclic, and* $H^0(\mathbb{F})$ *is* R*-finite projective*

1' $\mathbb{F}$ *is acyclic, and* $H^0(\mathbb{F})$ *is* R*-finite*

2 *For any* $\mathfrak{p} \in \operatorname{Spec} R$, *it holds that* $H^i(\mathbb{F} \otimes \kappa(\mathfrak{p})) = 0$ $(i > 0)$, *and* $\dim_{\kappa(\mathfrak{p})} H^0(\mathbb{F} \otimes \kappa(\mathfrak{p})) < \infty$ *is a locally constant function on* $\operatorname{Spec} R$.

Let k be an algebraically closed field. A k-group scheme G is said to be *reductive*, if it is non-trivial, an affine variety as a k-scheme, and the radical (maximal connected normal solvable subgroup of G) is a torus (a k-group scheme which is isomorphic to $\mathbb{G}_m^n$ for some $n \geq 0$), see [31].

Let R be a noetherian commutative ring again. An R-group scheme G is called *reductive*, if it is flat, affine and has reductive geometric fibers. Clearly, a reductive R-group scheme is R-smooth and has connected geometric fibers.

G is called *split reductive*, if it is reductive, and it has a closed subgroup T such that:

1 T is a split maximal torus of G. That is to say, $T \cong \mathbb{G}_m^n$ for some $n \geq 0$, and any geometric fiber $k \otimes T$ of T is a maximal torus of $k \otimes G$.

2 As G is R-smooth, $\operatorname{Lie} G := (I/I^2)^*$ is an R-finite projective G-module, induced by the adjoint action of G on itself. As T is a split torus, $\operatorname{Lie} G$ as a T-module is an $X(T)$-graded R-module. We set

$$\Sigma_G := \{\alpha \in X(T) \mid \alpha \neq 0, \quad (\operatorname{Lie} G)_\alpha \neq 0\}. \tag{4}$$

What we require is, $(\operatorname{Lie} G)_\alpha$ is (rank-one) R-free for $\alpha \in \Sigma_G$.

A reductive group over an algebraically closed field is split reductive. More generally, a reductive group over a strictly Henselian local ring (i.e., a Henselian local ring whose residue field is separably closed) is split reductive.

General linear group GL_n, special linear group SL_n, a split torus, symplectic group Sp_{2n}, special orthogonal group SO_n, and their direct products are typical examples of split reductive groups.

Let G be an R-split reductive group. Then, an element in Σ_G in (4) is called a *root* of G. The set of roots Σ_G of G is an abstract root system [31], and we fix a base $\Delta(G)$ of Σ_G. Then, the set of *dominant weights* $X_G^+ \subset X(T)$, positive roots Σ_G^+ are determined. Note that we have $\Sigma_G = \Sigma_G^+ \coprod \Sigma_G^-$, where $\Sigma_G^- := -\Sigma_G^+$ is the set of *negative roots*.

The set of weights $X(T)$ is an ordered set by the *dominant order*. We say that $\lambda \geq \mu$ for $\lambda, \mu \in X(T)$ if $\lambda - \mu$ is a linear combination of $\Delta(G)$, with coefficients in the set of non-negative integers. Note that $\Sigma_G^+ = \{\alpha \in \Sigma_G \mid \alpha > 0\}$.

For each $\alpha \in \Sigma_G$, the *root subgroup* U_α is determined [33]. We denote the subgroup of G generated by U_α for all $\alpha \in \Sigma_G^-$ by U. We set $B := TU$.

What is important is, not only that a split reductive group G is defined over the ring of integers $\mathbb{Z}$, but the construction above are all done over $\mathbb{Z}$, see [17, 33]. In particular, any split reductive group is IFP by Corollary 5.11. For any reductive group G over R and any geometric point x of $\operatorname{Spec} R$, there is an étale neighborhood $V = \operatorname{Spec} R'$ of x such that $R' \otimes G$ is R'-split. Thus, as remarked in [52], we have:

Proposition 11.3 *Assume that R is Nagata. Let G be a reductive R-group scheme, B a G-algebra which is of finite type over R, and $M \in {}_{G,B}\mathbb{M}_f$. Then, we have that the invariance B^G is an R-algebra of finite type, and M^G is a B^G-finite module.*

When $G = GL_n$, the only case what we need later for application, the description of items appeared above is done very explicitly as follows:

Example 11.4 Let $G = GL_n$.

1 As T, we may and shall take the subgroup of diagonal matrices in GL_n, so we have $X(T) \cong \mathbb{Z}^n$. With this identification, the character

$$\begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \cdots & \\ & & & t_n \end{bmatrix} \mapsto t^\lambda = t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_n^{\lambda_n}$$

corresponds to $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n$.

2 With the identification above, we have

$$\Sigma_G = \{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i, j \leq n, \quad i \neq j\},$$

where $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 at the ith entry. We may and shall fix the base

$$\Delta(G) = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1} \,|\, 1 \leq i < n\}.$$

Then, we have

$$\Sigma^+ = \{\varepsilon_i - \varepsilon_j \,|\, i < j\},$$

and

$$X^+ = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n \,|\, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}.$$

3 The subgroup U is the subgroup of unipotent lower triangular matrices (lower triangular matrices with all diagonal entries 1). Note that $B = UT$ is the subgroup of invertible lower triangular matrices.

4 In general, the Weyl group $W(G) := N_G(T)/T$ acts on $X(T)$. In our case $G = GL_n$, we have $W(G) = \mathfrak{S}_n$, the nth symmetric group, and the action on $X(T) = \mathbb{Z}^n$ is the canonical permutation. In general, $W(G)$ has the *longest element* w_0, which interchanges Σ_G^- and Σ_G^+. In our case, $w_0 \in \mathfrak{S}_n$ is given by $w_0(i) := n + 1 - i$.

We return to the general situation. As B is a semidirect product of U and T with U normal, any T-module is a B-module, letting U act trivially. The rank-one free B module obtained from $\lambda \in X(T)$ is denoted by R_λ.

Definition 11.5 For $\lambda \in X^+$, we denote $\mathrm{ind}_B^G R_\lambda$ by $\nabla_G(\lambda)$ and call it the *induced module* of highest weight λ, where $\mathrm{ind}_B^G = \mathrm{ind}_{R[B]}^{R[G]}$ is the right adjoint of the canonical restriction $\mathrm{res}_B^G = \mathrm{res}_{R[B]}^{R[G]}$, see section 2 and [33]. We denote $\nabla_G(-w_0\lambda)^*$ by $\Delta_G(\lambda)$, and call it the *Weyl module* of highest weight λ.

Lemma 11.6 *Let $\lambda \in X^+$. Then, the following hold.*

1 (Kempf's vanishing). *We have $R^i \mathrm{ind}_B^G(R_\lambda) = 0$ for $i > 0$.*

2 (Universal freeness). *We have $\nabla_G(\lambda)$ is R-finite free. If R' is a commutative R-algebra, then we have that the canonical map $R' \otimes \nabla_G(\lambda) \to \nabla_{R'\otimes G}(\lambda)$ is an isomorphism.*

For the proof, we may assume that $R = \mathbb{Z}$, and we utilize Lemma 11.2, using Theorem 5.9, Kempf's vanishing over a field, and Weyl's character formula over a field [33]. By the lemma, we have $\Delta_G(\lambda)$ is also universally free.

12 Donkin's Schur algebra and good filtrations over arbitrary base

In this section, we generalize some part of the theory of good filtrations of representations of reductive groups, including Ringel's approximation in the version of reductive group representations. Ringel's approximation, originally proved as a theorem on quasi-hereditary algebras, is another important example of (weak) AB contexts.

An ordered set is called a *poset*, for short. Let P be a poset. A subset Q of P is called a *poset ideal* of P if $\lambda \in Q$, $\mu \in P$, and $\mu \leq \lambda$ together imply $\mu \in Q$.

Lemma 12.1 *Let P be a poset. Then, the following are equivalent.*

1 *P is finite, or there is an order-preserving bijective map $f : P \to \mathbb{N}$, where $\mathbb{N}$ is the well-ordered set of positive integers.*

2 *P is countable, and $(-\infty, \lambda] := \{\mu \in P \mid \mu \leq \lambda\}$ is finite for any $\lambda \in P$.*

We say that a poset P is of ω-type, if the equivalent conditions in the lemma is satisfied. Any subset of a ω-type poset is of ω-type again.

Let R be a commutative noetherian ring, and G an R-split reductive group with a fixed data T and $\Delta(G) \subset X(T)$ defined over $\mathbb{Z}$. Restricting the order of $X(T)$ to X^+, X^+ is an ordered set. A *saturated subset* of X^+ is a poset ideal of X^+ by definition. Note that the poset X^+ is of ω-type [32].

Definition 12.2 Let V and W be G-modules. We say that W is *u-acyclic* with respect to V if for any R-module M, the following hold:

1 $\operatorname{Ext}^i_G(V, W \otimes M) = 0$ for $i > 0$

2 The canonical map $\operatorname{Hom}_G(V, W) \otimes M \to \operatorname{Hom}_G(V, W \otimes M)$ is an isomorphism.

We say that a G-module V is *u-good* if V is u-acyclic with respect to $\Delta_G(\lambda)$ for any $\lambda \in X^+$.

Note that if the condition in **2** in the definition holds, then $\operatorname{Hom}_G(V, W)$ is a pure R-submodule of $\operatorname{Hom}(V, W)$. The following is a sort of universal coefficient theorem.

Proposition 12.3 *Let V be an R-finite projective G-module, W an R-flat G-module, and M an R-module of finite flat dimension. Then, there is a spectral sequence*

$$E_2^{p,q} = \operatorname{Tor}^R_{-p}(M, \operatorname{Ext}^q_G(V, W)) \Rightarrow \operatorname{Ext}^{p+q}_G(V, M \otimes W).$$

Corollary 12.4 *Let V be an R-finite projective G-module, and W an R-flat G-module. If $\operatorname{Ext}^i_G(V, W) = 0$ for $i > 0$ and R is regular, then we have that W is u-acyclic with respect to V, and $\operatorname{Hom}_G(V, W)$ is R-flat.*

Lemma 12.5 *Let $\lambda, \mu \in X^+$. Then, the following hold:*

1 $\nabla_G(\mu)$ *is u-good.*

2 $\operatorname{Hom}_G(\Delta_G(\lambda), \nabla_G(\mu)) = 0$ *unless $\mu = \lambda$.*

3 $\operatorname{Hom}_G(\Delta_G(\mu), \nabla_G(\mu)) \cong R$*, as an R-module.*

This lemma is known as Cline-Parshall-Scott-v.d.Kallen vanishing over a field, see [33]. The general case is proved utilizing Lemma 11.2. Using this lemma, we can construct a nice family of subcoalgebras of $R[G]$.

Theorem 12.6 *There exists a unique family of R-subcoalgebras $(C(\pi))_\pi$ of $R[G]$, parameterized by all finite poset ideals of X^+, subject to the following conditions:*

a $C(\emptyset) = 0$

b *If $\pi' \supset \pi$, then $C(\pi') \supset C(\pi)$.*

c $\varinjlim C(\pi) = R[G]$

d *For any finite poset ideal π of X^+ and any maximal element λ of π, the cokernel of the inclusion map $C(\pi - \{\lambda\}) \hookrightarrow C(\pi)$ is isomorphic to $\nabla_G(-w_0\lambda) \otimes \nabla_G(\lambda)$ as a $G \times G$-module, where we regard $R[G]$ as a $G \times G$-module by $((g, g')f)(g'') := f(g^{-1}g''g')$ for $g, g', g'' \in G$ and $f \in R[G]$ (two-sided regular representation), and $C(\pi - \{\lambda\})$, $C(\pi)$ as its submodules.*

It is easy to see that $C(\pi)$ is R-finite free for any finite poset ideal π of X^+. We define the Schur algebra of G with respect to π as the dual algebra $C(\pi)^*$, and denote it by $S_G(\pi)$. The proof of the theorem depends on Donkin's construction over a field [19], and use Lemma 11.2 again (the construction is done over $\mathbb{Z}$, then we change the base). On the other hand, Donkin (loc. cit.) constructed the Schur algebra over arbitrary base using hyperalgebra method.

For a (possibly infinite) poset ideal π of X^+, we set $C(\pi) := \varinjlim C(\rho)$, where ρ runs through all finite poset ideals of π. It is easy to see that $C(\pi)$ is an R-subcoalgebra of $R[G]$. A $C(\pi)$-comodule, viewed as a G-module, is called a G-module which *belongs to* π.

For a poset ideal π of X^+, we set $\Delta_G(\pi) := \{\Delta_G(\lambda) \mid \lambda \in \pi\}$, and $\nabla_G(\pi) := \{\nabla_G(\lambda) \mid \lambda \in \pi\}$.

By the condition **d** in Theorem 12.6, we see that $C(\pi) \in \mathcal{F}(\nabla_G(\pi))$ for a finite poset ideal π of X^+. In particular, $C(\pi)$ is u-good. Combining this fact with a simple observation $\mathrm{ind}_{R[G]}^{C(\pi)} = \varinjlim \mathrm{ind}_{R[G]}^{C(\rho)}$ for any poset ideal π of X^+, we have that $C(\pi)$ is u-good for any poset ideal π of X^+.

Note that $\mathrm{ind}_{R[G]}^{C(\pi)} \cong \mathrm{Hom}_G(S_G(\pi), ?)$ for finite π. The following for the case R is a field are in [19].

Proposition 12.7 *Let π be a poset ideal of X^+, and M a $C(\pi)$-comodule. Then, we have $R^i \mathrm{ind}_{R[G]}^{C(\pi)}(M) = 0$ for $i > 0$.*

Corollary 12.8 *For any $C(\pi)$-comodules, the canonical map*

$$\mathrm{Ext}^i_{C(\pi)}(M, N) \to \mathrm{Ext}^i_G(M, N)$$

is an isomorphism.

Corollary 12.9 *If M and N are R-finite G-modules, then $\mathrm{Ext}^i_G(M, N)$ is R-finite.*

Let π be a finite poset ideal of X^+. Now it is not so difficult to show that $S_G(\pi)$ is a quasi-hereditary R-algebra of split type, in the sense of [16].

Let P be a finite poset. The rank of P, denoted by $\operatorname{rank} P$, is $r - 1$, where r is the maximum cardinality of totally ordered subsets in P. Utilizing the theory of quasi-hereditary algebras, we can show the following:

Corollary 12.10 *Let π be a finite poset ideal of X^+, r a non-negative integer, and M an R-finite $S_G(\pi)$-module with* $\operatorname{proj.dim}_R M \leq r$. *Then, we have* $\operatorname{proj.dim}_{S_G(\pi)} M \leq r + 2 \operatorname{rank} \pi$.

Proposition 12.11 *For a G-module V, the following are equivalent.*

1 *For any $\lambda \in X^+$, it holds* $\operatorname{Ext}^1_G(\Delta_G(\lambda), V) = 0$.

1' *For any $\lambda \in X^+$ and $i > 0$, it holds* $\operatorname{Ext}^i_G(\Delta_G(\lambda), V) = 0$.

2 *For any finite poset ideal π, it holds* $R^1 \operatorname{ind} R^{C(\pi)}_{R[G]} V = 0$.

2 *For any poset ideal π and $i > 0$, it holds* $R^i \operatorname{ind} R^{C(\pi)}_{R[G]} V = 0$.

3 *There exists a filtration $0 = V_0 \subset V_1 \subset \cdots$ of V such that:*

> **a** $\varinjlim V_i = V$
>
> **b** *There exists a bijective map $f : \mathbb{N} \to X^+$ such that f^{-1} preserves the ordering, and that for any $i \geq 1$, there is an isomorphism*
>
> $$V_i/V_{i-1} \cong \nabla_G(f(i)) \otimes \operatorname{Hom}_G(\Delta_G(f(i)), V).$$

We say that V is *good* if the equivalent conditions in the proposition hold. This notion was originated by S. Donkin, as modules with good filtrations. When R is a field, we say that V has a good filtrations if V is good and V is of countable dimension (cf. [22]).

Note that the category of good G-modules agrees with $\Delta_G(X^+)^\perp$ in ${}_G\mathbb{M}$. Obviously, a u-good module is good. An example shows that the converse is not true. However, if R is regular and V is an R-flat good module, then it is u-good, by Corollary 12.4. A good module is u-good if and only if the filtration in the proposition can be taken so that each V_i is an R-pure submodule of V.

The following is proved using Ringel's result and method in [48].

Proposition 12.12 *Let R be a field or a principal ideal domain. Let π be a finite poset ideal of X^+, λ a maximal element of π, and V an R-finite free G-module which is u-acyclic with respect to $\Delta_G(\mu)$ for any $\mu \in X^+ \setminus \pi$. Then, we have* $\operatorname{Ext}^i_G(\Delta_C(\lambda), V) = 0$ *for $i \geq 2$. Moreover, there is an exact sequence*

$$0 \to V \to V' \to D \to 0$$

of G-modules in which D is a finite direct sum of $\Delta_G(\lambda)$, and V' is u-acyclic with respect to any $\Delta_G(\mu)$ for $\mu \in X^+ \setminus (\pi \cup \{\lambda\})$.

Utilizing this proposition, we have the following:

Lemma 12.13 *Let R be a principal ideal domain or a field. For $\lambda \in X^+$, there is an exact sequence*

$$0 \to \Delta_G(\lambda) \to T \to D \to 0 \tag{5}$$

in which $D \in \mathcal{F}(\Delta_G((-\infty, \lambda)))$ and T is u-good. If R is local (a DVR or a field) moreover, then we can take T so that $k \otimes T$ is indecomposable as a $k \otimes G$-module for any field extension k of the residue field $R/\mathfrak{m}$ of R.

Note that we have the exact sequence of the form (5) over arbitrary R, as we can base change the one over $\mathbb{Z}$ to R.

Now we set $\mathcal{A}_G := {}_G\mathbb{M}_f$, the category of R-finite G-modules. We also set:

$$\mathcal{Y}_G := \{V \in \mathcal{A}_G \mid V \text{ is good, and } \operatorname{proj.dim}_R V < \infty\},$$

$\mathcal{X}_G := \operatorname{add} \mathcal{F}(\Delta_G(X^+))$, and $\omega_G := \mathcal{X}_G \cap \mathcal{Y}_G$.

Theorem 12.14 *The triple $(\mathcal{X}_G, \mathcal{Y}_G, \omega_G)$ is a weak Auslander-Buchweitz context in $\mathcal{A}_G$, and we have*

$$\hat{\mathcal{X}}_G = \{V \in \mathcal{A}_G \mid \operatorname{proj.dim}_R V < \infty\}.$$

In particular, $(\mathcal{X}_G, \mathcal{Y}_G, \omega_G)$ is an Auslander-Buchweitz context if and only if R is regular.

Corollary 12.15 *For an R-finite projective G-module V, the following are equivalent.*

1 *V is u-good.*

2 *V is good.*

3 *For any maximal ideal $\mathfrak{m}$ of R, $V \otimes \kappa(\mathfrak{m})$ is good as a $G \otimes \kappa(\mathfrak{m})$-module.*

4 *$V^* \in \mathcal{X}_G$.*

Definition 12.16 An object in ω_G is called a *tilting module* of G.

For the origin of the name 'tilting,' see [20]. If R is a field, then a G-module T is tilting if and only if T is a direct summand of a finite dimensional tilting-cotilting module (in the sense of [42]) of $S_G(\pi)$ for some finite poset ideal π of X^+, see [48].

Let $(R, \mathfrak{m})$ be local. Then, by Lemma 12.13, we have a base-changed $\mathcal{Y}_G$-hull (5) for any $\lambda \in X^+$, in which $\hat{R} \otimes T$ is indecomposable as an $\hat{R} \otimes G$-module, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R. It is easy to see that $\operatorname{End}_G T$ is local (we recall that T is u-good). This shows that (5) is minimal, and such a T is unique up to isomorphisms. We denote T in the exact sequence (5) by $T_G(\lambda)$, and call the indecomposable tilting module of highest weight λ. If $R \to R'$ is a local homomorphism of local rings, then we have $T_{R' \otimes G}(\lambda) \cong R' \otimes T_G(\lambda)$. Hence, $T_G(\lambda)$ depends only on the characteristic of the residue field, essentially.

Theorem 12.17 *Let $(R, \mathfrak{m})$ be a local ring. Then, $\hat{\mathcal{X}}_G$ has both minimal $\mathcal{X}_G$-approximations and minimal $\mathcal{Y}_G$-hulls. Any tilting G-module is uniquely a direct sum of $T_G(\lambda)$'s.*

The proof is reduced to the complete case, in which the Krull-Schmidt theorem holds and Proposition 10.2 can be used.

13 Relative Ringel's approximation

Let R be a commutative noetherian ring, and G an R-split reductive group. We assume that there is a fixed subgroup $\mathbb{G}_m \hookrightarrow Z(G)$, where $Z(G)$ is the center of G. By assumption, any G-module M is a $\mathbb{G}_m$-module, and we have a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$. By the assumption $\mathbb{G}_m \subset Z(G)$, we have M_i is a G-submodule of M for any i. We shall always consider the grading so obtained for a G-module M.

For a split reductive group G_0, when we set $G := G_0 \times \mathbb{G}_m$ and consider the natural inclusion $\mathbb{G}_m \hookrightarrow Z(G)$, then G is split reductive, and the category of G-modules is canonically isomorphic to the category of $\mathbb{Z}$-graded G_0-modules.

Let S be a G-algebra, flat of finite type over R. We assume that the graded R-algebra S is *positively graded.* That is to say, $S = \bigoplus_{i \geq 0} S_i$ and the canonical map $R \to S_0$ is isomorphic.

We say that a poset ideal π of X^+ is *t-closed* if $C(\pi)$ is an R-subalgebra (hence is a subbialgebra) of $R[G]$. We fix a t-closed poset ideal π of X^+.

We set $\mathcal{A}_{G,S}(\pi)$ to be the full-subcategory of ${}_{G,S}\mathbb{M}_f$ consisting of the objects which belong to π. We also define:

$$\begin{aligned} \mathcal{X}^{\mathrm{pro}}_{G,S}(\pi) &:= \{M \in \mathcal{A}_{G,S}(\pi) \mid M \text{ is } S\text{-projective and } \mathrm{Hom}_S(M,S) \text{ is good}\}, \\ \mathcal{Y}^{\mathrm{pro}}_{G,S}(\pi) &:= \{N \in \mathcal{A}_{G,S}(\pi) \mid N \text{ is good, and } \mathrm{proj.dim}_S N < \infty\}, \end{aligned}$$

and $\omega^{\mathrm{pro}}_{G,S}(\pi) := \mathcal{X}^{\mathrm{pro}}_{G,S}(\pi) \cap \mathcal{Y}^{\mathrm{pro}}_{G,S}(\pi)$. When $\pi = X^+$, we omit (π), and simply denote by $\mathcal{A}_{G,S}$, $\mathcal{X}^{\mathrm{pro}}_{G,S}$, and so on.

Theorem 13.1 *Assume that $S \in \omega^{\mathrm{pro}}_{G,S}(\pi)$, that is to say, S is good and belongs to π. Then, $(\mathcal{X}^{\mathrm{pro}}_{G,S}(\pi), \mathcal{Y}^{\mathrm{pro}}_{G,S}(\pi), \omega^{\mathrm{pro}}_{G,S}(\pi))$ is a weak Auslander-Buchweitz context. Moreover, we have:*

$$\begin{aligned} \hat{\mathcal{X}}^{\mathrm{pro}}_{G,S}(\pi) &= \{M \in \mathcal{A}_{G,S}(\pi) \mid \mathrm{proj.dim}_S M < \infty\}, \\ \omega^{\mathrm{pro}}_{G,S}(\pi) &= \{T \in \mathcal{A}_{G,S} \mid \exists\ T_0 \in \omega^{\mathrm{pro}}_{G,R}(\pi) \text{ such that } T \cong S \otimes T_0\}. \end{aligned}$$

In particular, this weak AB context is an AB context if and only if S is regular (by Theorem 9.1).

Note that Theorem 13.1 is a canonical extension, or a relative version of Theorem 12.14. For the proof, the following theorem due to O. Mathieu [37] is important.

Theorem 13.2 (O. Mathieu) *If V and W are u-good G-modules, then so is $V \otimes W$.*

It is easy to see that we may assume that R is an algebraically closed field, $V = \nabla_G(\lambda)$, and $W = \nabla_G(\mu)$, for the proof. This theorem was proved for the case the characteristic of R is not two, or G does not have any component of type E_7 or E_8 by S. Donkin [18], and finally proved by O. Mathieu completely. Note also that a characteristic-free explicit construction of a good filtration of $\nabla_G(\lambda) \otimes \nabla_G(\mu)$ was done for GL_n by G. Boffi [8].

Proposition 13.3 *Assume that R is local, and S as in Theorem 13.1. Then, any object in $\omega^{\mathrm{pro}}_{G,S}(\pi)$ is uniquely a direct sum of (G, S)-modules of the form $S \otimes T_G(\lambda)$ with $\lambda \in \pi$. Moreover, $\hat{\mathcal{X}}^{\mathrm{pro}}_{G,S}(\pi)$ has minimal $\mathcal{X}^{\mathrm{pro}}_{G,S}(\pi)$-approximations and minimal $\mathcal{Y}^{\mathrm{pro}}_{G,S}(\pi)$-hulls.*

For an S-module M, we denote $\operatorname{ht} \operatorname{ann} M$, the height of the annihilator of M, by $\operatorname{codim} M$. The following is a consequence of Theorem 10.4.

Theorem 13.4 *Let S be as in Theorem 13.1, h a non-negative integer, and $M \in \mathcal{A}_{G,S}$, with $0 \neq M$ and $\operatorname{codim} M \geq h$. Then, the following are equivalent:*

1 *M is a good G-module which belongs to π, M is perfect of codimension h as an S-module, and $\operatorname{Ext}^h_S(M, S)$ is good.*

2 $\omega^{\mathrm{pro}}_{G,S}(\pi)$ -resol.dim$(M) = h$.

If R is local and M satisfies the conditions **1** *and* **2** *moreover, then there exists a $\omega^{\mathrm{pro}}_{G,S}(\pi)$-resolution $\mathbb{F}$ of M which satisfies the condition*

3 *If $\mathbb{G}$ is a (possibly infinite) $\omega^{\mathrm{pro}}_{G,S}(\pi)$-resolution of M, then $\mathbb{F}$ is a direct summand of $\mathbb{G}$, as a (G, S)-complex.*

Such an $\mathbb{F}$ is unique up to isomorphisms of (G, S)-complexes, and has length h.

14 Resolutions of Buchsbaum-Rim type

Let R be a noetherian commutative ring, and t, m, n positive integers with $t \leq m \leq n$. We set $V := R^m$, $W := R^n$, $G := GL(V) \times GL(W)$, and $S := \operatorname{Sym}(V \otimes W)$, where Sym denotes the symmetric algebra. As $V \otimes W$ is a G-module by $(g, g')(v \otimes w) = gv \otimes g'w$, S is a G-algebra. When we fix bases of V and W, then we have an R-algebra isomorphism $S \cong R[x_{ij}]_{1 \leq i \leq m,\, 1 \leq j \leq n}$. The ideal of S, generated by all t-minors of the matrix (x_{ij}) is denoted by I_t, and called a *determinantal ideal.* This ideal I_t is a G-ideal of S, and is independent of the choice of the bases of V and W. We set $A := S/I_t$. M. Hochster and J. A. Eagon [29] proved that I_t is perfect of codimension $(m-t+1)(n-t+1)$.

Consider the following problem.

Problem 14.1 *Construct a finite free S-resolution*

$$0 \to F_h \to \cdots \to F_1 \to S \to A \to 0 \tag{6}$$

which is also a (G,S)-resolution.

We call such a resolution *G-equivariant finite free resolution.* An advantage of G-equivariance is explained as follows.

Let X be a noetherian R-scheme, $\mathcal{V}$ and $\mathcal{W}$ locally free coherent sheaves over X of well-defined ranks m and n, respectively. Although we have not mentioned anything about representations of group schemes over non-affine schemes, there is a canonical functor $E : {}_G\mathbb{M} \to {}_{GL(\mathcal{V})\times GL(\mathcal{W})}\mathbb{M}$. We denote $E(M)$ by $M(\mathcal{V},\mathcal{W})$. What is important here is, $M(\mathcal{V},\mathcal{W})$ is a quasi-coherent $\mathcal{O}_X$-module, and that if M is R-finite (resp. R-finite projective), then it is coherent (resp. locally free). Hence, the resolution (6) is lift to a locally free resolution

$$0 \to F_h(\mathcal{V},\mathcal{W}) \to \cdots F_1(\mathcal{V},\mathcal{W}) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_Y \to 0,$$

where $\tilde{X}$ is the vector bundle $\operatorname{Hom}(\mathcal{V},\mathcal{W}^*)$, and Y is the zero of $\bigwedge^t \phi$, where ϕ is the generic map of $\tilde{X}$.

As we have a canonical inclusion $\mathbb{G}_m \hookrightarrow Z(G)$ given by $t \mapsto (t1_V, t1_W)$, any (G,S)-resolution is graded with respect to the associated grading. Note also that S is positively graded, with each x_{ij} of degree one.

If R contains the rationals, then the graded minimal S-free resolution, which is also a (G,S)-complex exists, uniquely up to (G,S)-isomorphisms. Such a resolution was constructed explicitly by A. Lascoux [35] and Pragacz-Weyman [46].

When $R=\mathbb{Z}$, less is known on equivariant resolutions. A graded minimal S-free resolution of A admits a G-equivariant structure when $t=1$ (the Koszul complex) or $t=m$ (the Eagon-Northcott complex, see [21] and [11]). There is no graded minimal free resolution of A when $2 \le t \le m-3$ [27, 50]. It is not known if there is a G-equivariant structure of the graded minimal S-free resolution of A when $1, m-3 < t < m$, although there is a graded minimal S-free resolutions this case [1, 28].

Recently, D. A. Buchsbaum and J. Weyman posed the following problem [12].

Problem 14.2 *For any t with $1 \le t \le m$, construct a finite free S-resolution* (6) *which satisfies the conditions:*

1 *$\mathbb{F}$ is G-equivariant*

2 *h is minimal as possible, namely, $h = (m - t + 1)(n - t + 1)$.*

3 *For any $i \geq 1$, $F_i \cong S \otimes T(i)$ with $T(i)$ being a finite direct sum of tensor products of G-modules of the form $\bigwedge^i V \otimes \bigwedge^j W$.*

Although the Eagon-Northcott resolution for $t = m$ does not satisfy the condition 3 in general, the non-minimal resolution for $t = m$, constructed by D. A. Buchsbaum [10] satisfies **1,2** and **3**. This resolution is called the Buchsbaum-Rim resolution, see [13, 14].

We consider a little bit milder condition, instead of **3**.

3' For any $i \geq 1$, $F_i \cong S \otimes T(i)$ with $T(i)$ being a direct summand of a finite direct sum of tensor products of G-modules of the form $\bigwedge^i V \otimes \bigwedge^j W$.

This is because good properties such as vanishing of cohomology are inherited by direct summands, and because the following lemma holds.

Lemma 14.3 *For a G-module T, the following are equivalent.*

a *T is a direct summand of a finite direct sum of tensor products of G-modules of the form $\bigwedge^i V \otimes \bigwedge^j W$.*

b *$T \in \omega^{\text{pro}}_{G,R}(\pi)$, where π is the set of pairs of partitions (λ, μ) with $\lambda_{m+1} = 0$ and $\lambda_{n+1} = 0$ (note that $\pi \subset X^+_{GL(V)} \times X^+_{GL(W)} = X^+_G$).*

We call a finite free S-resolution (6) is of Buchsbaum-Rim type, if it satisfies the conditions **1,2** and **3'**.

A G-module belongs to π in the lemma if and only if it is a polynomial representation. Or equivalently, $C(\pi)$ is the coordinate ring of $\mathrm{End}(V) \times \mathrm{End}(W)$. Hence, π is t-closed.

Theorem 14.4 *The following hold:*

a *S and I_t are good, and S belongs to π.*

b *$\mathrm{Ext}^{h_0}_S(A, S)$ is good, where $h_0 = (m - t + 1)(n - t + 1)$.*

The assertion **a** has been well-known, as the straightening formula [2]. The assertion **b** is proved using Kempf's construction.

Now Theorem 13.4 is applicable, and we have

Theorem 14.5 *There is a Buchsbaum-Rim type resolution of $A = S/I_t$ for any $1 \leq t \leq m$. If R is local, then there is a Buchsbaum-Rim type resolution $\mathbb{F}$ of A which satisfies the condition*

4 *If $\mathbb{G}$ is an S-free resolution which satisfies* **1** *and* **3'**, *then $\mathbb{F}$ is a direct summand of $\mathbb{G}$ as a (G,S)-complex,*

uniquely up to isomorphisms of (G,S)-complexes.

This is merely an existence theorem, and so far I do not have any idea to connect the conditions **3** and **3'**.

15 Equivariant Cohen-Macaulay approximation

As in section 13, let R be a noetherian commutative ring, G an R-split reductive group with a fixed $\mathbb{G}_m \subset Z(G)$, and S a positively graded G-algebra which is R-flat of finite type. We also assume that S is good, as a G-module.

We assume that S is regular in this section. It follows that R is regular, and S is R-smooth. Let I be a G-ideal of S, which is perfect of codimension h. We set $K_A := \operatorname{Ext}^h_S(A, \omega_{S/R})$, so that K_A is pointwise dualizing.

We define:

$$\begin{aligned}\mathcal{X}_{G,A} &:= \{M \in \mathcal{A}_{G,A} \mid M \in \mathcal{X}_A^{\mathrm{cm}}, \text{ and } \operatorname{Hom}_A(M, K_A) \text{ is good}\},\\ \mathcal{Y}_{G,A} &:= \{N \in \mathcal{A}_{G,A} \mid N \in \mathcal{Y}_A^{\mathrm{cm}}, \text{ and } \operatorname{Hom}_A(K_A, N) \text{ is good}\},\end{aligned}$$

and $\omega_{G,A} := \mathcal{X}_{G,A} \cap \mathcal{Y}_{G,A}$.

Theorem 15.1 *If A and K_A are good, then we have $(\mathcal{X}_{G,A}, \mathcal{Y}_{G,A}, \omega_{G,A})$ is an Auslander-Buchweitz context. Any object in $\omega_{G,A}$ is of the form $K_A \otimes T_0$, with T_0 tilting.*

If A is regular, then this theorem is essentially Theorem 13.1. Note that the goodness conditions are void, if G has linearly reductive (i.e., ${}_G\mathbb{M} = {}_G\mathbb{M}^{\perp}$) geometric fibers. In particular, if $G = \mathbb{G}_m$, then it is a graded version of Cohen-Macaulay approximation. The assumption of the theorem is satisfied for the case R is regular, when we consider the situation of section 14. Moreover, Pfaffian ideals also satisfy the assumption of the theorem, by the characteristic-free plethysm formula [34, 9]. The author does not know any other non-trivial examples.

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